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Unitarily inequivalent local and global Fourier transforms in multipartite quantum systems

C. Lei[1] · A. Vourdas[1]

Abstract

A multipartite system comprised of n subsystems, each of which is described with 'local variables' in $\mathbb{Z}(d)$ and with a d-dimensional Hilbert space $H(d)$, is considered. Local Fourier transforms in each subsystem are defined and related phase space methods are discussed (displacement operators, Wigner and Weyl functions, etc). A holistic view of the same system might be more appropriate in the case of strong interactions, which uses 'global variables' in $\mathbb{Z}(d^n)$ and a d^n-dimensional Hilbert space $H(d^n)$. A global Fourier transform is then defined and related phase space methods are discussed. The local formalism is compared and contrasted with the global formalism. Depending on the values of d, n the local Fourier transform is unitarily inequivalent or unitarily equivalent to the global Fourier transform. Time evolution of the system in terms of both local and global variables, is discussed. The formalism can be useful in the general area of Fast Fourier transforms.

Keywords Finite quantum systems · Local and global Fourier transforms · Multipartite systems

1 Introduction

Entanglement and stronger than classical correlations in multipartite systems, are fundamental concepts in quantum mechanics (e.g., [1]). Even if the various components of the system are physically located far from each other, strong correlations and strong interactions between them, weaken the concept of separate identity for each component . This motivates a comparison between the formalism of a multipartite system, with a holistic formalism of the same system that uses global quantities.

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We consider a finite quantum system with variables in $\mathbb{Z}(d)$ where d is an odd integer, described by the d-dimensional Hilbert space $H(d)$ (e.g. [2, 3]). We also consider a multipartite system that consists of n of these systems (which are possibly located far from each other). In this system the positions and momenta take values in $[\mathbb{Z}(d)]^n = \mathbb{Z}(d) \times ... \times \mathbb{Z}(d)$. The system is described with the d^n-dimensional Hilbert space $\mathfrak{H} = H(d) \otimes ... \otimes H(d)$.

In the case of strong correlations and strong interactions between the n components we introduce a holistic approach and regard this as one system with variables in $\mathbb{Z}(d^n)$ and d^n-dimensional Hilbert space $H(d^n)$. We note that

- The Hilbert space $\mathfrak{H}$ is isomorphic to the $H(d^n)$, because they both have the same dimension.
- There is a bijective map between the sets $[\mathbb{Z}(d)]^n$ and $\mathbb{Z}(d^n)$ given below in Eq. (35) (in fact we can have many bijective maps between these two sets). However the $[\mathbb{Z}(d)]^n$ as a ring is not isomorphic to the ring $\mathbb{Z}(d^n)$ (see Eq. (36) below).

With this in mind, we study the following:

- We define a local Fourier transform F_L in the phase space $[\mathbb{Z}(d)]^n \times [\mathbb{Z}(d)]^n$ of the system when considered as n-component system. We also define a global Fourier transform F_G in the phase space $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$ of the system when considered as a single system. This has been introduced briefly in a different context in ref. [4], and here it is studied as a problem in its own right and in connection with a global phase space formalism. We show that depending on the values of d, n the local Fourier transform is unitarily inequivalent (unitarily equivalent) to the global Fourier transform. By that we mean that there exists no unitary transformation U (there exists such a transformation U) so that $F_G = U F_L U^\dagger$. This is discussed in Sect. 4.4 and in Proposition 4.4.
- Starting from an orthonormal basis of 'position states', we use local and global Fourier transforms to define local and global momentum states. Some of the local momentum states are the same as the global momentum states as discussed in Proposition 4.3. We also define local position and momentum operators, and also global position and momentum operators. We do numerical calculations of the time evolution for the case where the Hamiltonian is expressed in terms of local variables and also for the case where the Hamiltonian is expressed in terms of global variables (Sect. 5.2). For multipartite systems with strong interactions between the various components, it might be more appropriate to express the Hamiltonian in terms of the global variables.
- We define a local phase space formalism in $[\mathbb{Z}(d)]^n \times [\mathbb{Z}(d)]^n$ and a global phase space formalism in $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$. Displacements, Wigner and Weyl functions, etc, are defined in these two cases. Density matrices which have only diagonal elements with respect to the position basis, have the same local and global Wigner function (Proposition 5.3). The difference between local and global Wigner functions, is contained entirely in the off-diagonal elements.
- Deviations of a density matrix ρ from the corresponding factorisable density matrix $\mathfrak{R}(\rho)$ (defined in Eq. (16)) are described with the matrices R_L, $\widetilde{R}_L$ and R_G, $\widetilde{R}_G$. They describe classical and quantum correlations in the multipartite system described by ρ (Sect. 5.5).

- Understanding of the relationship between global and local Fourier transforms and related phase space methods, might be useful in other areas like fast Fourier transforms. For $n = 2$ we show that the global Fourier transform can be expressed in terms of many local Fourier transforms (Sect. 4.5). This is similar to the Cooley–Tukey formalism in fast Fourier transforms [5–7]. The general area of Fast Fourier transforms (in a quantum or even classical context) is a potential application of the present formalism.
- In the case that the local and global Fourier transform are unitarily inequivalent (Eq. (65) below), the concept of a multipartite system (and related concepts like entanglement) is fundamentally different from that of a single quantum system. But if they are unitarily equivalent (Eq. (64) below), the distinction between a multipartite system and a single system is weak. Unitary equivalence means that with a change of basis one concept is transformed to another, and consequently there is no fundamental difference between the two. In this case, further work is needed in order to clarify the correspondence between the two (especially of entanglement which is a concept applicable to a multipartite system but not to a single system).

In Sect. 2 we review briefly the phase-space formalism for systems with finite Hilbert space [2, 3]. In Sect. 3 we apply this to each component of a n-partite system, and this is the 'local formalism'. In Sect. 4 we define the global Fourier transform and discuss for which values of d, n it is unitarily inequivalent to the local Fourier transform. In Sect. 5 we present the global phase space formalism and compare and contrast it with the local formalism. In Sect. 6, we present examples. We conclude in Sect. 7 with a discussion of our results.

2 Background

We consider a quantum system (qudit) with variables in the ring $\mathbb{Z}(d)$ of integers modulo d where d is an odd integer. $H(d)$ is the d-dimensional Hilbert space describing this system. There are well known technical differences between quantum systems with odd dimension d and even dimension d (e.g., [8–10]). In this paper we consider systems with odd dimension d.

Let $|X; j\rangle$ where $j \in \mathbb{Z}(d)$ be an orthonormal basis in $H(d)$. The X in the notation is not a variable, it simply indicates 'position states'. The finite Fourier transform F is given by [11]

$$F = \frac{1}{\sqrt{d}} \sum_{j,k} \omega_d(jk)|X; j\rangle\langle X; k|; \quad \omega_d(\alpha) = \exp\left(i\frac{2\pi\alpha}{d}\right); \quad \alpha, j, k \in \mathbb{Z}(d)$$

$$F^4 = \mathbf{1}; \quad FF^\dagger = \mathbf{1}. \tag{1}$$

Its trace is [2]

$$d = 4m + 1 \rightarrow \text{Tr} F = 1;$$
$$d = 4m + 3 \rightarrow \text{Tr} F = i. \tag{2}$$

We act with F on position states and get the dual basis

$$|P; j\rangle = F|X; j\rangle. \tag{3}$$

The P in the notation is not a variable, it simply indicates 'momentum states'.

Using the relation

$$\frac{1}{d} \sum_k \omega_d[(j + \ell)k] = \delta(j, -\ell), \tag{4}$$

we show that F^2 is the parity operator around the origin:

$$F^2 = \frac{1}{d} \sum_{j,k,\ell} \omega_d[(j + \ell)k]|X; j\rangle\langle X; \ell| = \sum_j |X; j\rangle\langle X; -j|. \tag{5}$$

The phase space of this system is $\mathbb{Z}(d) \times \mathbb{Z}(d)$ and in it we introduce the displacement operators

$$X^\beta = \sum_j \omega_d(-j\beta)|P; j\rangle\langle P; j| = \sum_j |X; j + \beta\rangle\langle X; j|;$$

$$Z^\alpha = \sum_j |P; j + \alpha\rangle\langle P; j| = \sum_j \omega_d(\alpha j)|X; j\rangle\langle X; j| = FX^\alpha F^\dagger;$$

$$X^d = Z^d = \mathbf{1}; \quad X^\beta Z^\alpha = Z^\alpha X^\beta \omega_d(-\alpha\beta); \quad \alpha, \beta \in \mathbb{Z}(d). \tag{6}$$

General displacement operators are the unitary operators

$$D(\alpha, \beta) = Z^\alpha X^\beta \omega_d(-2^{-1}\alpha\beta); \quad [D(\alpha, \beta)]^\dagger = D(-\alpha, -\beta);$$

$$D(\alpha_1, \beta_1)D(\alpha_2, \beta_2) = D(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\omega_d[2^{-1}(\alpha_1\beta_2 - \alpha_2\beta_1)]. \tag{7}$$

The $2^{-1} = \frac{d+1}{2}$ is an integer in $\mathbb{Z}(d)$ with odd d, considered here. The $D(\alpha, \beta)\omega(\gamma)$ form a representation of the Heisenberg-Weyl group. We note that

$$D(\alpha, \beta)|X; j\rangle = \omega_d(2^{-1}\alpha\beta + \alpha j)|X; j + \beta\rangle;$$

$$D(\alpha, \beta)|P; j\rangle = \omega_d(-2^{-1}\alpha\beta - \beta j)|P; j + \alpha\rangle. \tag{8}$$

The

$$\mathcal{X} = -i\frac{d}{2\pi}\log(Z); \quad \mathcal{P} = i\frac{d}{2\pi}\log(X); \quad F\mathcal{P}F^\dagger = -\mathcal{X}. \tag{9}$$

are $d \times d$ matrices which can be interpreted as position and momentum operators. The commutator $[\mathcal{X}, \mathcal{P}]$ can be calculated (it is not $i\mathbf{1}$) but it has no mathematical significance because the Heisenberg-Weyl group in this context is discrete, and the concept of generators is non-applicable. Hamiltonians can be written as functions of these operators as $h(\mathcal{X}, \mathcal{P})$.

