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REPLACEMENT DECISIONS with MULTIPLE STOCHASTIC VALUES and DEPRECIATION

by

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HIGHLIGHTS

- **Consider operating costs, salvage value and tax depreciation.**

- **Earlier replacements if numerous replacement opportunities.**

- **Earlier replacement if salvage volatility increases.**

- **Deferred replacements if cost volatility increases.**

- **Our general model encompasses several other models.**

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REPLACEMENT DECISIONS with MULTIPLE STOCHASTIC VALUES and DEPRECIATION

Abstract

We develop an analytical real-option solution to the after-tax optimal timing boundary for a replaceable asset whose operating cost and salvage value deteriorate stochastically. We construct a general replacement model, from which seven other particular models can be derived, along with deterministic versions. We show that the presence of salvage value and tax depreciation significantly lowers the operating cost threshold that justifies (and thus hastens) replacement. Although operating cost volatility increases defer replacement, increases in the salvage value volatility hasten replacement, albeit modestly, while increases in the correlation between costs and salvage value defer replacement. Reducing the tax rate or depreciation lifetime, or allowing an investment tax credit, yield mixed results. These results are also compared with those of less complete models, and deterministic versions, showing that failure to consider several stochastic variables and taxation in the replacement process may lead to sub-optimal decisions.

1. Introduction

For assets with a significant second-hand market value, such as vehicles, earth moving equipment and aircraft, or a notable scrap value such as ships, salvage value may be a crucial ingredient to the replacement decision because of the cash flow implications. The analytical solution to the after-tax optimal timing boundary is developed for a replaceable asset characterized by a deteriorating and stochastic operating cost and salvage value. Since, at replacement, the after-tax salvage value for the incumbent plus any residual depreciation tax credits partly offset the re-investment cost, the replacement policy reflects the after-tax trade-off between the sacrificed value of the incumbent and the net benefits rendered by the succeeding asset.

From simulations on a deterministic NPV model, Robichek and Van Horne (1967) show that abandonment can significantly raise the project value because of the flexibility value embedded in the released funds. Enhancements are made by Dyl and Long (1969) by introducing a timing option, and by Gaumitz and Emery (1980) and Howe and McCabe (1983). In a stochastic dynamic programming formulation, Bonini (1977) models a stochastic operating cost and salvage value, explicitly.

There are various empirical studies of the parameter values used for replacement models. Rust (1987) derives the drifts of operating costs (including maintenance) from actual records, which are a deterministic function of time/age. Lai et al. (2000) fit various lifetime distributions (including normal) to maintenance records, but do not calibrate uncertainty. Keles and Hartman (2004) quantify operating costs, salvage value and investment costs discounted to time zero. Operating costs increase around 1-3% per year, salvage values decline linearly with age, investment costs are dependent on asset type, over a range of less than 8% for similar assets.

In a review of literature, Hartman and Tan (2014) note that there only a few models which consider stochastic deterioration in continuous-time. We observe that most real-option models which allow for stochastic variables, treat abandonment only implicitly. Salvage value and depreciation are interpreted by Mauer and Ott (1995) as functions of a stochastic operating cost as a way of reducing dimensionality to one, while Dobbs (2004) embeds the salvage value into a one-factor model. Ye (1990) allows combined maintenance and operating cost to follow an

arithmetic Brownian motion, with a fixed investment cost, no salvage value or depreciation. Yilmaz (2001) considers revenue produced by equipment to be stochastic, but maintenance to fix any equipment faults is deterministic. These simplifications yield an analytical solution, but any trade-offs or co-variation amongst the factors is entirely ignored.

A tractable solution for dealing with the two-stochastic-factor replacement model is developed by Adkins and Paxson (2011) that excludes depreciation and salvage value. Zambujal-Oliveira and Duque (2011) propose a two-factor model with stochastic operating cost and (autonomous) stochastic salvage value, with depreciation following a negative exponential function[1]. Two-factor models are proposed by Adkins and Paxson (2013a) and Adkins and Paxson (2013b), who consider the effect of three alternative depreciation schedules and technological progress on the replacement policy, respectively, but ignore salvage value.

In this paper, we formulate a three-factor real-option replacement model, based on operating cost, salvage value and depreciation, to investigate the effect on the replacement policy not only from including salvage value as a factor but also from their interactions. Since the solution to the replacement timing-boundary is quasi-analytical, it also solves the one- and two-factor derivative models. We show that changes in salvage value and tax depreciation can significantly alter the optimal replacement timing decision, but reductions in the tax rate or depreciation lifetime do not necessarily lead to earlier replacements. Moreover, while increases in the operating cost volatility and correlation with salvage value result in replacement deferral, an increase in salvage value volatility brings the replacement timing forward.

In summary, both S and D matter in replacements, which is intuitive, but C and S volatilities and correlation also matter, not considered in deterministic models. Our general model encompasses several other models, and enables easy comparisons of the results of different models. The number of possible replacements matters a lot, extending the approximate replacement timing from 25 to 38 years for our base case parameter values, as shown in Appendix, Part G.

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[1] Their negative exponential depreciation results in a more rapid depreciation schedule than a declining balance method, but it is not clear which tax systems allow this method. There is no allowance for recapture of any excess salvage value over the depreciated tax basis of the asset upon disposal, which could be significant for such a rapid schedule.

The rest of the paper is organized as follows. In Section 2, we develop a quasi-analytical method for identifying the after-tax optimal timing boundary for the three-factor replacement model. A numerical illustration provided in Section 3 reveals significant features of the model, which is extended through a sensitivity analysis in Section 4. Section 5 concludes and offers some suggestions for further research.

2. Replacement Opportunity with Salvage and Tax Depreciation

2.1 Valuation Function

We determine the real-option replacement policy for a durable productive asset, subject to input decay in a seemingly monopolistic situation whose output yields a constant revenue[2], assuming other flexibilities are inadmissible. Holding the asset remains optimal until, on an after-tax basis, the expected benefit of acquiring a successor net of replacement cost less any disposal value exceeds that from operating the incumbent. The relevant cash flows crucial to the replacement decision are those associated with the operating costs, the depreciation charge and the salvage value. While annual operating cost and salvage value, denoted by C and S, respectively, are treated as stochastic factors, the annual depreciation charge, denoted by D, is a deterministic factor. The replacement policy, represented by an optimal timing boundary separating the decision regions of continuance and replacement, is defined over a three-dimensional cost-salvage-depreciation (C-S-D) space. The tax rate τ is applicable to all cash flows, both positive and negative, and regardless of whether they represent income or capital gains. At replacement, the operating cost, salvage value and depreciation level for the newly installed succeeding asset are set to their known initial levels of C_I, S_I and D_I, respectively. The replacement re-investment cost is a known constant K. To avoid round-tripping, $S_I < K$. Asset re-investment is treated here as partly irreversible, since the firm recovers only a fraction of the original outlay if the asset is disposed at S. We assume that the revenue produced by the asset remains at a constant known level, with the restriction that it exceeds operating cost, thus insuring sufficient taxable income.

---

[2] It is straightforward to recast the model in terms of net revenue instead of operating costs.

The two uncertain factors are assumed to follow distinct geometric Brownian motion processes with drift. For $X \in \{C, S\}$:

$$dX = \alpha_X X dt + \sigma_X X dz_X, \qquad (1)$$

where α_X is the instantaneous drift rate, σ_X the instantaneous volatility rate, and dz_X is the increment of the standard Wiener process. Dependence between the two factors is described by the instantaneous covariance term $\rho\sigma_C\sigma_S$, $\mathrm{Cov}[dC, dS] = \rho\sigma_C\sigma_S CS dt$ with $|\rho| \leq 1$. As the asset efficiency deteriorates with usage and age, we assume that the expected operating cost change α_C is positive, measured as an annualized continuous rate; correspondingly, its salvage value declines with an expected change rate of $\alpha_S \leq, \geq 0$, depending on the salvage value characteristics. In contrast to previous formulations, salvage value is not directly tied to the revenue and/or operating cost of the asset, since different second-hand buyers in different countries may have little concern with the revenue or operating costs of the previous owner, and often salvage value reflects scrap value like in ships, rather than current use value.

The selected tax depreciation schedule is declining-balance, mainly because of its tractability[3]. This and alternative schedule forms are considered in a replacement setting by Adkins and Paxson (2013a). The depreciation level is described by the deterministic geometric process

$$dD = -\theta_D D dt, \qquad (2)$$

where $0 < \theta_D < 1$ is a known constant proportional depreciation rate. Being time dependent, the time elapsed since the last replacement, or the age of the incumbent, can be deduced directly from the value of D. The principal difference between the evolutionary forms of C and S compared with D is the absence of the volatility term in (2). If the re-investment cost K is fully

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[3] The MACRS (GDS) schedule in the U.S. is a declining-balance method until it is more beneficial to switch to straight line (when the asset is older, and may be considered for replacement). It is feasible (but complicated) to model alternative tax depreciation schedules such as straight-line and sum-of-years-digits.

depreciable for tax purposes[4], then $D_I = \theta_D K$. The after-tax capital gain/loss on disposal $(S - \tau(S - D/\theta_D))$, which is the gain/loss on S less the accumulated depreciation.[5]

The asset value together with its embedded replacement option depends on the prevailing factor levels and is denoted by $F_1 = F_1(C, S, D)$. By assuming complete markets, the risk-neutral valuation relationship is determined from standard contingent claims analysis, Dixit and Pindyck (1994), and expressed by:

$$
\begin{aligned}
&\tfrac{1}{2}\sigma_C^2 C^2 \frac{\partial^2 F_1}{\partial C^2} + \rho\sigma_C\sigma_S CS \frac{\partial^2 F_1}{\partial C \partial S} + \tfrac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 F_1}{\partial S^2} \\
&+ \theta_C C \frac{\partial F_1}{\partial C} + \theta_S S \frac{\partial F_1}{\partial S} - \theta_D D \frac{\partial F_1}{\partial D} - rF_1 + (P_I - C)(1 - \tau) + D\tau = 0,
\end{aligned}
\tag{3}
$$

where $r > 0$ is the constant risk-free rate of interest, P_I is the revenue assumed to be constant, and θ_C and θ_S are the respective risk-neutral drift rates, assumed to be equal to the α drift rates. We assume that $r - \theta_X > 0$. Following Adkins and Paxson (2011), the function satisfying (3) is:

$$
F_1 = A_1 C^{\eta_1} S^{\gamma_1} D^{\lambda_1} + \frac{P_I(1 - \tau)}{r} - \frac{C(1 - \tau)}{r - \theta_C} + \frac{D\tau}{r + \theta_D}.
\tag{4}
$$

In (4), the expression $A_1 C^{\eta_1} S^{\gamma_1} D^{\lambda_1} \geq 0$ represents the replacement option value, so $A_1 \geq 0$, while the remainder represents the asset value in the absence of optionality. Since the likelihood of exercise is positively related to individual increases in C, S, D, so all of the power parameter values $\eta_1, \gamma_1, \lambda_1$ are positive, see Adkins and Paxson (2011). Substituting (4) in (3) yields the characteristic root equation:

$$
\begin{aligned}
Q_1(\eta_1, \gamma_1, \lambda_1) = &\tfrac{1}{2}\sigma_C^2 \eta_1(\eta_1 - 1) + \rho\sigma_C\sigma_S \eta_1\gamma_1 + \tfrac{1}{2}\sigma_S^2 \gamma_1(\gamma_1 - 1) \\
&+ \theta_C\eta_1 + \theta_S\gamma_1 - \theta_D\lambda_1 - r = 0.
\end{aligned}
\tag{5}
$$

---

[4] This assumes there is no bonus or special depreciation, or investment tax credit, or requirement to estimate a residual salvage value (especially since that is stochastic), which could reduce the depreciation base.

[5] This is consistent with US tax on "excess salvage value" on certain assets, see Appendix, Part F. See Hartman and Hartman (2001) for certain other "trade-in" replacement assets, where the new K tax basis is reduced by the excess salvage value, creating a challenging context for analysis, since the net K is no longer constant.

Replacement is optimally triggered when the factor levels C, S, D attain their threshold levels $\hat{C}_1, \hat{S}_1, \hat{D}_1$, respectively, where $\hat{C}_1 \geq C_I, \hat{S}_1 \leq S_I, \hat{D}_1 \leq D_I$. This occurs when at exercise, the incumbent value and the successor value less the replacement cost net of salvage value and any depreciation recapture are in exact balance, eliminating the constant P_I from both sides:

$$F_1\left(\hat{C}_1, \hat{S}_1, \hat{D}_1\right) = F_1\left(C_I, S_I, D_I\right) + (1-\tau)\hat{S}_1 + \hat{D}_1\tau / \theta_D - K,$$

or explicitly, the value matching relationship is:

$$
\begin{aligned}
A_1 \hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} \hat{D}_1^{\lambda_1} &- \frac{\hat{C}_1(1-\tau)}{r-\theta_C} + \frac{\hat{D}_1\tau}{r+\theta_D} \\
&= A_1 C_I^{\eta_1} S_I^{\gamma_1} D_I^{\lambda_1} - \frac{C_I(1-\tau)}{r-\theta_C} + \frac{D_I\tau}{r+\theta_D} + (1-\tau)\hat{S}_1 + \hat{D}_1\tau / \theta_D - K.
\end{aligned}
\tag{6}
$$

Optimality is assured by the smooth-pasting conditions, one for each factor C, S, D, which can be expressed in a reduced form by:

$$A_1 \hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} \hat{D}_1^{\lambda_1} = \frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)} > 0, \tag{7}$$

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)} = \frac{\hat{S}_1(1-\tau)}{\gamma_1} > 0, \tag{8}$$

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)} = \frac{\hat{D}_1\tau r}{\lambda_1\theta_D(r+\theta_D)} > 0. \tag{9}$$

(7), (8) and (9) demonstrate our conjecture that $\eta_1, \gamma_1, \lambda_1 \geq 0$. Using (7) to eliminate A_1, (6) becomes:

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)}\left[\eta_1 + \gamma_1 + \lambda_1 - 1 + \frac{C_I^{\eta_1} S_I^{\gamma_1} D_I^{\lambda_1}}{\hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} \hat{D}_1^{\lambda_1}}\right] = K + \frac{C_I(1-\tau)}{r-\theta_C} - \frac{D_I\tau}{r+\theta_D}. \tag{10}$$

The C-S-D model is composed of four simultaneous equations: the reduced form value matching relationship (10), two reduced form smooth pasting conditions (8) and (9), and the characteristic

root equation (5). The optimal timing boundary, denoted by $G_1\left(\hat{C}_1, \hat{S}_1, \hat{D}_1\right) = 0$, can be determined by eliminating the parameters η_1, γ_1 and λ_1 from the composite model made up of these four equations, or alternatively solving simultaneously four equations for $\hat{C}_1, \eta_1$, γ_1 and λ_1, given assumptions about $\hat{S}_1$ and $\hat{D}_1$ (see the Appendix, Part G).

If we conjecture that both γ_1, λ_1 are small, $C_I^{\eta_1} S_I^{\gamma_1} D_I^{\lambda_1} < \hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} \hat{D}_1^{\lambda_1}$, and then (10) becomes:

$$\frac{\left(\hat{C}_1 - C_I\right)(1-\tau)}{r - \theta_C} + \frac{\left(D_I - \hat{D}_1\right)\tau}{r + \theta_D} \geq K - (1-\tau)\hat{S}_1 - \hat{D}_1\tau / \theta_D. \tag{11}$$

This implies that for the C-S-D model, an optimal replacement occurs only if on an after-tax basis, the net benefits from acquiring a successor exceed the replacement cost net of salvage value and recapture of depreciation.

2.2 Single Replacement Opportunity

If there exists only one available remaining replacement opportunity, then the value-matching relationship for the multiple replacement model has to be amended to exclude replacement option value for the succeeding asset to become:

$$\begin{aligned} A_{1s} \hat{C}_{1s}^{\eta_{1s}} \hat{S}_{1s}^{\gamma_{1s}} \hat{D}_{1s}^{\lambda_{1s}} &- \frac{\hat{C}_{1s}(1-\tau)}{r - \theta_C} + \frac{\hat{D}_{1s}\tau}{r + \theta_D} \\ &= -\frac{C_I(1-\tau)}{r - \theta_C} + \frac{D_I\tau}{r + \theta_D} + (1-\tau)\hat{S}_{1s} + \hat{D}_{1s}\tau / \theta_D - K, \end{aligned} \tag{12}$$

where the subscript s refers to the single replacement opportunity. Since the smooth-pasting conditions are identical to (7)-(9), except for the inclusion of the subscript s, the reduced form value-matching relationship is obtained by eliminating A_{1s} from (12):

$$\frac{\hat{C}_{1s}(1-\tau)}{\eta_{1s}(r-\theta_C)}[\eta_{1s}+\gamma_{1s}+\lambda_{1s}-1]=K+\frac{C_I(1-\tau)}{r-\theta_C}-\frac{D_I\tau}{r+\theta_D}. \tag{13}$$

This reveals that for a single replacement to be economically justified, the after-tax operating cost threshold has to exceed the re-investment cost plus the after-tax operating cost value for the replica less its depreciation tax shield value. In theory the difference between multiple and single replacement thresholds comparing (10) with (13), so $\hat{C}_{1s} \geq \hat{C}_1$, is due to $\frac{C_I^{\eta_1} S_I^{\gamma_1} D_I^{\lambda_1}}{\hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} \hat{D}_1^{\lambda_1}}<1$ in (10).

Note the real option value upon replacement $A_1 C_I^{\eta_1} S_I^{\gamma_1} D_I^{\lambda_1}$ from the (6) multiple model does not appear on the RHS of (12), the value matching equation for the single replacement.

For any salvage value threshold, the operating cost threshold for the multiple opportunity replacement model is always less than that for the single opportunity model, since its re-investment cost can be recouped over multiple replacements instead of only one.

The single replacement policy is determined by solving the four equations (i) the reduced form value-matching relationship (13), (ii) and (iii) two reduced form smooth-pasting conditions, modified, (8) and (9), and (iv) the characteristic root equation $Q_1(\eta_{1s},\gamma_{1s},\lambda_{1s})=0$ (5) for $\hat{C}_{1s}$, η_{1s}, γ_{1s} and λ_{1s}, given assumptions about $\hat{S}_{1s}$ and $\hat{D}_{1s}$.