2.1 Wigner and Weyl functions

The parity operator (around the point (γ, δ)) is defined as

$$\mathfrak{P}(\{\gamma, \delta\}) = D(\gamma, \delta)F^2[D(\gamma, \delta)]^\dagger; \quad [\mathfrak{P}(\{\gamma, \delta\})]^2 = \mathbf{1}. \tag{10}$$

It is related to the displacement operators through the Fourier transform

$$\mathfrak{P}(\gamma, \delta) = \frac{1}{d}\sum_{\alpha,\beta} D(\alpha, \beta)\omega_d(\beta\gamma - \alpha\delta);$$

$$D(\alpha, \beta) = \frac{1}{d}\sum_{\gamma,\delta} \mathfrak{P}(\gamma, \delta)\omega_d(-\beta\gamma + \alpha\delta). \tag{11}$$

If ρ is a density matrix, we define the Wigner function $W(\gamma, \delta)$ and the Weyl function $\widetilde{W}(\alpha, \beta)$ as:

$$W(\gamma, \delta) = \text{Tr}[\rho\mathfrak{P}(\gamma, \delta)]; \quad \widetilde{W}(\alpha, \beta) = \text{Tr}[\rho D(\alpha, \beta)]. \tag{12}$$

From Eq. (11) follows immediately that they are related to each other through the Fourier transform:

$$W(\gamma, \delta) = \frac{1}{d}\sum_{\alpha,\beta} \widetilde{W}(\alpha, \beta)\omega_d(\beta\gamma - \alpha\delta);$$

$$\widetilde{W}(\alpha, \beta) = \frac{1}{d}\sum_{\gamma,\delta} W(\gamma, \delta)\omega_d(-\beta\gamma + \alpha\delta). \tag{13}$$

The following marginal properties of the Wigner function are well known for odd values of the dimension d (e.g., [2]):

$$\frac{1}{d}\sum_{\gamma} W(\gamma, \delta) = \langle X; \delta|\rho|X; \delta\rangle;$$

$$\frac{1}{d}\sum_{\delta} W(\gamma, \delta) = \langle P; \gamma|\rho|P; \gamma\rangle;$$

$$\frac{1}{d}\sum_{\gamma,\delta} W(\gamma, \delta) = 1. \tag{14}$$

3 Local phase space methods

3.1 Local Fourier transforms

We consider a n-partite system comprised of n components each of which is a qudit. This system is described with the d^n-dimensional Hilbert space $\mathfrak{H} = H(d)\otimes...\otimes H(d)$.

Positions and momenta take values in $[\mathbb{Z}(d)]^n = \mathbb{Z}(d) \times ... \times \mathbb{Z}(d)$. If ρ is the density matrix of the system, we use the notation

$$\check{\rho}_r = \text{Tr}_{i \neq r} \rho; \quad r = 0, ..., n-1, \tag{15}$$

for the reduced density matrix describing the r-component of the system. We also define the corresponding factorisable density matrix

$$\mathfrak{R}(\rho) = \check{\rho}_0 \otimes ... \otimes \check{\rho}_{n-1}; \quad \text{Tr}\mathfrak{R}(\rho) = 1, \tag{16}$$

and the correlator

$$\mathfrak{C}(\rho) = \rho - \mathfrak{R}(\rho); \quad \text{Tr}\mathfrak{C}(\rho) = 0. \tag{17}$$

For factorisable density matrices $\mathfrak{R}(\rho) = \rho$ and $\mathfrak{C}(\rho) = 0$. Below we compare quantities for ρ with the corresponding quantities for $\mathfrak{R}(\rho)$.

We consider the basis

$$|X; j_0, ..., j_{n-1}\rangle = |X; j_0\rangle \otimes ... \otimes |X; j_{n-1}\rangle; \quad j_r \in \mathbb{Z}(d). \tag{18}$$

called basis of position states. We also consider the local Fourier transforms:

$$F_L = F \otimes ... \otimes F; \quad F_L^4 = \mathbf{1}; \quad F_L F_L^\dagger = \mathbf{1}. \tag{19}$$

The index L in the notation stands for local. Acting with F_L on the basis $|X; j_0, ..., j_{n-1}\rangle$ we get the 'local momentum states':

$$|P_L; j_0, ..., j_{n-1}\rangle = F_L|X; j_0, ..., j_{n-1}\rangle = \frac{1}{\sqrt{d^d}} \bigotimes_{r=0}^{n-1} \left[\sum_{k_r=0}^{d-1} \omega_d(j_r k_r)|X; k_r\rangle \right]$$
$$= |P; j_0\rangle \otimes ... \otimes |P; j_{n-1}\rangle. \tag{20}$$

F_L^2 is a parity operator in the sense that

$$F_L^2|X; j_0, ..., j_{n-1}\rangle = |X; -j_0, ..., -j_{n-1}\rangle. \tag{21}$$

For later use we define the matrix elements of the correlator $\mathfrak{C}(\rho)$:

$$C(X; j_0, ..., j_{n-1}) = \langle X; j_0, ..., j_{n-1}|\mathfrak{C}(\rho)|X; j_0, ..., j_{n-1}\rangle$$
$$= \langle X; j_0, ..., j_{n-1}|\rho|X; j_0, ..., j_{n-1}\rangle - \prod_{r=0}^{n-1}\langle X; j_r|\check{\rho}_r|X; j_r\rangle, \tag{22}$$

and

$$\mathcal{C}(P_L; j_0, ..., j_{n-1}) = \langle P_L; j_0, ..., j_{n-1} | \mathfrak{C}(\rho) | P_L; j_0, ..., j_{n-1} \rangle$$

$$= \langle P_L; j_0, ..., j_{n-1} | \rho | P_L; j_0, ..., j_{n-1} \rangle - \prod_{r=0}^{n-1} \langle P; j_r | \breve{\rho}_r | P; j_r \rangle.$$

$$(23)$$

Then

$$\text{Tr}\mathfrak{C}(\rho) = \sum_{j_0, ..., j_{n-1}} \mathcal{C}(X; j_0, ..., j_{n-1}) = \sum_{j_0, ..., j_{n-1}} \mathcal{C}(X; j_0, ..., j_{n-1}) = 0. \quad (24)$$

For factorisable density matrices $\mathcal{C}(X; j_0, ..., j_{n-1}) = \mathcal{C}(P_L; j_0, ..., j_{n-1}) = 0$.

3.2 Displacements in $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$

The phase space of the system is $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$ and local displacement operators in it are defined as

$$\mathfrak{X}_L(\{\beta_r\}) = \sum_{j_r} \omega_d(-\beta_0 j_0 - ... - \beta_{n-1} j_{n-1}) | P_L; j_0, ..., j_{n-1} \rangle \langle P_L; j_0, ..., j_{n-1} |$$

$$= \sum_{j_r} | X; j_0 + \beta_0, ..., j_{n-1} + \beta_{n-1} \rangle \langle X; j_0, ..., j_{n-1} |$$

$$= X^{\beta_0} \otimes ... \otimes X^{\beta_{n-1}}, \quad (25)$$

where $r = 0, ..., n-1$, and

$$\mathfrak{Z}_L(\{\alpha_r\}) = \sum_{j_r} | P_L; j_0 + \alpha_0, ..., j_{n-1} + \alpha_{n-1} \rangle \langle P_L; j_0, ..., j_{n-1} |$$

$$= \sum_{j_r} \omega_d(\alpha_0 j_0 + ... + \alpha_{n-1} j_{n-1}) | X; j_0, ..., j_{n-1} \rangle \langle X; j_0, ..., j_{n-1} |$$

$$= Z^{\alpha_0} \otimes ... \otimes Z^{\alpha_{n-1}} = F_L \mathfrak{X}_L(\{\alpha_r\}) F_L^{\dagger}. \quad (26)$$

Since $\mathfrak{Z}_L(\{\alpha_r\})\mathfrak{Z}_L(\{\gamma_r\}) = \mathfrak{Z}_L(\{\alpha_r + \gamma_r\})$, the $\mathfrak{Z}_L(\{\alpha_r\})$ form a representation of $[\mathbb{Z}(d)]^n$ as an additive group The same is true for the $\mathfrak{X}_L(\{\beta_r\})$. Also

$$\mathfrak{X}_L(\{\beta_r\})\mathfrak{Z}_L(\{\alpha_r\}) = \mathfrak{Z}_L(\{\alpha_r\})\mathfrak{X}_L(\{\beta_r\})\omega_d[-(\alpha_0\beta_0 + ... + \alpha_{n-1}\beta_{n-1})]$$

$$[\mathfrak{X}_L(\{\beta_r\})]^d = [\mathfrak{Z}_L(\{\alpha_r\})]^d = \mathbf{1}. \quad (27)$$

Using the notation

$$\{\alpha_r, \beta_r\} = \{a_0, ..., a_{n-1}, \beta_0, ..., \beta_{n-1}\}, \quad (28)$$

general local displacement operators are defined as

$$
\begin{aligned}
D_L(\{\alpha_r, \beta_r\}) &= \mathfrak{Z}_L(\{\alpha_r\})\mathfrak{X}_L(\{\beta_r\})\omega_d[-2^{-1}(\alpha_0\beta_0 + \ldots + \alpha_{n-1}\beta_{n-1})] \\
&= D(\alpha_0, \beta_0) \otimes \ldots \otimes D(\alpha_{n-1}, \beta_{n-1}) \quad \alpha_r, \beta_r \in \mathbb{Z}(d).
\end{aligned}
\tag{29}
$$

The $D_L(\{\alpha_r, \beta_r\})\omega(\{\gamma_r\})$ form a representation of the Heisenberg-Weyl group of displacements in the phase space $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$.

The local parity operator (around the point $\{\gamma_r, \delta_r\}$ in the phase space $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$) is defined as

$$
\begin{aligned}
\mathfrak{P}_L(\{\gamma_r, \delta_r\}) &= D_L(\{\gamma_r, \delta_r\})F_L^2[D_L(\{\gamma_r, \delta_r\})]^\dagger \\
&= \mathfrak{P}(\gamma_0, \delta_0) \otimes \ldots \otimes \mathfrak{P}(\gamma_{n-1}, \delta_{n-1}) \\
[\mathfrak{P}_L(\{\gamma_r, \delta_r\})]^2 &= \mathbf{1}.
\end{aligned}
\tag{30}
$$

It is related to the local displacement operators through the Fourier transform

$$
\begin{aligned}
\mathfrak{P}_L(\{\gamma_r, \delta_r\}) &= \frac{1}{d^n} \sum_{\{\alpha_r, \beta_r\}} D_L(\{\alpha_r, \beta_r\})\omega_d\left[\sum_{r=0}^{n-1}(\beta_r\gamma_r - \alpha_r\delta_r)\right]; \\
D_L(\{\alpha_r, \beta_r\}) &= \frac{1}{d^n} \sum_{\{\gamma_r, \delta_r\}} \mathfrak{P}_L(\{\gamma_r, \delta_r\})\omega_d\left[\sum_{r=0}^{n-1}(-\beta_r\gamma_r + \alpha_r\delta_r)\right].
\end{aligned}
\tag{31}
$$

The proof of this follows easily from Eq. (11).