2.3 Model Variants

A merit of a quasi-analytical method is its capacity for reproducing particular forms of the C-S-D model. In the Appendix, Part A, we derive the two- and one-factor variants, the replacement salvage C-S model, the replacement depreciation C-D model, and the replacement C model, as well as the deterministic solution. As an illustration, if the salvage value is zero, then the C-S-D model converts to C-D with both $\hat{S}_1$ and γ_1 set to equal zero. By omitting depreciation, the C-S model would be appropriate for asset where depreciation is not relevant, such as those fully expensed for tax purposes at installation, or in countries with no income tax, or tax holidays, while the exclusion of salvage value in the C-D model would be relevant for assets having no alternative use or disposal value, such as obsolete fittings and equipment. A full listing of all

10

models and their constituent equations is presented in Table 1, subscript 2 represents C-S, 3 C-D and 4 C .

Table 1

Constituent Equations for the Various Models

Multiple Opportunity C-S-D Model 1

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)}\left[\eta_1+\gamma_1+\lambda_1-1+\frac{C_I^{\eta_1}S_I^{\gamma_1}D_I^{\lambda_1}}{\hat{C}_1^{\eta_1}\hat{S}_1^{\gamma_1}\hat{D}_1^{\lambda_1}}\right]=K+\frac{C_I(1-\tau)}{r-\theta_C}-\frac{D_I\tau}{r+\theta_D}$$

Two Reduced Form Smooth-Pasting Conditions:

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)}=\frac{\hat{S}_1(1-\tau)}{\gamma_1}$$

$$\frac{\hat{C}_1(1-\tau)}{\eta_1(r-\theta_C)}=\frac{\hat{D}_1\tau r}{\lambda_1\theta_D(r+\theta_D)}$$

Characteristic Root Equation:

$$Q_1(\eta_1,\gamma_1,\lambda_1)=\tfrac{1}{2}\sigma_C^2\eta_1(\eta_1-1)+\rho\sigma_C\sigma_S\eta_1\gamma_1+\tfrac{1}{2}\sigma_S^2\gamma_1(\gamma_1-1)$$
$$+\theta_C\eta_1+\theta_S\gamma_1-\theta_D\lambda_1-r=0$$

Single Opportunity C-S-D Model 1s

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_{1s}(1-\tau)}{\eta_{1s}(r-\theta_C)}[\eta_{1s}+\gamma_{1s}+\lambda_{1s}-1]=K+\frac{C_I(1-\tau)}{r-\theta_C}-\frac{D_I\tau}{r+\theta_D}$$

Two Reduced Form Smooth-Pasting Conditions:

$$\frac{\hat{C}_{1s}(1-\tau)}{\eta_{1s}(r-\theta_C)}=\frac{\hat{S}_{1s}(1-\tau)}{\gamma_{1s}}$$

$$\frac{\hat{C}_{1s}(1-\tau)}{\eta_{1s}(r-\theta_C)}=\frac{\hat{D}_{1s}\tau r}{\lambda_{1s}\theta_D(r+\theta_D)}$$

Characteristic Root Equation:

$$Q_1\left(\eta_{1s}, \gamma_{1s}, \lambda_{1s}\right) = \tfrac{1}{2}\sigma_C^2 \eta_{1s}\left(\eta_{1s} - 1\right) + \rho\sigma_C\sigma_S\eta_{1s}\gamma_{1s} + \tfrac{1}{2}\sigma_S^2\gamma_{1s}\left(\gamma_{1s} - 1\right)$$
$$+ \theta_C\eta_{1s} + \theta_S\gamma_{1s} - \theta_D\lambda_{1s} - r = 0$$

Multiple Opportunity C-S Model 2

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_2\left(1-\tau\right)}{\eta_2\left(r-\theta_C\right)}\left[\eta_2 + \gamma_2 - 1 + \frac{C_I^{\eta_2}S_I^{\gamma_2}}{\hat{C}_2^{\eta_2}\hat{S}_2^{\gamma_2}}\right] = K + \frac{C_I\left(1-\tau\right)}{r-\theta_C}$$

Reduced Form Smooth-Pasting Condition:

$$\frac{\hat{C}_2\left(1-\tau\right)}{\eta_2\left(r-\theta_C\right)} = \frac{\hat{S}_2\left(1-\tau\right)}{\gamma_2}$$

Characteristic Root Equation:

$$Q_2\left(\eta_2, \gamma_2\right) = \tfrac{1}{2}\sigma_C^2\eta_2\left(\eta_{21} - 1\right) + \rho\sigma_C\sigma_S\eta_2\gamma_2 + \tfrac{1}{2}\sigma_S^2\gamma_2\left(\gamma_2 - 1\right) + \theta_C\eta_2 + \theta_S\gamma_2 - r = 0$$

Single Opportunity C-S Model 2s

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_{2s}\left(1-\tau\right)}{\eta_{2s}\left(r-\theta_C\right)}\left[\eta_{2s} + \gamma_{2s} - 1\right] = K + \frac{C_I\left(1-\tau\right)}{r-\theta_C}$$

Reduced Form Smooth-Pasting Condition:

$$\frac{\hat{C}_{2s}\left(1-\tau\right)}{\eta_{2s}\left(r-\theta_C\right)} = \frac{\hat{S}_{2s}\left(1-\tau\right)}{\gamma_{2s}}$$

Characteristic Root Equation:

$$Q_2\left(\eta_{2s}, \gamma_{2s}\right) = \tfrac{1}{2}\sigma_C^2\eta_{2s}\left(\eta_{2s} - 1\right) + \rho\sigma_C\sigma_S\eta_{2s}\gamma_{2s} + \tfrac{1}{2}\sigma_S^2\gamma_{2s}\left(\gamma_{2s} - 1\right)$$
$$+ \theta_C\eta_{2s} + \theta_S\gamma_{2s} - r = 0$$

Multiple Opportunity C-D Model 3

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_3\left(1-\tau\right)}{\eta_3\left(r-\theta_C\right)}\left[\eta_3 + \lambda_3 - 1 + \frac{C_I^{\eta_3}D_I^{\lambda_3}}{\hat{C}_1^{\eta_3}\hat{D}_1^{\lambda_3}}\right] = K + \frac{C_I\left(1-\tau\right)}{r-\theta_C} - \frac{D_I\tau}{r+\theta_D}$$

Reduced Form Smooth-Pasting Condition:

$$\frac{\hat{C}_3(1-\tau)}{\eta_3(r-\theta_C)} = \frac{\hat{D}_3 \tau r}{\lambda_3 \theta_D (r+\theta_D)}$$

Characteristic Root Equation:

$$Q_3(\eta_3, \lambda_3) = \tfrac{1}{2}\sigma_C^2 \eta_3(\eta_3 - 1) + \theta_C \eta_3 - \theta_D \lambda_3 - r = 0$$

Single Opportunity C-D Model 3s

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_{3s}(1-\tau)}{\eta_{3s}(r-\theta_C)}[\eta_{3s} + \lambda_{3s} - 1] = K + \frac{C_I(1-\tau)}{r-\theta_C} - \frac{D_I \tau}{r+\theta_D}$$

Reduced Form Smooth-Pasting Condition:

$$\frac{\hat{C}_{3s}(1-\tau)}{\eta_{3s}(r-\theta_C)} = \frac{\hat{D}_{3s}\tau r}{\lambda_{3s}\theta_D(r+\theta_D)}$$

Characteristic Root Equation:

$$Q_3(\eta_{3s}, \lambda_{3s}) = \tfrac{1}{2}\sigma_C^2 \eta_{3s}(\eta_{3s}-1) + \theta_C \eta_{3s} - \theta_D \lambda_{3s} - r = 0$$

Multiple Opportunity C Model 4

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_4(1-\tau)}{\eta_4(r-\theta_C)}\left[\eta_4 - 1 + \frac{C_I^{\eta_4}}{\hat{C}_1^{\eta_4}}\right] = K + \frac{C_I(1-\tau)}{r-\theta_C}$$

Characteristic Root Equation:

$$Q_4(\eta_4) = \tfrac{1}{2}\sigma_C^2 \eta_4(\eta_4 - 1) + \theta_C \eta_4 - r = 0$$

Single Opportunity C Model 4s

Reduced Form Value-Matching Relationship:

$$\frac{\hat{C}_{4s}(1-\tau)}{\eta_{4s}(r-\theta_C)}[\eta_{4s} - 1] = K + \frac{C_I(1-\tau)}{r-\theta_C}$$

Characteristic Root Equation:

$$Q_4(\eta_{4s}) = \tfrac{1}{2}\sigma_C^2 \eta_{4s}(\eta_{4s}-1) + \theta_C \eta_{4s} - r = 0$$

Note that $\eta_{4s} = \eta_4$

"Mauer and Ott (1995), Dobbs (2004) and Adkins and Paxson (2011) are more or less the C model, Zambujal-Olivera and Duque (2011) the C-S-D model but with an uncorrelated S and an unusual D, Adkins and Paxson (2013 a,b) are the C-D model. Adkins and Paxson (2015) is the C-S model, but with a stochastic net revenue rather than C. The traditional C-S-D **deterministic model is in Lutz and Lutz (1951). There are numerous deterministic articles using C-S, or C-D, or C, which are cited in surveys such as Hartman and Tan (2014).**

3. Illustrative Results for the Timing Boundary

Since the optimal timing boundary reflects a trade-off amongst the thresholds for each of the three factors, the boundary occupies a three dimensional space, even if one[6] of the factors is deterministic. While an element of choice exists for displaying the three dimensional optimal timing boundary, we depict the boundary as a representative set within a two-dimensional space. Specifically, a salvage value threshold is initially pre-specified, and then for this given threshold level, the optimal timing boundary is found for the two remaining factors, operating costs and depreciation, by varying the depreciation threshold level. This procedure is repeated several times for alternative pre-specified salvage value thresholds. In this way, a representative set of optimal timing boundaries can be constructed. Since depreciation is a deterministic variable, its threshold level implies a certain timing level or asset age, $\hat{T}_1$, that is: $\hat{T}_1 = \dfrac{1}{\theta_D} \ln\left(\dfrac{D_I}{\hat{D}_1}\right)$. Because time seems to be perceptibly a more natural quantity than depreciation, the optimal timing boundaries are expressed in a two dimensional space of operating costs and time. The numerical illustrations are computed using the base data set presented in Table 2. We assume that the whole amount of the re-investment cost K is allowable for depreciation, so $D_I = \theta_D K$. The merit of using this condition is that changes in either the declining balance rate or the re-investment cost are automatically converted into the initial level of depreciation.

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[6] If two of the factors in a three-factor model are deterministic, their threshold levels would be related through a time variable and the three dimensional optimal timing boundary could be fully represented by a two dimensional function.

14

Table 2

Base Case Parameter Values

Description	Parameter	Value
Replacement re-investment cost	K	100
Initial operating cost for replica	C_I	10
Initial salvage value for replica	S_I	60
Initial depreciation value for replica	D_I	10
Risk-neutral operating cost drift rate	θ_C	4%
Risk-neutral salvage value drift rate	θ_S	-5%
Depreciation rate	θ_D	10%
Operating cost volatility	σ_C	25%
Salvage value volatility	σ_S	25%
Operating cost and salvage value correlation coefficient	ρ	0%
Risk-free interest rate	r	7%
Tax rate	τ	30%

This table reports our choice of base case parameter values[7], assumed to be constant over time.

3.1 Multiple *Opportunity* C-S-D Boundary

Various optimal timing boundaries for the **C-S-D** model (and subsets) are illustrated in Figure 1. These are depicted by the generic curves AB, where the subscript refers to a particular salvage threshold level, from its minimum of 0 to its maximum of 60, in steps of 20. The **C-D** model is obtained for $\hat{S}_1 = 0$, so its boundary is depicted by A_0B_0. Figure 1 also illustrates the generic boundaries CD for when $\hat{D}_1 = 0$, so C_0D_0 depicts the after-tax one factor solution for the **C** model, and $C_{20+}D_{20+}$ depicts the after-tax one factor solution for the **C-D** model. Representative values along each boundary are shown in Table 3. The continuance and replacement decision

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[7] See Appendix, Part E, for some evidence that these parameter values may be in a reasonable range for some equipment replacements.

regions lie below and above each boundary, respectively. When the prevailing salvage value equals the indicated threshold level, replacement is the optimal decision if the pair of prevailing operating cost and asset age belongs to the replacement region, and continuance if otherwise.

Figure 1

Optimal Timing Boundary for the Multiple Opportunity C-S-D Model
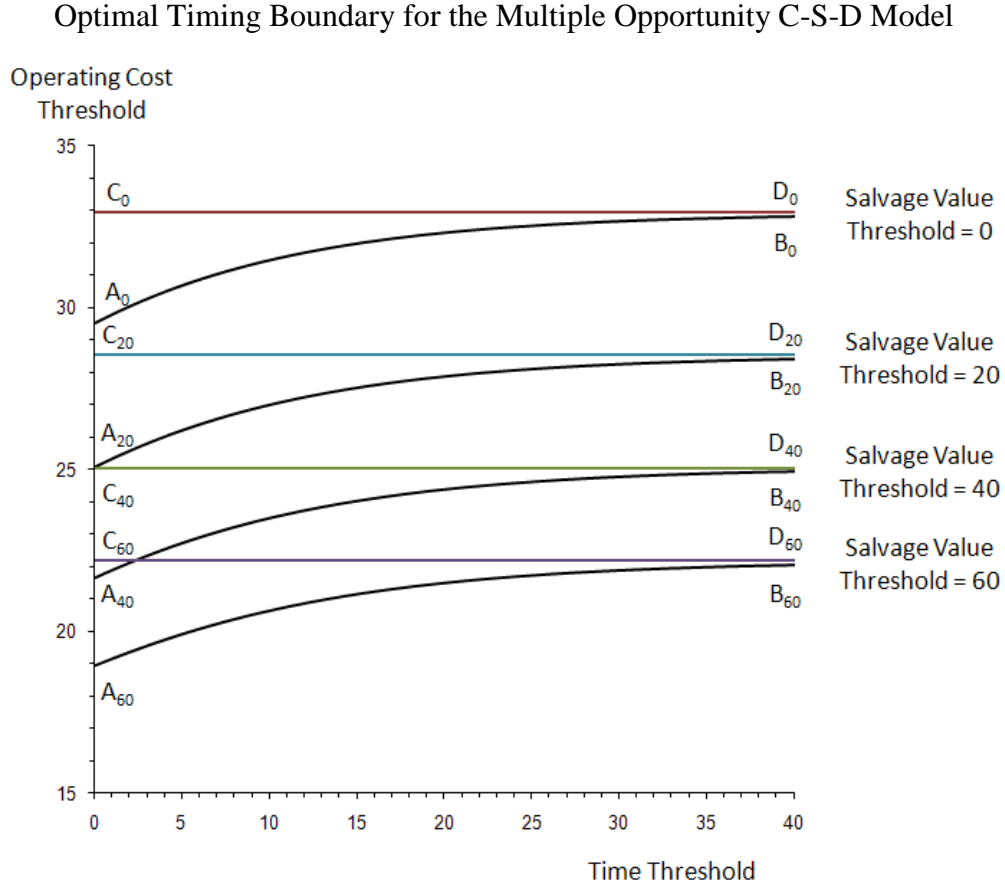
The operating cost threshold levels $\hat{C}_1$ depicted in this figure by the boundaries AB are evaluated using the Table 2 values, for the recorded salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I / \hat{D}_1\right)/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10), see also Table 3. The threshold levels for zero depreciation $\hat{D}_1 = 0$ are depicted by CD. The continuance and replacement regions lie below and above each boundary, respectively.

The slopes for the boundaries AB are all positive but non-linear. For a given salvage value, the operating cost threshold increases with asset age and younger assets are replaced at lower operating cost thresholds than older assets. As the asset ages, the operating cost threshold

16

increases, but at a decreasing rate, with the rate declining to zero for an infinitely aged asset. The increase in the operating cost threshold as the asset ages is due to the need to capture more value from the incumbent because of the fall in the residual depreciation tax shield. Also, the boundaries AB are vertically stacked, with those for lower salvage value thresholds lying above those for higher levels. As the salvage value threshold level falls, the operating cost threshold for a given asset age increases owing to the need to capture more value from the incumbent because of the rise in the net re-investment cost. As a result, the effect of omitting either the salvage value or the depreciation charge from the model is to significantly raise the operating cost threshold, and their inclusion tends to hasten the next act of replacement.

Previously, we conjecture that for the after-tax value improvement rendered by the re-investment to exceed the net re-investment cost, both γ_1 and λ_1 have to be small. From Table 3, we observe that this requirement is met for our data set, and moreover, that the values of γ_1 and λ_1 respectively decline towards zero as $\hat{S}_1$ and $\hat{D}_1$ tend to zero such that both $S_I^{\gamma_1}/\hat{S}_1^{\gamma_1}$ and $D_I^{\gamma_1}/\hat{D}_1^{\gamma_1}$ remain close to one. This suggests that it is the operating cost that exerts the greatest pressure on the differential option value.

Table 3

Representative Values along the Multiple Opportunity C-S-D Boundaries

	$\hat{T}_1=0$	$\hat{T}_1=2.5$	$\hat{T}_1=5$	$\hat{T}_1=10$	$\hat{T}_1=20$	$\hat{T}_1=40$	$\hat{T}_1=\infty$
Discriminatory boundary values for $\hat{S}_1=0$							
$\hat{C}_1$	29.540	30.176	30.700	31.478	32.322	32.818	32.919
η_1	1.3895	1.3832	1.3785	1.3722	1.3664	1.3636	1.3632
γ_1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
λ_1	0.02490	0.01890	0.01442	0.00849	0.00303	0.00041	0.00000
Discriminatory boundary values for $\hat{S}_1=20$							
$\hat{C}_1$	25.080	25.699	26.219	27.008	27.889	28.424	28.537
η_1	1.4230	1.4147	1.4084	1.4001	1.3923	1.3886	1.3879

17

γ_1	0.03404	0.03303	0.03223	0.03110	0.02995	0.02931	0.02918
λ_1	0.03004	0.02270	0.01725	0.01010	0.00358	0.00048	0.00000

Discriminatory boundary values for $\hat{S}_1 = 40$

$\hat{C}_1$	21.648	22.225	22.722	23.493	24.378	24.932	25.052
η_1	1.4659	1.4550	1.4468	1.4358	1.4254	1.4203	1.4194
γ_1	0.08126	0.07856	0.07641	0.07334	0.07017	0.06836	0.06799
λ_1	0.03585	0.02699	0.02045	0.01190	0.00419	0.00056	0.00000