3.3 Local Wigner and local Weyl functions in $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$

If ρ is a density matrix, we define the local Wigner function $W_L(\{\gamma_r, \delta_r\}|\rho)$ and the local Weyl function $\widetilde{W}_L(\{\alpha_r, \beta_r\}|\rho)$ as:

$$
W_L(\{\gamma_r, \delta_r\}|\rho) = \mathrm{Tr}[\rho\mathfrak{P}_L(\{\gamma_r, \delta_r\})]; \quad \widetilde{W}_L(\{\alpha_r, \beta_r\}|\rho) = \mathrm{Tr}[\rho D_L(\{\alpha_r, \beta_r\})]
\tag{32}
$$

From Eq. (31) follows immediately that they are related to each other through the Fourier transform:

$$
\begin{aligned}
W_L(\{\gamma_r, \delta_r\}|\rho) &= \frac{1}{d^n} \sum_{\{\alpha_r, \beta_r\}} \widetilde{W}_L(\{\alpha_r, \beta_r\}|\rho)\omega_d\left[\sum_{r=0}^{n-1}(\beta_r\gamma_r - \alpha_r\delta_r)\right]; \\
\widetilde{W}_L(\{\alpha_r, \beta_r\}|\rho) &= \frac{1}{d^n} \sum_{\{\gamma_r, \delta_r\}} W_L(\{\gamma_r, \delta_r\}|\rho)\omega_d\left[\sum_{r=0}^{n-1}(-\beta_r\gamma_r + \alpha_r\delta_r)\right].
\end{aligned}
\tag{33}
$$

4 Global Fourier tarnsforms

4.1 A bijective map between the non-isomorphic rings $[\mathbb{Z}(d)]^n$ and $\mathbb{Z}(d^n)$

We consider a bijective map between $[\mathbb{Z}(d)]^n$ and $\mathbb{Z}(d^n)$ as follows. We first take each $j_r \in \mathbb{Z}(d)$ and $\widehat{j} \in \mathbb{Z}(d^n)$ in the 'periods'

$$\left[-\frac{d-1}{2}, \frac{d-1}{2}\right]; \quad \left[-\frac{d^n-1}{2}, \frac{d^n-1}{2}\right], \tag{34}$$

correspondingly (for odd d). We introduce the bijective map

$$j = (j_0, ..., j_{d-1}) \leftrightarrow \widehat{j} = j_0 + j_1 d + ... + j_{n-1} d^{n-1}. \tag{35}$$

We then take each j_r modulo d and the $\widehat{j}$ modulo d^n, and we get a bijective map from $[\mathbb{Z}(d)]^n$ to $\mathbb{Z}(d^n)$. Numbers in $\mathbb{Z}(d^n)$ will be denoted with a 'hat', so that it is clear whether a number belongs to $\mathbb{Z}(d)$ or to $\mathbb{Z}(d^n)$.

The Hilbert space $\mathfrak{H}$ is isomorphic to $H(d^n)$ (a d^n-dimensional Hilbert space describing systems with variables in $\mathbb{Z}(d^n)$). But the $[\mathbb{Z}(d)]^n$ as a ring (with addition and multiplication componentwise), is not isomorphic to the ring $\mathbb{Z}(d^n)$ because addition and multiplication is different, and consequently our 'local formalism' is different from our 'global formalism'. Indeed

$$\widehat{j} + \widehat{k} \neq \widehat{j+k}; \quad \widetilde{\widehat{jk}} \neq \widehat{j}\widehat{k}. \tag{36}$$

The sum is different because $\widehat{j} + \widehat{k}$ in $\mathbb{Z}(d^n)$ has the 'carry' rule and the r-component might be $j_r + k_r + 1$ rather than $j_r + k_r$. In contrast, there is no 'carry' rule in $[\mathbb{Z}(d)]^n$:

$$j + k = (j_0 + k_0, ..., j_{d-1} + k_{d-1}) \leftrightarrow \widehat{j+k} = (j_0 + k_0) + (j_1 + k_1)d + ...$$
$$+ (j_{n-1} + k_{n-1})d^{n-1}. \tag{37}$$

Also multiplication in $\mathbb{Z}(d^n)$ is

$$\widetilde{\widehat{jk}} = j_0 k_0 + d(j_1 k_0 + k_1 j_0) + ... + d^{n-1}(j_0 k_{n-1} + ... + j_{n-1} k_0). \tag{38}$$

The corresponding multiplication in $[\mathbb{Z}(d)]^n$ is

$$(j_0, ..., j_{n-1}) \cdot (k_0, ..., k_{n-1}) = (j_0 k_0, ..., j_{n-1} k_{n-1}), \tag{39}$$

and with the bijective map in Eq. (35) this corresponds to

$$\widehat{jk} = j_0 k_0 + d(j_1 k_1) + ... + d^{n-1}(j_{n-1} k_{n-1}). \tag{40}$$

It is seen that in general $\widetilde{\widehat{jk}} \neq \widehat{j}\widehat{k}$ (but $\widehat{1k} = \widehat{k}$).

Example 4.1 We consider the elements of $\mathbb{Z}(3)$ in the 'period' $[-1, 1]$ and the elements of $\mathbb{Z}(9)$ in the 'period' $[-4, 4]$. A bijective map between $[\mathbb{Z}(3)]^2$ and $\mathbb{Z}(9)$ is as follows

$$\widehat{(-1, -1)} = \widehat{-4}; \quad \widehat{(0, -1)} = \widehat{-3}; \quad \widehat{(1, -1)} = \widehat{-2}; \quad \widehat{(-1, 0)} = \widehat{-1}; \quad \widehat{(0, 0)} = \widehat{0};$$
$$\widehat{(1, 0)} = \widehat{1}; \quad \widehat{(-1, 1)} = \widehat{2}; \quad \widehat{(0, 1)} = \widehat{3}; \quad \widehat{(1, 1)} = \widehat{4}. \tag{41}$$

An example of addition that confirms Eq. (36) is the following:

$$\widehat{(1, 1)} + \widehat{(1, 1)} = \widehat{4} + \widehat{4} = \widehat{-1};$$
$$\widehat{(1, 1) + (1, 1)} = \widehat{(-1, -1)} = \widehat{-4}. \tag{42}$$

An example of multiplication that confirms Eq. (36) is the following:

$$\widehat{(-1, 0)} \cdot \widehat{(0, 1)} = \widehat{-1} \cdot \widehat{3} = \widehat{-3};$$
$$\widehat{(-1, 0) \cdot (0, 1)} = \widehat{(0, 0)} = \widehat{0}. \tag{43}$$

Remark 4.2 If $d_1, ..., d_n$ are coprime to each other, then the $\mathbb{Z}(d_1) \times ... \times \mathbb{Z}(d_n)$ is isomorphic to $\mathbb{Z}(d_1 \times ... \times d_n)$. We can define a bijective map

$$(j_0, ..., j_{n-1}) \leftrightarrow j; \quad j_r \in \mathbb{Z}(d_r); \quad j \in \mathbb{Z}(d_1 \times ... \times d_n), \tag{44}$$

such that

$$(j_0 + k_0, ..., j_{n-1} + k_{n-1}) \leftrightarrow j + k;$$
$$(j_0 k_0, ..., j_{n-1} k_{n-1}) \leftrightarrow jk. \tag{45}$$

This is based on the Chinese remainder theorem, and has been used by Good [12] in the context of fast Fourier transforms (see also [5–7]). In a quantum context it has been use in [2, 13] for factorisation of a quantum system into subsystems. Here we consider the case $d_1 = ... = d_n$ and then the bijective map of Eq. (35) does not establish an isomorphism between the ring $[\mathbb{Z}(d)]^n$ and the ring $\mathbb{Z}(d^n)$ (because of Eq. (36)).

4.2 Dual notation

We use the following dual notation for position states, based on the bijective map in Eq. (35):

$$|X; j_0, ..., j_{n-1}\rangle = |X; \widehat{j}\rangle. \tag{46}$$

When local operators act on them we use addition and multiplication in $[\mathbb{Z}(d)]^n$, in connection with the phase space $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$. When global operators (defined below) act on them we use addition and multiplication in $\mathbb{Z}(d^n)$, in connection with the phase space $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$.

Analogous dual notation is used for all quantities. For example, the displacement operators in Eq. (29) can be denoted as

$$D_L(\{\alpha_r, \beta_r\}) = D_L(\widehat{\alpha}, \widehat{\beta}).$$
(47)

In some equations both notations appear together.

4.3 Global Fourier transforms

The global Fourier transform in $\mathfrak{H}$ is defined as:

$$F_G = \frac{1}{\sqrt{d^n}} \sum_{\widehat{j}, \widehat{k}} \omega_{d^n}(\widehat{\widehat{jk}}) |X; j_0, ..., j_{n-1}\rangle\langle X; k_0, ..., k_{n-1}|.$$
(48)

The index G in the notation stands for global. It is easily seen that

$$F_G^4 = \mathbf{1}; \quad F_G F_G^\dagger = \mathbf{1}; \quad F_G \neq F_L.$$
(49)

Acting with F_G on the basis $|X; j_0, ..., j_{n-1}\rangle$ (which we also denote as $|X; \widehat{j}\rangle$) we get the 'global momentum states':

$$|P_G; \widehat{j}\rangle = |P_G; j_0, ..., j_{n-1}\rangle = F_G |X; \widehat{j}\rangle$$
$$= \frac{1}{\sqrt{d^n}} \bigotimes_{r=0}^{n-1} \left[\sum_{k_r=0}^{d-1} \omega_{d^n}[(j_0 d^r + .. + j_{n-r-1} d^{n-1}) k_r] |X; k_r\rangle \right]. \quad (50)$$