Discriminatory boundary values for $\hat{S}_1 = 60$

$\hat{C}_1$	18.963	19.478	19.932	20.654	21.509	22.060	22.182
η_1	1.5180	1.5043	1.4938	1.4795	1.4660	1.4591	1.4579
γ_1	0.14410	0.13901	0.13490	0.12894	0.12268	0.11906	0.11830
λ_1	0.04238	0.03184	0.02407	0.01395	0.00489	0.00066	0.00000

The operating cost threshold levels $\hat{C}_1$ presented in this table are evaluated using the information shown in Table 2 for the recorded salvage threshold level $\hat{S}_1$ and time threshold level $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I / \hat{D}_1\right)/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

3.2 Single Opportunity C-S-D Model

The various timing boundaries for the single opportunity C-S-D model are illustrated in Figure 2. These have a similar shape as the multiple opportunity boundaries, except that their locations are very different, as the y axis runs from 30-55, instead of 15-35 (Figure 1). A comparison of Figures 1 and 2 reveals that the operating cost threshold for any given pair of salvage value and asset age is highest for the single opportunity model, so any trajectory starting from the initial levels is always bound to hit the multiple opportunity boundary first. This is because the re-investment cost in the multiple opportunity model is partially offset by the option value of

several future replacements, while for the single opportunity model there is no option of another replacement.

Figure 2

Optimal Timing Boundary for the Single Opportunity C-S-D Model
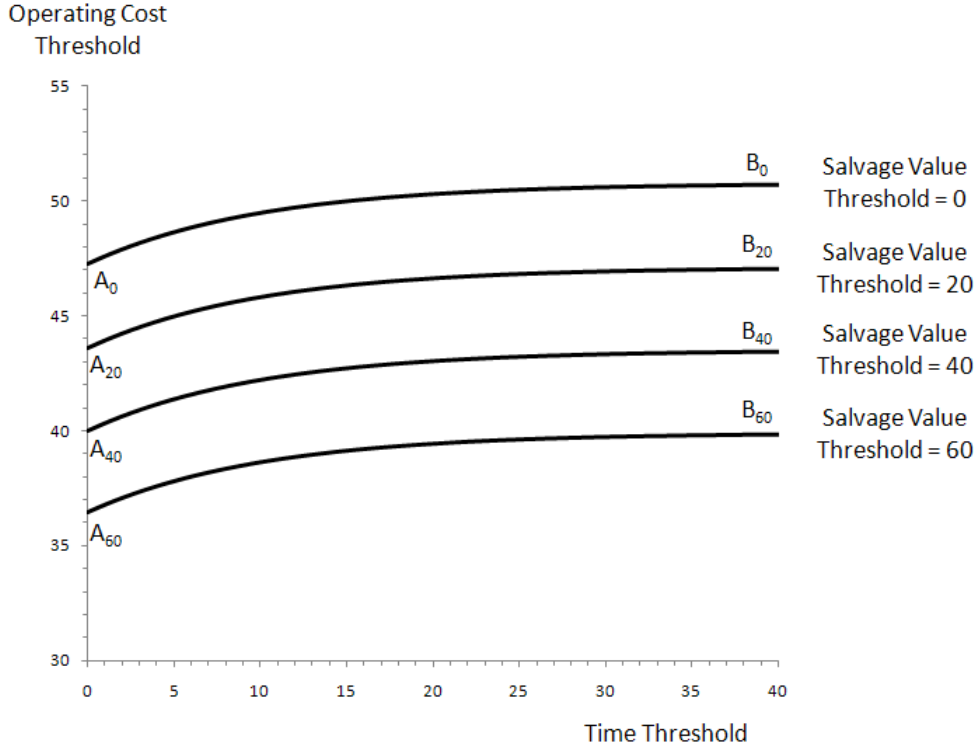
The operating cost threshold levels $\hat{C}_{1s}$ depicted in this figure by the boundaries AB are evaluated using the Table 2 values, for the recorded salvage threshold level $\hat{S}_{1s}$ and time threshold levels $\hat{T}_{1s}$, where $\hat{T}_{1s} = \ln\left(D_I / \hat{D}_{1s}\right) / \theta_D$. The threshold levels are determined by eliminating η_{1s}, γ_{1s} and λ_{1s} from (5), (8), (9), after the inclusion of the subscript s, and (13). The continuance and replacement regions lie below and above each boundary, respectively.

3.3 The C-S Model

The C-S boundary for the multiple and single opportunity variants, depicted respectively by AB and DE, over the range for $0 \le \hat{S}_2 \le S_I$, is shown in Figure 3, since unlike the C-D boundary, it is not directly observable from Figure 1. Each boundary separates the continuance region below the curve from the replacement region above. The nearly linear negative slope of the boundaries

19

reflects the nature of the trade-off between the two thresholds. This arises due to the way that an increase in salvage value, through reductions in the net re-investment cost, partially offsets a decrease in the operating cost because there is less need to capture value from the incumbent. It also reveals that higher rather than lower salvage values have a greater significance in deciding between continuance and replacement, and its importance in making the decision wanes as the salvage value approaches zero. We also observe the single opportunity boundary DE lying entirely above the multiple opportunity boundary AB, again for the reason that for the former representation, the re-investment cost has to be fully recouped over only one occasion. Representative values along AB are provided in Table 4.
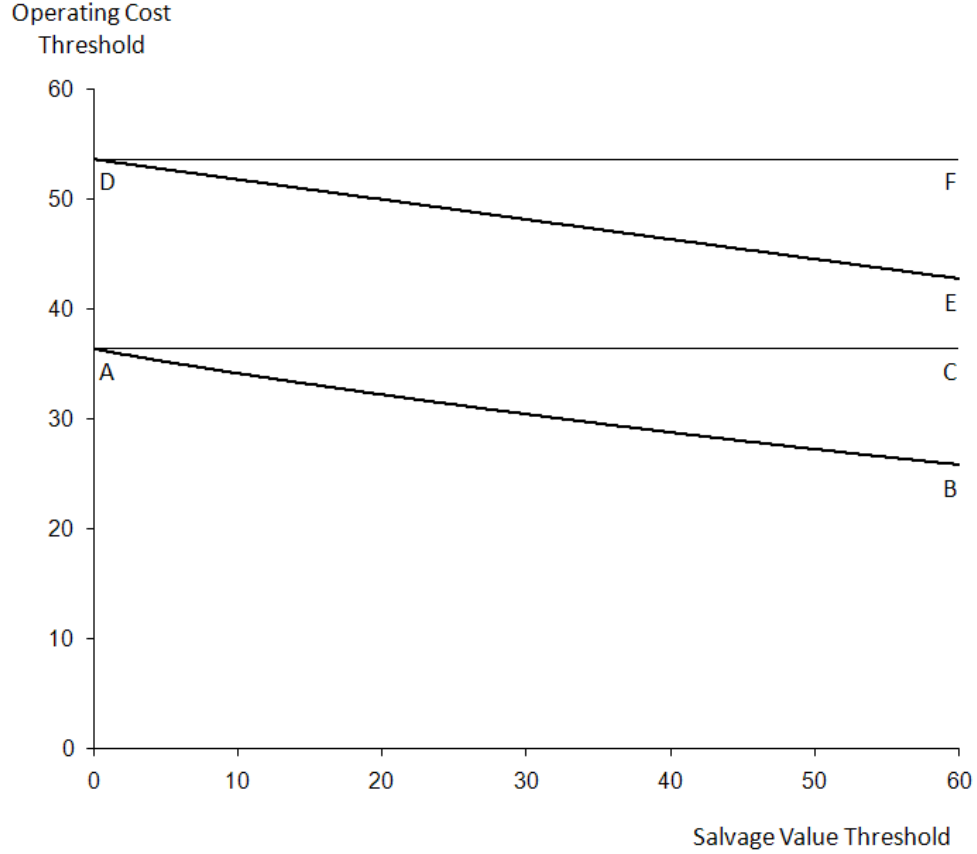
Table 4

Representative Values along the Multiple Opportunity C-S Boundary

$\hat{S}_2$	$\hat{C}_2$	η_2	γ_2
60.0	25.812	1.4447	0.10075
50.0	27.223	1.4278	0.07867
40.0	28.755	1.4122	0.05893
30.0	30.409	1.3980	0.04138
20.0	32.193	1.3851	0.02582
10.0	34.132	1.3736	0.01207
0.0	36.397	1.3632	0.00000

The operating cost threshold levels $\hat{C}_2$ presented in this table are evaluated using the information shown in Table 2 for the recorded salvage threshold level $\hat{S}_2$. The threshold levels are determined by eliminating η_2 and γ_2 from (5), (A.18) and (A.19).

Figure 3

Boundaries for the Multiple and Single Opportunity C-S Model

The operating cost threshold levels $\hat{C}_2$ depicted in this figure by the boundaries AB for the multiple replacement opportunity and by DE for the single replacement opportunity model are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_2$. The threshold levels for the multiple opportunity model are determined by eliminating η_2 and γ_2 from $Q_2(\eta_2, \gamma_2, \lambda = 0) = 0$, (5), (A.18) and (A.19); the respective equations for the single opportunity model are reported in Table 1. The profiles AC and DF depict the boundaries AB and DE, respectively for a zero salvage level $S = 0$; the respective operating cost thresholds are 36.397 and 53.619. The continuance and replacement regions lie below and above each boundary, respectively.

Figure 3 also illustrates the boundaries, AC and DF, for the multiple and single opportunity variants when the salvage value is zero, which is identical to the after-tax version of the solution for a one-factor (cost) model. The boundary AC lies entirely above AB, and DF above DE, except when the salvage value threshold is zero. This establishes the importance of not omitting

a non-zero salvage value from the formulation. Ignoring the salvage value tends to defer replacement, more than is ideally required, and this deferment grows in magnitude with the extent of the salvage value at replacement.

3.4 Summary

Our numerical illustration has shown the importance of the salvage value and the tax depreciation allowance in the replacement policy, since their mistaken omission produces an overly conservative policy whereby the incumbent is retained for longer than is economically justified. Further, comparing the C-S and C-S-D models provide a useful framework for evaluating the consequences of using less complete models.

4. Sensitivity Analysis

 In this section, we investigate the effects of changes in some of the parameter values in Table 2 on the timing boundary for the C-S-D and the C-D (where S=0) models, since no new insights were obtained from sensitivity analysis for the C-S model[8]. All sensitivities display the same sign for every model, with the notable exception of the effect of the salvage value volatility on the operating cost threshold for the C-S-D and C-S models.

4.1 Initial Levels

The effect of variations in the re-investment cost and the initial operating cost, respectively, for salvage value thresholds of 0 and 60 results are as expected, and shown in the Appendix, Part B. In response to an increase in either the re-investment cost or the initial operating cost, there is a significant accompanying rise in the operating cost threshold, since more value V_i has to be captured from the incumbent to compensate for the unfavorable change. The effect of a change in the initial salvage value on the operating cost threshold is similar but weak, except of course that an increase in the initial level means that less value has to be captured from the incumbent, and there is a resulting fall in the operating cost threshold. If this weak response is universal,

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[8] Full results are available from the corresponding author.

then in their negotiations with suppliers, prospective owners should devote more attention to obtaining an improved offer on the initial operating cost and re-investment cost rather than the initial salvage value, since the salvage value is only realized at the next act of replacement and discounted owing to the time value of money.
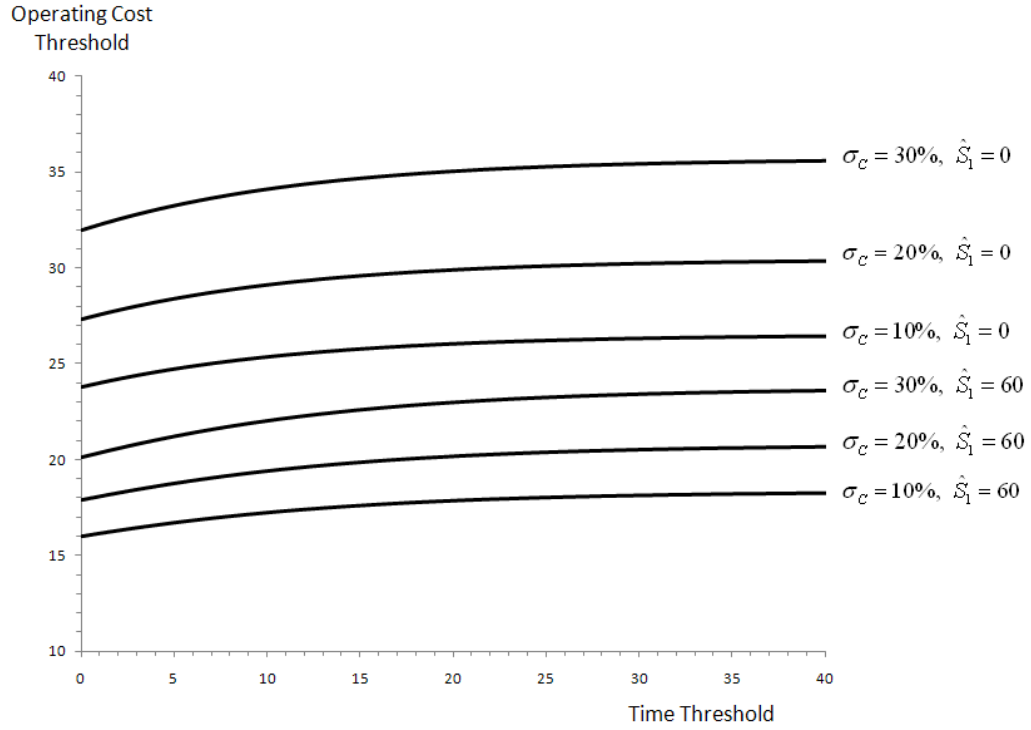
4.2 Volatility

The timing boundary response to variations in the operating cost volatility is illustrated in Figure 4, for salvage value thresholds of 0[9] and 60. This reveals the expected result that a volatility increase produces a rise in the operating cost threshold because of the positive effect of uncertainty on the value of waiting, which has to be compensated by extracting more value from the incumbent. Further, we observe that the magnitude of the operating cost threshold due to the volatility increase is least when the salvage value threshold is high, or to a lesser extent when the age threshold is low. Not only does a higher salvage value threshold level reduce the operating cost threshold level, but it also mitigates against a rise in the operating cost volatility. Figure 5 illustrates the effect of salvage value volatility changes on the operating cost threshold, only for a salvage value threshold of 60 because when $\hat{S}_1 = 0$, then $\gamma_1 = 0$, and so volatility changes have no impact on the timing boundary. The surprising result is that a salvage value volatility increase produces a fall in the operating cost threshold for all asset ages, even though the effect is quite modest. In this case, uncertainty has a negative effect on the value of waiting, or a positive effect on the value of hastening, if $\rho_{C,S} \leq .5$. By hastening the act of replacement, less value is being captured from the incumbent, which suggests that the replacement option also decreases in value. One way of interpreting this finding is that it represents a form of protection policy, since over the next time interval, a fall in cash flow due to an operating cost deterioration may be accompanied by a fall in the salvage value whose severity intensifies for increases in the salvage value volatility. The effect of increases of salvage volatility on the C threshold is limited because the salvage volatility affects only S upon replacement every 30 years or so, while the cost volatility affects a periodic operating cost.

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[9] The model composition changes radically for a zero volatility, so its full consideration is given in Appendix, Part C.
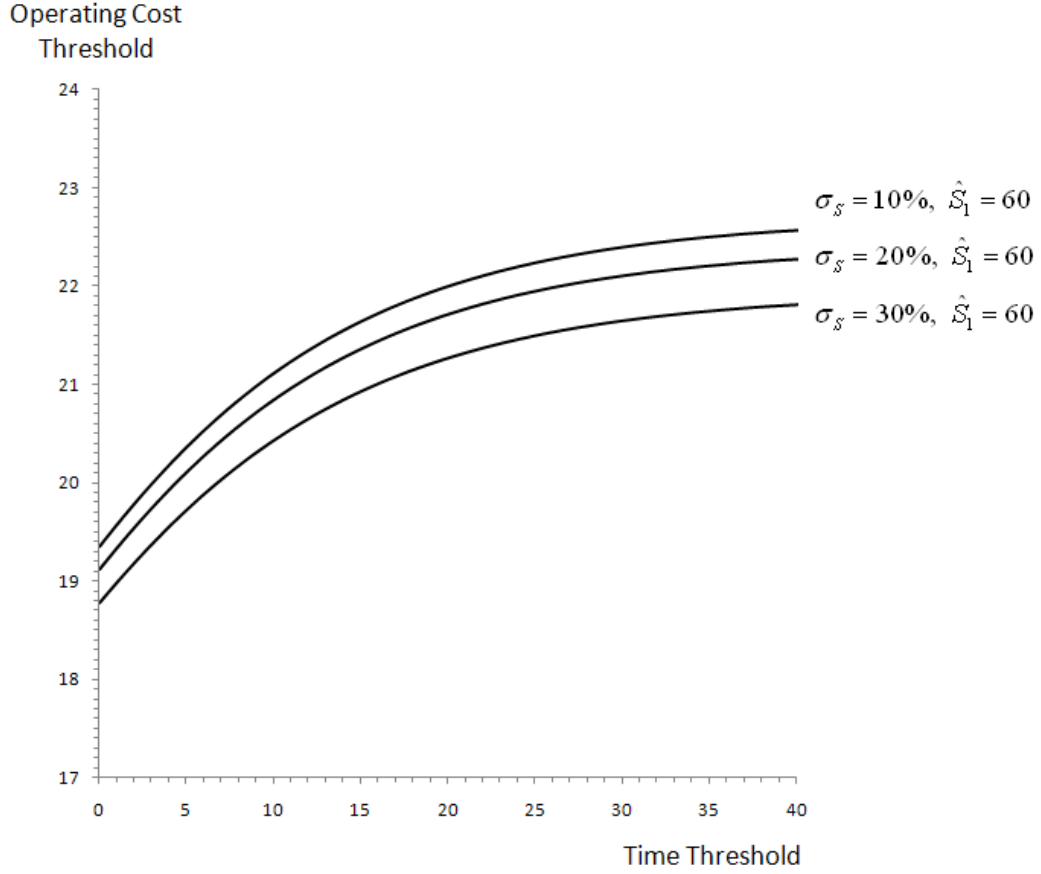
Figure 4