In the states $|P_G; j_0, ..., j_{n-1}\rangle$, the coefficients $\omega_{d^n}[(j_0 d^r + .. + j_{n-r-1} d^{n-1}) k_r]$ in the r-component depend on all $j_0, ..., j_{n-1}$, and the term 'global' refers to this. Information from all components is needed, in order to determine these coefficients. In the local Fourier transform of Eq. (21), the coefficients $\omega_d(j_r k_r)$ in the r-component depend only on j_r. We note that

$$\langle P_L; \ell_0, ..., \ell_{n-1}|P_G; j_0, ..., j_{n-1}\rangle = \langle X; \ell_0, ..., \ell_{n-1}|F_L^\dagger F_G|X; j_0, ..., j_{n-1}\rangle$$
$$= \frac{1}{d^n} \sum_{\widehat{k}} \omega_{d^n}(\widehat{\widehat{jk}}) \omega_d[-(\ell_0 k_0 + ... + \ell_{d-1} k_{n-1})].$$
(51)

and that

$$|\langle X; \ell_0, ..., \ell_{n-1}|P_L; j_0, ..., j_{n-1}\rangle|^2 = |\langle X; \ell_0, ..., \ell_{n-1}|P_G; j_0, ..., j_{n-1}\rangle|^2 = \frac{1}{d^n}.$$
(52)

Proposition 4.3 *We take the elements of $\mathbb{Z}(d)$ and the elements of $\mathbb{Z}(d^n)$ in the 'periods' of Eq. (34). Then*

(1) The parity operator around the origin is the same in both the local and global formalism:

$$F_G^2 = F_L^2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}. \tag{53}$$

Here the matrix is in the position basis.
(2) For any n

$$|P_G; \widehat{-d^{n-1}}\rangle = |P_L; -1, 0, ..., 0\rangle; \quad |P_G; 0\rangle = |P_L; 0, 0, ..., 0\rangle; \quad |P_G; \widehat{d^{n-1}}\rangle$$
$$= |P_L; 1, 0, ..., 0\rangle. \tag{54}$$

(3) For $n = 2$ we have the stronger result

$$|P_G; \widehat{d\lambda}\rangle = |P_L; \lambda, 0\rangle; \quad \lambda = -\frac{d-1}{2}, ..., \frac{d-1}{2}. \tag{55}$$

At least d of the global momentum states are equal to d of the local momentum states.

Proof (1) For $\mathbb{Z}_{d^n}$ Eq. (4) becomes

$$\frac{1}{d^n} \sum_{\widehat{k}} \omega_{d^n}[(\widehat{j} + \widehat{\ell})\widehat{k}] = \delta(\widehat{j} + \widehat{\ell}, 0); \quad \widehat{j}, \widehat{\ell} \in \mathbb{Z}_{d^n}. \tag{56}$$

The $\widehat{j} + \widehat{\ell} = 0$ implies $j_r + \ell_r = 0$, and we prove that

$$F_G^2 = \frac{1}{d^d} \sum_{\widehat{j}, \widehat{k}, \widehat{\ell}} \omega_{d^n}[(\widehat{j} + \widehat{\ell})\widehat{k}] |j_0, ..., j_{n-1}\rangle\langle\ell_0, ..., \ell_{n-1}|$$
$$= \sum_{j_0, ..., j_{n-1}} |j_0, ..., j_{n-1}\rangle\langle-j_0, ..., -j_{n-1}| = F_L^2. \tag{57}$$

(2) Using Eq. (51) we get

$$\langle P_L; 1, ..., 0 | P_G; \widehat{d^{n-1}}\rangle = \frac{1}{d^n} \sum_{\widehat{k}} \omega_{d^n}(\widehat{d^{n-1}}\widehat{k})\omega_d(-k_0)$$
$$= \frac{1}{d^n} \sum_{\widehat{k}} \omega_{d^n}(d^{n-1}k_0)\omega_d(-k_0) = \frac{1}{d^n} \sum_{\widehat{k}} 1 = 1. \tag{58}$$

In a similar way we prove that

$$|P_G; \widehat{-d^{n-1}}\rangle = |P_L; -1, 0, ..., 0\rangle. \tag{59}$$

(3) Using Eq. (51) with $n = 2$ and $\widehat{k} = k_0 + dk_1$ we get

$$\langle P_L; \lambda, 0 | P_G; \widehat{d\lambda}\rangle = \frac{1}{d^2} \sum_{\widehat{k}} \omega_{d^2}(\widehat{d\lambda k}) \omega_d(-\lambda k_0) = \frac{1}{d^2} \sum_{\widehat{k}} \omega_{d^2}(\widehat{d\lambda k} - d\lambda k_0)$$

$$= \frac{1}{d^2} \sum_{\widehat{k}} \omega_{d^2}(d^2 \lambda k_1) = 1. \tag{60}$$

$d\lambda$ takes values between $-\frac{d^2-1}{2}$ and $\frac{d^2-1}{2}$ and consequently λ takes the values in Eq. (55).

$\square$

For later use we define the matrix elements of the correlator $\mathfrak{C}(\rho)$:

$$\mathcal{C}(X; \widehat{j}) = \langle X; \widehat{j} | \mathfrak{C}(\rho) | X; \widehat{j}\rangle; \quad \mathcal{E}(P_G; \widehat{j}) = \langle P_G; \widehat{j} | \mathfrak{C}(\rho) | P_G; \widehat{j}\rangle. \tag{61}$$

In both the 'local formalism' and the 'global formalism' the position states are the same and the momentum states are different. Consequently the $\mathcal{E}(P_G; \widehat{j})$ is different from the corresponding $\mathcal{C}(P_L; j_0, ..., j_{n-1})$.

Then

$$\mathrm{Tr}\mathfrak{C}(\rho) = \sum_{\widehat{j}} \mathcal{C}(X; \widehat{j}) = \sum_{\widehat{j}} \mathcal{E}(P_G; \widehat{j}) = 0. \tag{62}$$

For factorisable density matrices $\mathcal{C}(X; \widehat{j}) = \mathcal{E}(P_G; \widehat{j}) = 0$.

4.4 Unitarily inequivalent local and global Fourier transforms

In this paper we use the following definition of unitary equivalence. Two square matrices A, B are called unitarily equivalent if there exists a unitary matrix U such that $A = UBU^\dagger$. Unitary equivalence is an equivalence relation, i.e., matrices which are unitarily inequivalent belong to different equivalence classes. It is known [14, 15] that two normal $d \times d$ matrices A, B are unitarily equivalent if and only if

$$||A|| = ||B||; \quad \mathrm{Tr}(A^\eta) = \mathrm{Tr}(B^\eta); \quad \eta = 1, ..., d; \quad ||A|| = \sqrt{\sum_{i,j} |A_{ij}|^2}. \tag{63}$$

We note that Specht's general theorem for unitary equivalence (e.g., [15]) reduces easily to the above criteria for the Fourier matrices which are unitary.

Some authors call the above unitary similarity, and they use the term unitary equivalence for the case where there exist two unitary matrices U, V such that $A = UBV^\dagger$.

Proposition 4.4 *In an n-partite system that has Hilbert space with dimension d^n, the $d^n \times d^n$ matrices F_G, F_L are unitarily equivalent in the cases*

$$
\begin{aligned}
&d = 4m + 1; \\
&d = 4m + 3 \text{ and } n = 4N; \\
&d = 4m + 3 \text{ and } n = 4N + 1.
\end{aligned}
\tag{64}
$$

The matrices F_G, F_L are unitarily inequivalent in the cases

$$
\begin{aligned}
&d = 4m + 3 \text{ and } n = 4N + 2; \\
&d = 4m + 3 \text{ and } n = 4N + 3.
\end{aligned}
\tag{65}
$$

Proof The matrices F_G and F_L are unitary and therefore normal, and we use the criterion in Eq. (63). We first note that

$$
||F_G|| = ||F_L|| = d^n.
\tag{66}
$$

We next compare $\text{Tr}(F_G^\eta)$ with $\text{Tr}(F_L^\eta)$ for $\eta = 1, ..., d^n$. But

$$
\begin{aligned}
\eta = 4\epsilon &\rightarrow F_G^\eta = F_L^\eta = \mathbf{1}; \\
\eta = 4\epsilon + 1 &\rightarrow F_G^\eta = F_G; \quad F_L^\eta = F_L; \\
\eta = 4\epsilon + 2 &\rightarrow F_G^\eta = F_L^\eta; \\
\eta = 4\epsilon + 3 &\rightarrow F_G^\eta = F_G^\dagger; \quad F_L^\eta = F_L^\dagger.
\end{aligned}
\tag{67}
$$

Therefore if $\text{Tr}(F_G) = \text{Tr}(F_L)$ the F_G, F_L are unitarily equivalent, and if $\text{Tr}(F_G) \neq \text{Tr}(F_L)$ the F_G, F_L are unitarily inequivalent.

For an *n*-partite system with dimension d^n we get $\text{Tr} F_L = (\text{Tr} F)^n$ and using Eq. (2) we get

$$
\begin{aligned}
&d = 4m + 1 \rightarrow \text{Tr} F_L = 1; \\
&d = 4m + 3 \rightarrow \text{Tr} F_L = i^n.
\end{aligned}
\tag{68}
$$

For the $\text{Tr} F_G$ if $d = 4m + 1$, the $d^n = (4m + 1)^n = 4M_1 + 1$ and we get

$$
d = 4m + 1 \rightarrow \text{Tr} F_G = 1.
\tag{69}
$$

If $d = 4m + 3$ we consider two cases where n is an even number and an odd number. For even n we get $d^n = (4m+3)^n = 4M_2 + 1$ and for odd n we find $d^n = (4m+3)^n = 4M_3 + 3$. Therefore

$$
\begin{aligned}
&d = 4m + 3 \text{ and } n = \text{even} \rightarrow \text{Tr} F_G = 1; \\
&d = 4m + 3 \text{ and } n = \text{odd} \rightarrow \text{Tr} F_G = i.
\end{aligned}
\tag{70}
$$

Comparison of Eq. (68) with Eqs. (69, 70) proves the proposition. $\qquad\square$

In the case of Eq. (64) there exists a $d^n \times d^n$ unitary matrix U such that $F_G = U F_L U^\dagger$, i.e.,

$$\omega_{d^n}(\widehat{\widetilde{ij}}) = \sum_{\widehat{j},\widehat{\ell}} U(\widehat{i},\widehat{k}) U(\widehat{j},\widehat{\ell}) \omega_d(\ell_0 k_0 + ... + \ell_{d-1} k_{n-1}). \tag{71}$$

So if instead of the basis $|X; \widehat{j}\rangle$ we choose the basis $U|X; \widehat{j}\rangle$ as position states, then the local Fourier transform with respect to the new basis is the global Fourier transform with respect to the old basis $F_G = U F_L U^\dagger$. So in this case the global Fourier transform is not a new concept. However U is in general a global transformation (it cannot be written as $U_1 \otimes ... \otimes U_n$) and for this reason there is some merit in the study of the global Fourier transform even in this case.