Effect of Operating Cost Volatility on the Multiple Opportunity C-S-D Boundary

Except for the variations in the operating cost volatility level σ_C, the operating cost threshold levels $\hat{C}_1$, shown in this figure, are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I / \hat{D}_1\right)/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

Figure 5

Effect of Salvage Value Volatility on the Multiple Opportunity C-S-D Boundary

Except for the variations in the salvage value volatility level σ_S, the operating cost threshold levels $\hat{C}_1$, shown in this figure, are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I/\hat{D}_1\right)/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

4.3 Correlation

Figure 6 illustrates the effects of variations in the correlation coefficient ρ between C and S on the timing boundary, but only for a salvage value threshold of 60. Normally, we would expect

ρ to be negative since poorly performing assets with an increasing operating cost can only command a low salvage value. Figure 6 reveals that as the correlation coefficient increases the operating cost threshold increases, which suggests a deferral of the replacement act and the need to capture more value from the incumbent. This finding is counter-intuitive since if ρ is negative, then an operating cost rise is likely to be accompanied with a salvage value fall, which leads to a rise in net re-investment cost. Now, these outcomes are unfortunate not only due to the simultaneous operating cost rise and the salvage value fall, but also because they are not mutually compensatory. A plausible explanation for the decrease in operating cost threshold due to a fall in the correlation coefficient is that it is signaling the likely adverse consequences arising from changes in the operating cost and salvage value. The fall in the operating cost threshold observed from a drop in the correlation coefficient is interpreted as acting as a kind of protection policy against suffering from both types of loss.

Figure 6

Effect of the Correlation Coefficient on the Multiple Opportunity C-S-D Boundary
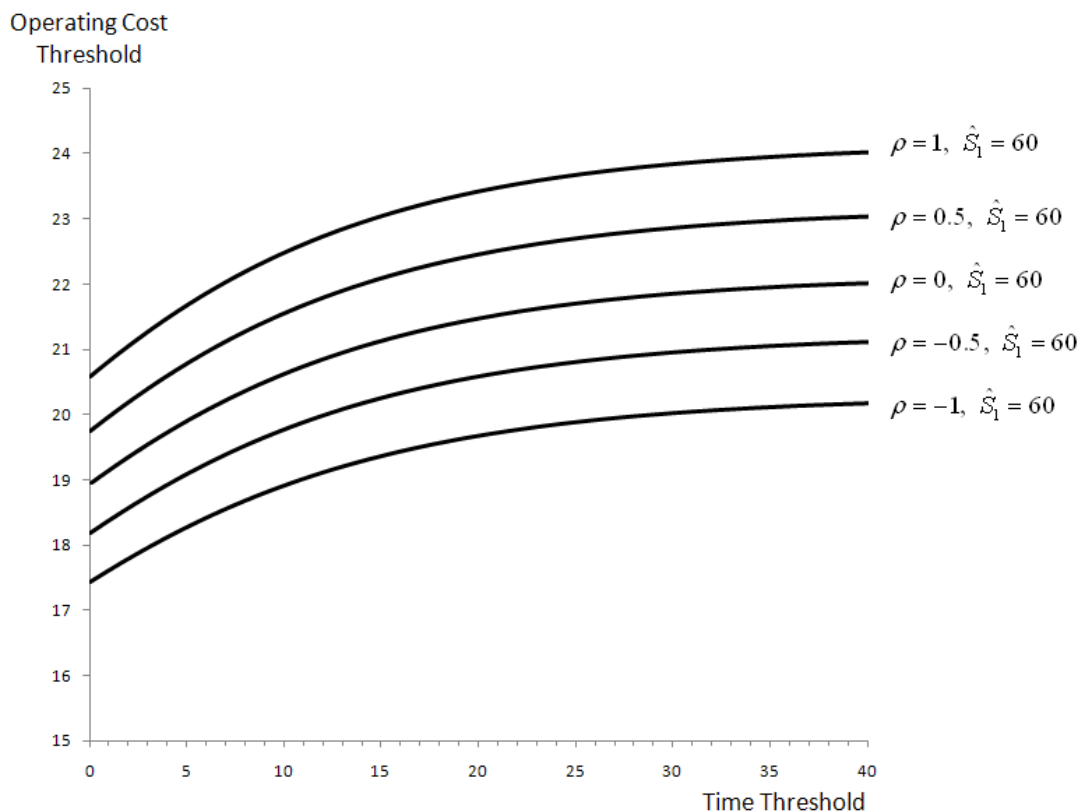
Except for the variations in the operating cost salvage value correlation level ρ, the operating cost threshold levels $\hat{C}_1$, shown in this figure, are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I / \hat{D}_1\right) / \theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

4.4 Tax Attributes

Tax authorities have at least four basic policy variables that can be altered to affect investment expenditures: the type of allowable depreciation, the depreciation rate, the overall tax rate, and the investment tax credit. We study the middle two policy variables, and indirectly the last variable. The amount of tax incurred from operating the asset is influenced directly by the tax rate τ, and indirectly by the expected depreciation lifetime, given by $1/\theta_D$. We consider the indirect effect first.

By increasing the depreciation tax shield, we would expect any increase in the declining balance rate θ_D to hasten the replacement act. Figure 7 illustrates the effects of changes in θ_D on the timing boundary for only $\hat{S}_1 = 60$, since the profiles for other salvage value thresholds are very similar except for a vertical shift. The profiles for the three different declining balance rates depicted in Figure 7 reveal a mixed picture. An increase in the declining balance rate does yield a fall in the operating cost threshold for older assets $\left(\hat{T}_1 \geq 20\right)$ and for newly installed assets; but for assets having an in-between age, the operating cost threshold rises. This contrasting behavior can be explained through considering the value-matching relationship (6). If the threshold for the depreciation charge is close to its initial level and the asset is young, then the tax shield for the incumbent neutralizes the tax shield for the replica, so the re-investment cost is reduced by an amount equaling the residual depreciation tax shield. Similarly, if the depreciation threshold is close to zero and the asset is old, the incumbent tax shield and the residual tax shield neutralize each other, so the re-investment cost is reduced by an amount equaling the replica depreciation shield. These two effects are less intense for assets of in-between years, and so the reduction in the re-investment cost is less, and this is reflected in a lower operating cost threshold.

Figure 7

27

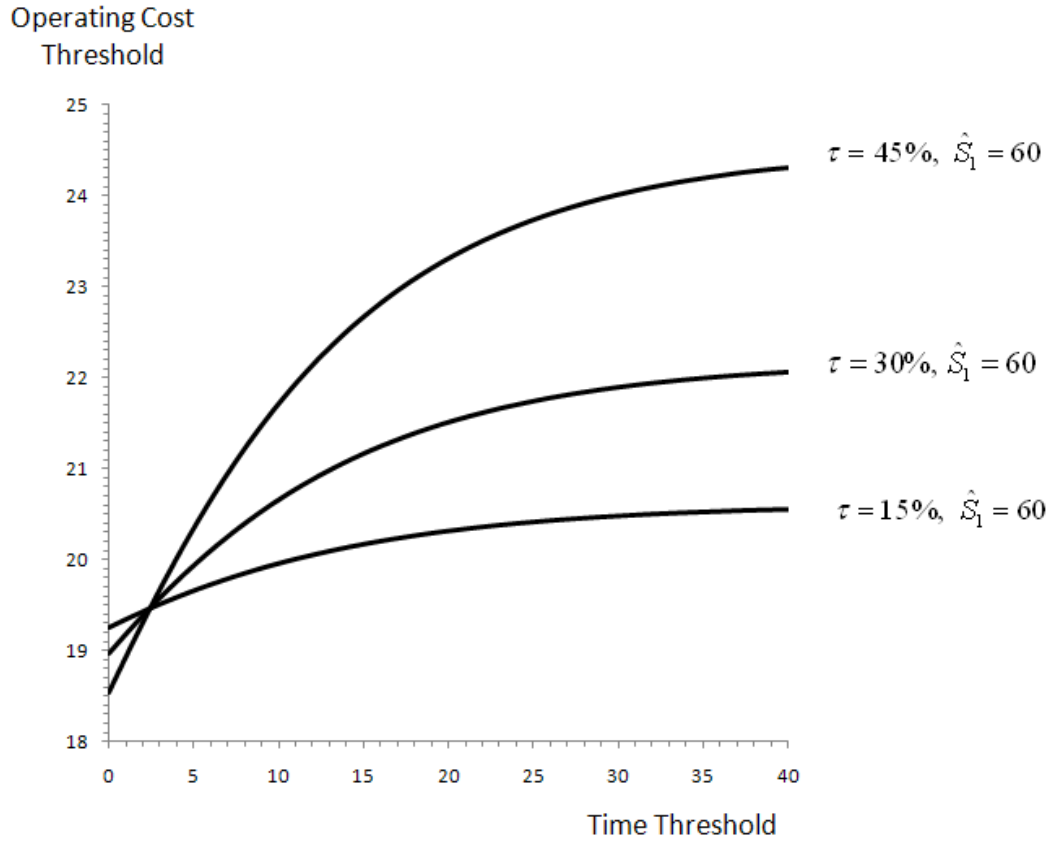## Effect of Depreciation Rate on the Multiple Opportunity C-S-D Boundary
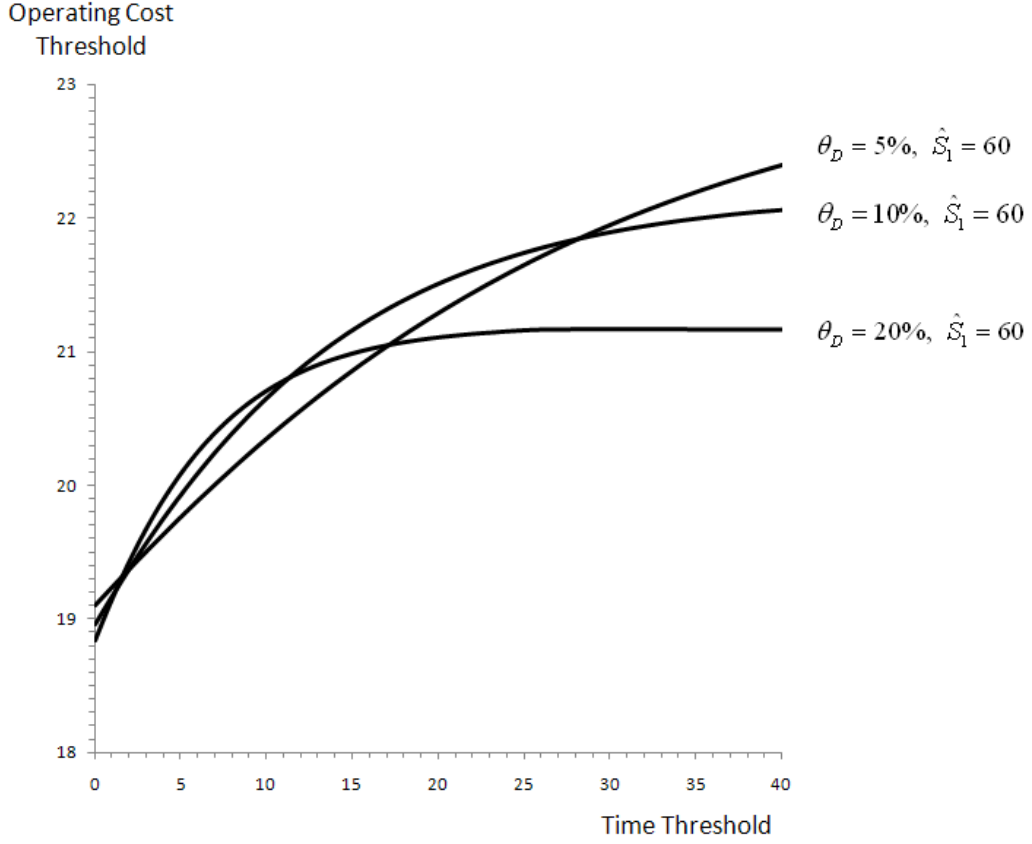
Except for the variations in the depreciation rate θ_D, the operating cost threshold levels $\hat{C}_1$, shown in this figure, are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I/\hat{D}_1\right)\big/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

Figure 8 illustrates the effect of a tax rate change on the replacement policy for $\hat{S}_1 = 60$. Although we expect a tax rate increase to make the asset less attractive and to raise the operating cost because of the need to capture additional value from the incumbent, this finding is observed only for assets older than 2½ years. For newly installed assets, the operating cost threshold falls for a tax rate increase, since the increase raises the residual depreciation tax and so lowers the net re-investment cost, while the depreciation tax shields for the incumbent and the succeeding asset largely cancel each other out. As the asset ages, the residual depreciation tax shield becomes less

28

significant and the replacement policy is influenced more by the operating cost. A decline in the tax rate until zero is accompanied by an operating cost threshold fall for a given salvage value threshold only for older rather than very young assets.

Figure 8

Effect of Tax Rate on the Multiple Opportunity C-S-D Boundary

Except for the variations in the tax rate τ, the operating cost threshold levels $\hat{C}_1$, shown in this figure, are evaluated using the Table 2 values for the specified salvage threshold level $\hat{S}_1$ and time threshold levels $\hat{T}_1$, where $\hat{T}_1 = \ln\left(D_I/\hat{D}_1\right)\Big/\theta_D$. The threshold levels are determined by eliminating η_1, γ_1 and λ_1 from (5), (8), (9), and (10).

Altering the tax rate τ to zero makes the depreciation charge irrelevant as far as the replacement policy is concerned. If $\tau = 0$, then the depreciation term is eliminated from the value matching relationship for the C-S-D model, so (6) becomes:

$$A_1 \hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} - \frac{\hat{C}_1}{r - \theta_C} = A_1 C_I^{\eta_1} S_I^{\gamma_1} - \frac{C_I}{r - \theta_C} + \hat{S}_1 - K \,. \tag{14}$$

Now, (14) is similar in form to the value-matching relationship for the C-S model, (A.17). If the capital expenditure could be fully expensed for tax purposes (or in the US, a 100% deduction of the investment cost for certain new assets from pre-tax income, proposed for 2011), then (14) would be expressed as:

$$A_1 \hat{C}_1^{\eta_1} \hat{S}_1^{\gamma_1} - \frac{\hat{C}_1 (1 - \tau)}{r - \theta_C} = A_1 C_I^{\eta_1} S_I^{\gamma_1} - \frac{C_I (1 - \tau)}{r - \theta_C} + (1 - \tau)\hat{S}_1 - (1 - \tau)K \,. \tag{15}$$

It is easy to demonstrate that the solutions for the two revised expressions, (14) and (15), are identical. This implies that the timing boundary is invariant to the tax rate when the re-investment cost is fully expensed for tax purposes, so the replacement policy is not distorted by the tax rate level. This finding for the stochastic replacement model endorses the conclusion of Smith (1963) that is based on a deterministic NPV evaluation. The operating cost threshold for the case where the re-investment cost for the succeeding asset is fully expensed for tax purposes is lower provided that the incumbent is retired at half its expected life or later. Except for assets that are retired at a very young age, a tax policy based on fully expensing the entire capital expenditure leads to a more generous outcome for owners, who now become incentivized to hasten asset replacement[10].

5   Conclusion

We apply a quasi-analytical method to find the after-tax timing boundary for replacing an incumbent asset when both its operating cost and salvage value deteriorate and are stochastic. The method copes with three factors in the absence of a dimension reducing transformation, and without recourse to onerous numerical methods, and has the versatility for reproducing particularized forms of the three-factor model and also for delivering general findings.

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[10] In Appendix, Part D, we discuss the effects of variation in the drift and interest rates on the thresholds that justify immediate replacement.

There are advantages from including the salvage value as a distinct factor in the replacement model. First, since the presence of the salvage value in the formulation significantly lowers the operating cost threshold, its mistaken omission unnecessarily prolongs the lifetime of the incumbent and produces a replacement policy that is uneconomic, ignored by many authors such as Adkins and Paxson (2011, 2013a). Second, the actual optimal replacement policy depends on which factors are uncertain, so understanding the operational context is critical to making a proper decision. Even though a positive variation in the operating cost volatility yields the recognized outcome that deferral has value, the operating cost threshold responds negatively to either a positive change in the salvage value volatility, albeit by a modest amount, or a negative change in the correlation coefficient. We refer to this finding as the value of hastening the act of replacement. Its existence is due to the presence of the salvage value as a distinct uncertain factor as well as the interaction between the two uncertain factors, and is explained as a type of protection policy that helps to guard against the double disadvantage of an operating cost increase and a salvage value decrease simultaneously occurring.

The omission of depreciation in replacement models leads to uneconomic decisions, and also including depreciation in a capital budgeting model allows consideration of the impact of tax policy changes on the operating cost threshold. Although a higher declining balance rate or a lower tax rate is expected to lower the operating cost threshold because of the benefits in advancing the cash flow, this is not universal. A tax depreciation rate increase only produces a lower operating cost threshold if the asset is very young or very old, while a tax rate fall only lowers the threshold provided that the asset is not young. An alteration in tax policy designed to motivate re-investment is not going to be uniformly effective. Finally, if the entire re-investment cost is allowed to be fully expensed for tax purposes, then this results in an accelerated replacement policy except for the youngest assets. This simple rule has the merit of incentivizing owners to replace more frequently, balanced by the government sacrifice of deferring tax revenues.

There are numerous qualifications in the proposed replacement methodology, and analysis: investment costs are considered constant or deterministic; replacements are assumed to be identical so not allowing for technical innovation; no account has been given to alternative evolutionary processes; the possibility of sudden failure has been ignored; the replacement

decision for the asset under consideration is examined in isolation from the other assets of the firm; results have not been shown for the value of the asset in place, including the real replacement option value (so optimal replacement policies are not necessarily identical for maximizing the value of assets in place); alternatives from infinite multiple to single replacements have not been explicitly considered; competition among firms has been ignored; and no empirical comparisons have been made with actual replacement decisions, for specific firms or for industries.

Further research is required to investigate these matters, to examine the feasibility of a quasi-analytical method for overcoming these shortcomings, such as stochastic investment costs, technological innovation and/or failure, strategic considerations, and the real value of assets in place, and the possibility of revealing new insights in optimal replacement policy, at the risk of raising model complexity and lowering transparency.

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