The case of Eq. (65) where global and local Fourier transforms are unitarily inequivalent is clearly the most interesting one. Then the global Fourier transform is a new concept. In any case, the formalism below is the same for both cases in Eqs. (64, 65).

4.5 The global Fourier transform in terms of local Fourier transforms and applications in Fast Fourier transforms

The general idea of Fast Fourier transforms is to express the 'large' Fourier transform in a large Hilbert space, as an 'appropriate' combination of 'small' Fourier transforms in smaller Hilbert spaces. Performing the 'small' Fourier transforms instead of the 'large' Fourier transform, is computationally beneficial. The general formalism of this paper can be helpful in this direction.

As an example, we express in this section the global Fourier transform in terms of many local Fourier transforms. This is similar to the Cooley–Tukey formalism in fast Fourier transforms [5–7]. We only consider the special case $n = 2$, and we do not discuss complexity issues. But we point out that understanding of the relationship between global and local Fourier transforms and related phase space methods, can be useful in other areas like fast Fourier transforms.

For the case $n = 2$ we get

$$\omega_{d^2}(\widehat{\widetilde{jk}}) = \omega_{d^2}(j_0 k_0) \omega_d(j_0 k_1 + j_1 k_0). \tag{72}$$

Let $|s\rangle$ be a quantum state in $\mathfrak{H} = H(d) \otimes H(d)$ and $s(k_0, k_1) = \langle X; k_0, k_1|s\rangle$. Then

$$\begin{aligned}
\langle X; j_0, j_1|F_G|s\rangle &= \frac{1}{d} \sum_{k_0,k_1} \omega_{d^2}(\widehat{\widetilde{jk}}) s(k_0, k_1) \\
&= \frac{1}{d} \sum_{k_0} \omega_d(j_1 k_0) \omega_{d^2}(j_0 k_0) \sum_{k_1} \omega_d(j_0 k_1) s(k_0, k_1) \\
&= \frac{1}{\sqrt{d}} \sum_{k_0} \omega_d(j_1 k_0) \left[\omega_{d^2}(j_0 k_0) \widetilde{s}(k_0, j_0) \right], \tag{73}
\end{aligned}$$

where

$$\widetilde{s}(k_0, j_0) = \frac{1}{\sqrt{d}} \sum_{k_1} \omega_d(j_0 k_1) s(k_0, k_1). \tag{74}$$

In this way the Fourier transform in a d^2-dimensional space reduces to two Fourier transforms in d-dimensional spaces.

5 Global phase space methods

5.1 Global displacements in $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$

The phase space is defined by the Fourier transform and for global Fourier transforms is $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$. Displacement operators in it are defined as

$$\mathfrak{X}_G(\widehat{\beta}) = \sum_{\widehat{j}} \omega_{d^n}(-\widehat{j}\widehat{\beta})|P_G; \widehat{j}\rangle\langle P_G; \widehat{j}| = \sum_{\widehat{j}} |X; \widehat{j} + \widehat{\beta}\rangle\langle X; \widehat{j}|, \tag{75}$$

and

$$\mathfrak{Z}_G(\widehat{\alpha}) = \sum_{\widehat{j}} |P_G; \widehat{j} + \widehat{\alpha}\rangle\langle P_G; \widehat{j}| = \sum_{\widehat{j}} \omega_{d^n}(\widehat{\alpha}\widehat{j})|X; \widehat{j}\rangle\langle X; \widehat{j}| = F_G \mathfrak{X}_G(\widehat{\alpha}) F_G^\dagger. \tag{76}$$

Addition in $\mathbb{Z}(d^n)$ is used in these two equations, in contrast to Eqs. (26, 25) where we have addition in $[\mathbb{Z}(d)]^n$. The $\mathfrak{X}_L(\{\beta_r\})$ in Eq. (25) can also be written as

$$\mathfrak{X}_L(\widehat{\beta}) = \sum_{\widehat{j}} |X; \widehat{j + \beta}\rangle\langle X; \widehat{j}|. \tag{77}$$

We have explained that $\widehat{j + \beta} \neq \widehat{j} + \widehat{\beta}$, and consequently Eqs. (75, 77) are an example of the difference between the local and global formalism. Also the $\mathfrak{Z}_L(\{\alpha_r\})$ in Eq. (25) can also be written as

$$\mathfrak{Z}_L(\widehat{\alpha}) = \sum_{j_r} \omega_d(\alpha_0 j_0 + ... + \alpha_{n-1} j_{n-1})|X; \widehat{j}\rangle\langle X; \widehat{j}|. \tag{78}$$

Comparison of Eqs. (76, 78) again shows the difference between the local and global formalism.

Since $\mathfrak{Z}_G(\widehat{\alpha})\mathfrak{Z}_G(\widehat{\gamma}) = \mathfrak{Z}_G(\widehat{\alpha} + \widehat{\gamma})$, the $\mathfrak{Z}_G(\widehat{\alpha})$ form a representation of $\mathbb{Z}(d^n)$ as an additive group (which is not isomorphic to $[\mathbb{Z}(d)]^n$). The same is true for the $\mathfrak{X}_G(\widehat{\beta})$.

Also

$$\mathfrak{X}_G(\widehat{\beta})\mathfrak{Z}_G(\widehat{\alpha}) = \mathfrak{Z}_G(\widehat{\alpha})\mathfrak{X}_G(\widehat{\beta})\omega_{d^n}(-\widehat{\alpha}\widehat{\beta});$$
$$[\mathfrak{X}_G(\widehat{\beta})]^{d^n} = [\mathfrak{Z}_G(\widehat{\alpha})]^{d^n} = \mathbf{1}.$$

(79)

These relations should be compared and contrasted with Eq. (27) for the local formalism. Global displacement operators are defined as

$$D_G(\widehat{\alpha}, \widehat{\beta}) = \mathfrak{Z}_G(\widehat{\alpha})\mathfrak{X}_G(\widehat{\beta})\omega_{d^n}(-2^{-1}\widehat{\alpha}\widehat{\beta}).$$

(80)

Here $2^{-1} = \frac{d^n+1}{2}$ is an element of $\mathbb{Z}(d^n)$. The $D_G(\widehat{\alpha}, \widehat{\beta})\omega_{d^n}(\widehat{\gamma})$ form a representation of the Heisenberg-Weyl group of displacements in the phase space $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$. We note that

$$D_G(\widehat{\alpha}, \widehat{\beta})|X; \widehat{j}\rangle = \omega_{d^n}(2^{-1}\widehat{\alpha}\widehat{\beta} + \widehat{\alpha}\widehat{j})|X; \widehat{j} + \widehat{\beta}\rangle;$$
$$D_G(\widehat{\alpha}, \widehat{\beta})|P_G; \widehat{j}\rangle = \omega_{d^n}(-2^{-1}\widehat{\alpha}\widehat{\beta} - \widehat{\beta}\widehat{j})|P_G; \widehat{j} + \widehat{\alpha}\rangle.$$

(81)

Also

$$D_G(\widehat{\alpha}, \widehat{\beta})|P_L; \widehat{j}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\widehat{k}} \omega_d(j_0 k_0 + ... + j_{n-1}k_{n-1})D_G(\widehat{\alpha}, \widehat{\beta})|X; \widehat{k}\rangle$$

$$= \frac{1}{\sqrt{d^n}} \sum_{\widehat{k}} \omega_d(j_0 k_0 + ... + j_{n-1}k_{n-1})\omega_{d^n}(2^{-1}\widehat{\alpha}\widehat{\beta} + \widehat{\alpha}\widehat{k})|X; \widehat{k} + \widehat{\beta}\rangle.$$

(82)

These relations should be compared and contrasted to

$$D_L(\widehat{\alpha}, \widehat{\beta})|X; \widehat{j}\rangle = \omega_d[2^{-1}(\alpha_0\beta_0 + ... + \alpha_{n-1}\beta_{n-1})$$
$$+ (\alpha_0 j_0 + ... + \alpha_{n-1}j_{n-1})]|X; \widehat{j + \beta}\rangle;$$
$$D_L(\widehat{\alpha}, \widehat{\beta})|P_L; \widehat{j}\rangle = \omega_d[-2^{-1}(\alpha_0\beta_0 + ... + \alpha_{n-1}\beta_{n-1})$$
$$- (\beta_0 j_0 + ... + \beta_{n-1}j_{n-1})]|P_L; \widehat{j + \alpha}\rangle.$$

(83)

Also

$$D_L(\widehat{\alpha}, \widehat{\beta})|P_G; \widehat{j}\rangle = \frac{1}{\sqrt{d^n}} \sum_{\widehat{k}} \omega_{d^n}(\widehat{jk})D_L(\widehat{\alpha}, \widehat{\beta})|X; \widehat{k}\rangle$$

$$= \frac{1}{\sqrt{d^n}} \sum_{\widehat{k}} \omega_{d^n}(\widehat{jk})\omega_d[2^{-1}(\alpha_0\beta_0 + ... + \alpha_{n-1}\beta_{n-1})$$

$$+ (\alpha_0 k_0 + ... + \alpha_{n-1}k_{n-1})]|X; \widehat{k + \beta}\rangle.$$

(84)

The global parity operator (around the point $(\widehat{\gamma}, \widehat{\delta})$ in the phase space $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$) is

$$\mathfrak{P}_G(\widehat{\gamma}, \widehat{\delta}) = D_G(\widehat{\gamma}, \widehat{\delta}) F_G^2 [D_G(\widehat{\gamma}, \widehat{\delta})]^\dagger; \quad [\mathfrak{P}_G(\widehat{\gamma}, \widehat{\delta})]^2 = \mathbf{1}. \tag{85}$$

In analogy to Eq. (11) we find that the global parity operator is related to the global displacement operators through the Fourier transform

$$\mathfrak{P}_G(\widehat{\gamma}, \widehat{\delta}) = \frac{1}{d^n} \sum_{\widehat{\alpha}, \widehat{\beta}} D_G(\widehat{\alpha}, \widehat{\beta}) \omega_{d^n}(\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta});$$

$$D_G(\widehat{\alpha}, \widehat{\beta}) = \frac{1}{d^n} \sum_{\widehat{\gamma}, \widehat{\delta}} \mathfrak{P}_G(\widehat{\gamma}, \widehat{\delta}) \omega_{d^n}(-\widehat{\beta}\widehat{\gamma} + \widehat{\alpha}\widehat{\delta}). \tag{86}$$

Example 5.1 We consider the case $d = 3$ and $n = 2$. In this case the global Fourier transform is unitarily inequivalent to the local Fourier transform. We work in the 'periods' of Eq. (34).

Let $|X; k_0, k_1\rangle$ be the basis of position states. The globally Fourier transformed basis is

$$|P_G; j_0, j_1\rangle = \frac{1}{3} \sum_{k_0, k_1} \omega_9[j_0 k_0 + 3(j_1 k_0 + j_0 k_1)]|X; k_0, k_1\rangle. \tag{87}$$

The j_r, k_r take the values $-1, 0, 1$. The locally Fourier transformed basis is

$$|P_L; j_0, j_1\rangle = \frac{1}{3} \sum_{k_0, k_1} \omega_3(j_0 k_0 + j_1 k_1)|X; k_0, k_1\rangle. \tag{88}$$

Then

$$\langle P_L; \ell_0, \ell_1 | P_G; j_0, j_1 \rangle = \frac{1}{9}\{1 + \omega_9(3 j_0 + 9 j_1)\omega_3(-\ell_1) + \omega_9(j_0 + 3 j_1)\omega_3(-\ell_0)$$
$$+ \omega_9(4 j_0 + 12 j_1)\omega_3(-\ell_0 - \ell_1)]\}. \tag{89}$$

We next consider the local displacement operator $\mathfrak{X}_L(-1, 1)$ which acts on the states $|X; 1, 0\rangle$ and $|P_L; 1, 0\rangle$ as follows:

$$\mathfrak{X}_L(-1, 1)|X; 1, 0\rangle = |X; 0, 1\rangle, \tag{90}$$

and

$$\mathfrak{X}_L(-1, 1)|P_L; 1, 0\rangle = \omega_3(1)|P_L; 1, 0\rangle. \tag{91}$$

$\mathfrak{X}_L(-1, 1)$ acts on the state $|P_G; 1, 0\rangle$ as follows:

$$\mathfrak{X}_L(-1, 1)|P_G; 1, 0\rangle = \mathfrak{X}_L(-1, 1) \sum_{j_1, j_0} |X; j_0, j_1\rangle\langle X; j_0, j_1|P_G; 1, 0\rangle$$
$$= \frac{1}{3} \sum_{j_1, j_0} \omega_9(\widehat{j_0 + 3 j_1})|X; j_0 - 1, j_1 + 1\rangle. \tag{92}$$

We also consider the corresponding global displacement operator $\mathfrak{X}_G(-\widehat{1 + 3 \cdot 1}) = \mathfrak{X}_G(\widehat{2})$ which acts on the states $|X; 1, 0\rangle = |X; \widehat{1}\rangle$ and $|P_G; 1, 0\rangle = |P_G; \widehat{1}\rangle$ as follows:

$$\mathfrak{X}_G(\widehat{2})|X; \widehat{1}\rangle = |X; \widehat{3}\rangle = |X; 0, 1\rangle, \tag{93}$$

and

$$\mathfrak{X}_G(\widehat{2})|P_G; \widehat{1}\rangle = \omega_9(-\widehat{2})|P_G; \widehat{1}\rangle. \tag{94}$$

$\mathfrak{X}_G(\widehat{5})$ acts on the state $|P_L; 1, 0\rangle$ as follows:

$$\mathfrak{X}_G(\widehat{2})|P_L; 1, 0\rangle = \mathfrak{X}_G(\widehat{2}) \sum_{j_1, j_0} |X; j_0, j_1\rangle\langle X; j_0, j_1|P_L; 1, 0\rangle$$
$$= \frac{1}{3}\mathfrak{X}_G(\widehat{2}) \sum_{j_1, j_0} |X; \widehat{j_0 + 3 j_1}\rangle \omega_3(j_0) = \frac{1}{3} \sum_{j_1, j_0} \omega_3(j_0)|X; \widehat{2 + j_0 + 3 j_1}\rangle$$
$$= \frac{1}{3}\{\omega_3(-1)[|X; \widehat{-2}\rangle + |X; \widehat{1}\rangle + |X; \widehat{4}\rangle] + [|X; \widehat{-1}\rangle + |X; \widehat{2}\rangle + |X; \widehat{5}\rangle]$$
$$+\omega_3(1)[|X; \widehat{0}\rangle + |X; \widehat{3}\rangle + |X; \widehat{6}\rangle]\}. \tag{95}$$

Eqs. (90, 91, 92) involve local displacements and should be compared and contrasted to Eqs. (93, 95,91) correspondingly, that involve global displacements.

5.2 Local and global position and momentum operators and time evolution

We define local position and local momentum operators for the r-component of the system as

$$\mathcal{X}_L^{(r)} = \mathbf{1} \otimes ... \otimes \mathbf{1} \otimes \mathcal{X} \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}; \quad r = 0, ..., n - 1;$$
$$\mathcal{P}_L^{(r)} = \mathbf{1} \otimes ... \otimes \mathbf{1} \otimes \mathcal{P} \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}; \quad F_L \mathcal{P}_L F_L^\dagger = -\mathcal{X}_L. \tag{96}$$

The $\mathcal{X}, \mathcal{P}$ have been defined in Eq. (9). We can also define global position and global momentum operators as

$$\mathcal{X}_G = -i\frac{d^n}{2\pi} \log[\mathfrak{Z}_G(\widehat{1})]; \quad \mathcal{P}_G = i\frac{d^n}{2\pi} \log[\mathfrak{X}_G(\widehat{1})]; \quad F_G \mathcal{P}_G F_G^\dagger = -\mathcal{X}_G. \tag{97}$$

They all are $d^n \times d^n$ matrices which can be interpreted as position and momentum operators. In a multipartite system with weak interaction between the various parties, it can be argued that the local variables $\mathcal{X}_L^{(r)}, \mathcal{P}_L^{(r)}$ are more physical operators and the Hamiltonian should be expressed in terms of them. But in the case of strong interactions between the parties, the global variables $\mathcal{X}_G, \mathcal{P}_G$ might be better for a holistic simple description of the physical Hamiltonian with a good approximation.

Example 5.2 We consider the case $d = 3, n = 2$ and the quantum state

$$|s\rangle = \frac{1}{\sqrt{84}}(|X; 1\rangle + 2|X; 0\rangle - 3|X; -1\rangle) \otimes (i|X; 1\rangle + |X; 0\rangle - 2i|X; -1\rangle). \quad (98)$$

We also consider time evolution with the Hamiltonians

$$h_1 = \frac{1}{2}[\mathcal{P}_G^2 + \mathcal{X}_G^2];$$

$$h_2 = \frac{1}{2}(\mathcal{P}^2 \otimes \mathbf{1}) + \frac{1}{2}(\mathcal{X}^2 \otimes \mathbf{1}) + \frac{1}{2}(\mathbf{1} \otimes \mathcal{P}^2) + \frac{1}{2}(\mathbf{1} \otimes \mathcal{X}^2) + \mathcal{X} \otimes \mathcal{X}. \quad (99)$$

The first uses the global momentum and position, and the second uses the local momenta and positions. Here (in the position basis)

$$\mathcal{X} = |X; 1\rangle\langle X; 1| - |X; -1\rangle\langle X; -1|; \quad \mathcal{P} = -F^\dagger \mathcal{X} F \quad (100)$$

Using both notations $\mathcal{X}_G$ and $\mathcal{P}_G$ are

$$\mathcal{X}_G = \sum_{a=-1}^{1} \sum_{b=-1}^{1} (a + 3b)|X; a, b\rangle\langle X; a, b|$$

$$= \sum_{a=-1}^{1} \sum_{b=-1}^{1} (a + 3b)|X; \widehat{a + 3b}\rangle\langle X; \widehat{a + 3b}|$$

$$\mathcal{P}_G = -F_G^\dagger \mathcal{X}_G F_G. \quad (101)$$

At time $t = 1$ the state becomes

$$\exp(ith_1)|s\rangle = \sum_{a=-1}^{1} \sum_{b=-1}^{1} \lambda_1(a, b)|X; a, b\rangle;$$

$$\exp(ith_2)|s\rangle = \sum_{a=-1}^{1} \sum_{b=-1}^{1} \lambda_2(a, b)|X; a, b\rangle \quad (102)$$

where

$$\lambda_1(-1, -1) = -0.2337 - 0.3556i; \quad \lambda_1(-1, 0) = 0.0259 - 0.1315i;$$
$$\lambda_1(-1, 1) = 0.3254 + 0.4898i;$$
$$\lambda_1(0, -1) = 0.1160 + 0.0430i; \quad \lambda_1(0, 0) = 0.2836 - 0.0628i;$$
$$\lambda_1(0, 1) = -0.3090 + 0.1367i;$$
$$\lambda_1(1, -1) = 0.0671 + 0.0943i; \quad \lambda_1(1, 0) = -0.3539 - 0.0845i;$$
$$\lambda_1(1, 1) = 0.3014 - 0.0675i; \tag{103}$$

and

$$\lambda_2(-1, -1) = -0.2911 - 0.7197i; \quad \lambda_2(-1, 0) = -0.1491 - 0.2668i;$$
$$\lambda_2(-1, 1) = 0.1502 - 0.0831i;$$
$$\lambda_2(0, -1) = 0.3700 - 0.0854i; \quad \lambda_2(0, 0) = 0.1218 + 0.1319i;$$
$$\lambda_2(0, 1) = -0.2208 - 0.0247i;$$
$$\lambda_2(1, -1) = 0.0872 - 0.1268i; \quad \lambda_2(1, 0) = -0.0032 + 0.0910i;$$
$$\lambda_2(1, 1) = -0.0340 - 0.1243i. \tag{104}$$

5.3 Global Wigner and global Weyl functions in $\mathbb{Z}(d^n) \times \mathbb{Z}(d^n)$

If ρ is a density matrix, we define the global Wigner function $W_G(\widehat{\gamma}, \widehat{\delta}|\rho)$ and the global Weyl function $\widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho)$ as:

$$W_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \text{Tr}[\rho \mathfrak{P}_G(\widehat{\gamma}, \widehat{\delta})]; \quad \widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho) = \text{Tr}[\rho D_G(\widehat{\alpha}, \widehat{\beta})]. \tag{105}$$

From Eq. (86) it follows that they are related to each other through the Fourier transform:

$$W_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \frac{1}{d^n} \sum_{\widehat{\alpha}, \widehat{\beta}} \widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho) \omega_{d^n}(\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta});$$

$$\widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho) = \frac{1}{d^n} \sum_{\widehat{\gamma}, \widehat{\delta}} W_G(\widehat{\gamma}, \widehat{\delta}|\rho) \omega_{d^n}(-\widehat{\beta}\widehat{\gamma} + \widehat{\alpha}\widehat{\delta}). \tag{106}$$

The following marginal properties of the Wigner function follow immediately from Eq. (14) (for odd values of the dimension d):

$$\frac{1}{d^n} \sum_{\{\gamma_r\}} W_L(\{\gamma_r, \delta_r\}|\rho) = \frac{1}{d^n} \sum_{\widehat{\gamma}} W_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \langle X; \delta_0, ..., \delta_{n-1}|\rho|X; \delta_0, ..., \delta_{n-1}\rangle$$

$$= \langle X; \widehat{\delta}|\rho|X; \widehat{\delta}\rangle;$$

$$\frac{1}{d^n} \sum_{\{\delta_r\}} W_L(\{\gamma_r, \delta_r\}|\rho) = \langle P_L; \gamma_0, ..., \gamma_{n-1}|\rho|P_L; \gamma_0, ..., \gamma_{n-1}\rangle;$$

$$\frac{1}{d^n} \sum_{\widehat{\delta}} W_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \langle P_G; \widehat{\gamma}|\rho|P_G; \widehat{\gamma}\rangle;$$

$$\frac{1}{d^n} \sum_{\{\gamma_r, \delta_r\}} W_L(\{\gamma_r, \delta_r\}|\rho) = \frac{1}{d^n} \sum_{\widehat{\gamma}, \widehat{\delta}} W_G(\widehat{\gamma}, \widehat{\delta}|\rho) = 1. \tag{107}$$

We have already emphasized that in both the 'local formalism' and the 'global formalism' the position states are the same and the momentum states are different. Consequently $\langle P_L; \gamma_0, ..., \gamma_{n-1}|\rho|P_L; \gamma_0, ..., \gamma_{n-1}\rangle$ is different from $\langle P_G; \widehat{\gamma}|\rho|P_G; \widehat{\gamma}\rangle$ and the marginal properties in the second and third of these equations are different.

5.4 The difference between the local and global Wigner functions

We first consider states for which the local Wigner function is the same as the global Wigner function.

Proposition 5.3 *We consider the following separable density matrix that contains only diagonal elements with respect to the basis of position states:*

$$\sigma = \sum_{\widehat{j}} p(\widehat{j})|X; \widehat{j}\rangle\langle X; \widehat{j}|; \quad \sum_{\widehat{j}} p(\widehat{j}) = 1. \tag{108}$$

Here the $p(\widehat{j})$ are probabilities. In this case the local and global Wigner functions are equal to each other, they are non-negative and they do not depend on $\widehat{\alpha}$:

$$W_G(\widehat{\alpha}, \widehat{\beta}|\sigma) = W_L(\{\alpha_r, \beta_r\}|\sigma) = p(\widehat{\beta}). \tag{109}$$

Proof For the position states

$$\sigma_0(\widehat{j}) = |X; \widehat{j}\rangle\langle X; \widehat{j}| = |X; j_0, ..., j_{n-1}\rangle\langle X; j_0, ..., j_{n-1}|, \tag{110}$$

we get

$$W_G(\widehat{\alpha}, \widehat{\beta}|\sigma_0(\widehat{j})) = W_L(\{\alpha_r, \beta_r\}|\sigma_0(\widehat{j})) = \delta(\widehat{j}, \widehat{\beta}). \tag{111}$$

Indeed

$$
\begin{aligned}
W_G(\widehat{\alpha}, \widehat{\beta}|\sigma_0(\widehat{j})) &= \mathrm{Tr}\left[\mathfrak{P}_G(\widehat{\alpha}, \widehat{\beta})|X; \widehat{j}\rangle\langle X; \widehat{j}|\right] \\
&= \mathrm{Tr}\left\{F_G^2[D_G(\widehat{\alpha}, \widehat{\beta})]^\dagger |X; \widehat{j}\rangle\langle X; \widehat{j}|D_G(\widehat{\alpha}, \widehat{\beta})\right\} \\
&= \mathrm{Tr}\left[F_G^2|X; \widehat{j} - \widehat{\beta}\rangle\langle X; \widehat{j} - \widehat{\beta}|\right] = \delta(\widehat{j}, \widehat{\beta}),
\end{aligned} \tag{112}
$$

and also

$$
\begin{aligned}
W_L(\{\alpha_r, \beta_r\}|\sigma_0(\widehat{j})) &= \mathrm{Tr}[\mathfrak{P}_L(\{\alpha_r, \beta_r\})|X; j_0, ..., j_{n-1}\rangle\langle X; j_0, ..., j_{n-1}|] \\
&= \mathrm{Tr}\left\{F_L^2[D_L(\{\alpha_r, \beta_r\})]^\dagger |X; j_0, ..., j_{n-1}\rangle\langle X; j_0, ..., j_{n-1}|D_L(\{\alpha_r, \beta_r\})\right\} \\
&= \mathrm{Tr}\left[F_L^2|X; j_0 - \beta_0, ..., j_{n-1} - \beta_{n-1}\rangle\langle X; j_0 - \beta_0, ..., j_{n-1} - \beta_{n-1}|\right] \\
&= \delta(j_0, \beta_0)...\delta(j_{n-1}, \beta_{n-1}).
\end{aligned} \tag{113}
$$

This proves Eq. (111). Then

$$
W_G(\widehat{\alpha}, \widehat{\beta}|\sigma) = \sum_{\widehat{j}} p(\widehat{j})W_G(\widehat{\alpha}, \widehat{\beta}|\sigma_0(\widehat{j})) = \sum_{\widehat{j}} p(\widehat{j})\delta(\widehat{j}, \widehat{\beta}) = p(\widehat{\beta}). \tag{114}
$$

and also

$$
\begin{aligned}
W_L(\{\alpha_r, \beta_r\}|\sigma) &= \sum_{j_0, ..., j_{n-1}} p(j_0, ..., j_{n-1})W_L(\{\alpha_r, \beta_r\}|\sigma_0(\widehat{j})) \\
&= \sum_{j_0, ..., j_{n-1}} p(j_0, ..., j_{n-1})\delta(j_0, \beta_0)...\delta(j_{n-1}, \beta_{n-1}) = p(\widehat{\beta}).
\end{aligned} \tag{115}
$$

$\square$

An arbitrary density matrix ρ can be written in the basis of position states, as the sum of a separable density matrix $\sigma(\rho)$ that contains the d^n diagonal elements (as in Eq. (108)), and a Hermitian matrix $\tau(\rho)$ with trace zero that contains the $d^{2n} - d^n$ off-diagonal elements:

$$
\rho = \sigma(\rho) + \tau(\rho); \quad \mathrm{Tr}[\tau(\rho)] = 0. \tag{116}
$$

$\tau(\rho)$ is not a density matrix but using Eqs. (32, 105) we can define 'Wigner-like' functions for it. Then the Wigner function is written as a sum of two terms that correspond to the diagonal and off-diagonal part:

$$
\begin{aligned}
W_L(\{\alpha_r, \beta_r\}|\rho) &= W_L(\{\alpha_r, \beta_r\}|\sigma(\rho)) + A_L(\{\alpha_r, \beta_r\}|\tau(\rho)); \quad A_L(\{\alpha_r, \beta_r\}|\tau(\rho)) \\
&= \mathrm{Tr}[\tau(\rho)\mathfrak{P}_L(\{\alpha_r, \beta_r\})];
\end{aligned} \tag{117}
$$

and

$$W_G(\widehat{\alpha}, \widehat{\beta}|\rho) = W_G(\widehat{\alpha}, \widehat{\beta}|\sigma(\rho)) + A_G(\widehat{\alpha}, \widehat{\beta}|\tau(\rho));$$
$$A_G(\widehat{\alpha}, \widehat{\beta}|\tau(\rho)) = \text{Tr}[\tau(\rho)\mathfrak{P}_G(\widehat{\alpha}, \widehat{\beta})]. \tag{118}$$

Then

$$W_L(\{\alpha_r, \beta_r\}|\rho) - W_G(\widehat{\alpha}, \widehat{\beta}|\rho) = A_L(\{\alpha_r, \beta_r\}|\tau(\rho)) - A_G(\widehat{\alpha}, \widehat{\beta}|\tau(\rho)). \tag{119}$$

The difference between local and global Wigner functions, is related only to the off-diagonal elements of the density matrix (with respect to the position basis).

Proposition 5.4 *We consider the following separable density matrices*

$$q_L = \sum_{\widehat{j}} p(\widehat{j})|P_L; \widehat{j}\rangle\langle P_L; \widehat{j}|; \quad q_G = \sum_{\widehat{j}} p(\widehat{j})|P_G; \widehat{j}\rangle\langle P_G; \widehat{j}|;$$
$$F_L^\dagger q_L F_L = F_G^\dagger q_G F_G; \sum_{\widehat{j}} p(\widehat{j}) = 1. \tag{120}$$

Here the $p(\widehat{j})$ are probabilities. Then

$$W_G(\widehat{\alpha}, \widehat{\beta}|q_G) = W_L(\{\alpha_r, \beta_r\}|q_L) = p(\widehat{\alpha}). \tag{121}$$

Proof We first consider the density matrices

$$q_{L0}(\widehat{j}) = |P_L; \widehat{j}\rangle\langle P_L; \widehat{j}| = |P_L; j_0, ..., j_{n-1}\rangle\langle P_L; j_0, ..., j_{n-1}|;$$
$$q_{G0}(\widehat{j}) = |P_G; \widehat{j}\rangle\langle P_G; \widehat{j}|. \tag{122}$$

and prove that

$$W_G(\widehat{\alpha}, \widehat{\beta}|q_{G0}(\widehat{j})) = W_L(\{\alpha_r, \beta_r\}|q_{L0}(\widehat{j})) = \delta(\widehat{j}, \widehat{\alpha}). \tag{123}$$

indeed

$$W_G(\widehat{\alpha}, \widehat{\beta}|q_{G0}(\widehat{j})) = \text{Tr}\left[\mathfrak{P}_G(\widehat{\alpha}, \widehat{\beta})|P_G; \widehat{j}\rangle\langle P_G; \widehat{j}|\right]$$
$$= \text{Tr}\left\{F_G^2[D_G(\widehat{\alpha}, \widehat{\beta})]^\dagger|P_G; \widehat{j}\rangle\langle P_G; \widehat{j}|D_G(\widehat{\alpha}, \widehat{\beta})\right\}$$
$$= \text{Tr}\left[F_G^2|P_G; \widehat{j} - \widehat{\alpha}\rangle\langle P_G; \widehat{j} - \widehat{\alpha}|\right] = \delta(\widehat{j}, \widehat{\alpha}), \tag{124}$$

and

$$W_L(\{\alpha_r, \beta_r\}|q_{L0}(\widehat{j})) = \text{Tr}[\mathfrak{P}_L(\{\alpha_r, \beta_r\})|P_L; j_0, ..., j_{n-1}\rangle\langle P_L; j_0, ..., j_{n-1}|]$$

$$= \text{Tr}\left\{F_L^2[D_L(\{\alpha_r, \beta_r\})]^\dagger|P_L; j_0, ..., j_{n-1}\rangle\langle P_L; j_0, ..., j_{n-1}|D_L(\{\alpha_r, \beta_r\})\right\}$$

$$= \text{Tr}\left[F_L^2|P_L; j_0 - \alpha_0, ..., j_{n-1} - \alpha_{n-1}\rangle\langle P_L; j_0 - \alpha_0, ..., j_{n-1} - \alpha_{n-1}|\right]$$

$$= \delta(j_0, \alpha_0)...\delta(j_{n-1}, \alpha_{n-1}). \tag{125}$$

This proves Eq. (123). Then

$$W_G(\widehat{\alpha}, \widehat{\beta}|q_G) = \sum_{\widehat{j}} p(\widehat{j})W_G(\widehat{\alpha}, \widehat{\beta}|q_{G0}(\widehat{j})) = \sum_{\widehat{j}} p(\widehat{j})\delta(\widehat{j}, \widehat{\alpha}) = p(\widehat{\alpha}). \tag{126}$$

and also

$$W_L(\{\alpha_r, \beta_r\}|q_L) = \sum_{j_0, ..., j_{n-1}} p(\widehat{j})W_L(\{\alpha_r, \beta_r\}|q_{L0}(\widehat{j}))$$

$$= \sum_{j_0, ..., j_{n-1}} p(j_0, ..., j_{n-1})\delta(j_0, \alpha_0)...\delta(j_{n-1}, \alpha_{n-1}) = p(\widehat{\alpha}).$$

$$\tag{127}$$

$$\square$$

Example 5.5 We consider the density matrices

$$\rho_L = |P_L; \widehat{j}\rangle\langle P_L; \widehat{j}|; \quad \rho_G = |P_G; \widehat{j}\rangle\langle P_G; \widehat{j}|; \quad \rho_0 = \frac{1}{d^n}\mathbf{1}_{d^n}. \tag{128}$$

Then

$$\sigma(\rho_L) = \sigma(\rho_G) = \sigma(\rho_0) = \rho_0 = \frac{1}{d^n}\mathbf{1}_{d^n};$$

$$\tau(\rho_L) = |P_L; \widehat{j}\rangle\langle P_L; \widehat{j}| - \frac{1}{d^n}\mathbf{1}_{d^n}; \quad \tau(\rho_G) = |P_L; \widehat{j}\rangle\langle P_L; \widehat{j}| - \frac{1}{d^n}\mathbf{1}_{d^n}; \quad \tau(\rho_0) = 0. \tag{129}$$

In this case

$$W_G(\widehat{\alpha}, \widehat{\beta}|\rho_0) = W_L(\{\alpha_r, \beta_r\}|\rho_0) = \frac{1}{d^n}. \tag{130}$$

From Proposition 5.4 it follows that

$$W_G(\widehat{\alpha}, \widehat{\beta}|\rho_G) = W_L(\widehat{\alpha}, \widehat{\beta}|\rho_L) = \delta(\widehat{j}, \widehat{\alpha}). \tag{131}$$

We next calculate numerically the $W_L(\widehat{\alpha}, \widehat{\beta}|\rho_G)$, $W_G(\widehat{\alpha}, \widehat{\beta}|\rho_L)$, for an example. We consider the case $d = 3$ and $n = 2$, and the density matrices ρ_L, ρ_G with $\widehat{j} = 4$ (which is an element of $\mathbb{Z}(9)$). Results for the $W_L(\widehat{\alpha}, \widehat{\beta}|\rho_G)$, $W_G(\widehat{\alpha}, \widehat{\beta}|\rho_L)$, given in Tables 1 and 2 correspondingly.

5.5 The R_L, R_G matrices: indicators of classical and quantum correlations

In this section we compare quantities for ρ with the corresponding quantities for $\mathfrak{R}(\rho)$ (in Eq. (16)).

Definition 5.6 If ρ is a density matrix, R_L, $\widetilde{R}_L$ are $d^n \times d^n$ matrices with elements

$$
\begin{aligned}
R_L(\{\gamma_r, \delta_r\}|\rho) &= W_L(\{\gamma_r, \delta_r\}|\rho) - W_L(\{\gamma_r, \delta_r\}|\mathfrak{R}(\rho)); \\
\widetilde{R}_L(\{\alpha_r, \beta_r\}|\rho) &= \widetilde{W}_L(\{\alpha_r, \beta_r\}|\rho) - \widetilde{W}_L(\{\alpha_r, \beta_r\}|\mathfrak{R}(\rho)).
\end{aligned}
\tag{132}
$$

Also R_G, $\widetilde{R}_G$ are $d^n \times d^n$ matrices with elements

$$
\begin{aligned}
R_G((\widehat{\gamma}, \widehat{\delta}|\rho) &= W_G(\widehat{\gamma}, \widehat{\delta}|\rho) - W_G(\widehat{\gamma}, \widehat{\delta}|\mathfrak{R}(\rho)); \\
\widetilde{R}_G(\widehat{\alpha}, \widehat{\beta}|\rho) &= \widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho) - \widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\mathfrak{R}(\rho)).
\end{aligned}
\tag{133}
$$

Proposition 5.7 *(1) For factorisable density matrices $R_L = \widetilde{R}_L = R_G = \widetilde{R}_G = 0$.*
(2) The R_L and $\widetilde{R}_L$ are related through a local Fourier transform:

$$
\begin{aligned}
R_L(\{\gamma_r, \delta_r\}|\rho) &= \frac{1}{d^n} \sum_{\{\alpha_r, \beta_r\}} \widetilde{R}_L(\{\alpha_r, \beta_r\}|\rho)\omega_d\left[\sum_{r=0}^{n-1}(\beta_r\gamma_r - \alpha_r\delta_r)\right]; \\
\widetilde{R}_L(\{\alpha_r, \beta_r\}|\rho) &= \frac{1}{d^n} \sum_{\{\gamma_r, \delta_r\}} R_L(\{\gamma_r, \delta_r\}|\rho)\omega_d\left[\sum_{r=0}^{n-1}(-\beta_r\gamma_r + \alpha_r\delta_r)\right]
\end{aligned}
\tag{134}
$$

(3) The R_G and $\widetilde{R}_G$ are related through a global Fourier transform:

$$
\begin{aligned}
R_G(\widehat{\gamma}, \widehat{\delta}|\rho) &= \frac{1}{d^n} \sum_{\widehat{\alpha}, \widehat{\beta}} \widetilde{R}_G(\widehat{\alpha}, \widehat{\beta}|\rho)\omega_{d^n}(\widehat{\beta}\widehat{\gamma} - \widehat{\alpha}\widehat{\delta}); \\
\widetilde{R}_G(\widehat{\alpha}, \widehat{\beta}|\rho) &= \frac{1}{d^n} \sum_{\widehat{\gamma}, \widehat{\delta}} R_G(\widehat{\gamma}, \widehat{\delta}|\rho)\omega_{d^n}(-\widehat{\beta}\widehat{\gamma} + \widehat{\alpha}\widehat{\delta}).
\end{aligned}
\tag{135}
$$

Table 1 The local Wigner function $W_L(\{\alpha_0, \alpha_1; \beta_0, \beta_1\}|\rho_G)$ for the density matrix ρ_G in Eq. (128) with $\widehat{j} = 4$ and $d = 3$, $n = 2$

α_0,α_1	β_0,β_1								
	$\{-1,-1\}$	$\{0,-1\}$	$\{1,-1\}$	$\{-1,0\}$	$\{0,0\}$	$\{1,0\}$	$\{-1,1\}$	$\{0,1\}$	$\{1,1\}$
$\{-1,-1\}$	0	0	0	0	0	0	0	0	0
$\{0,-1\}$	0	0	0	0	0	0	0	0	0
$\{1,-1\}$	0	0	0	0	0	0	0	0	0
$\{-1,0\}$	0	0	0	0	0	0	0	0	0
$\{0,0\}$	0	0	0	0	0	0	0	0	0
$\{1,0\}$	0	0	0	0	0	0	0	0	0
$\{-1,1\}$	0.4491	-0.2931	0.4491	0.4491	-0.2931	0.4491	0.4491	-0.2931	0.4491
$\{0,1\}$	-0.2931	0.844	-0.2931	-0.2931	0.844	-0.2931	-0.2931	0.844	-0.2931
$\{1,1\}$	0.844	0.4491	0.844	0.844	0.4491	0.844	0.844	0.4491	0.844

Table 2 The global Wigner function $W_G(\widehat{\alpha}, \widehat{\beta}|\rho_L)$ for the density matrix ρ_L in Eq. (128), with $\widehat{j} = 4$ and $d = 3, n = 2$

$\widehat{\alpha} = (\alpha_0, \alpha_1)$	$\widehat{\beta} = (\beta_0, \beta_1)$								
	$\widehat{-4}=(-1,-1)$	$\widehat{-3}=(0,-1)$	$\widehat{-2}=(1,-1)$	$\widehat{-1}=(1,0)$	$\widehat{0}=(0,0)$	$\widehat{1}=(1,0)$	$\widehat{2}=(-1,1)$	$\widehat{3}=(0,1)$	$\widehat{4}=(1,1)$
$\widehat{-4}=(-1,-1)$	0	0	0	0	0	0	0	0	0
$\widehat{-3}=(0,-1)$	0	0	0	0	0	0	0	0	0
$\widehat{-2}=(1,-1)$	-0.2931	0.844	-0.2931	-0.2931	0.844	-0.2931	-0.2931	0.844	-0.2931
$\widehat{-1}=(-1,0)$	0	0	0	0	0	0	0	0	0
$\widehat{0V}=(0,0)$	0	0	0	0	$\widehat{0}$	0	0	0	0
$\widehat{1}=(1,0)$	0.4491	-0.2931	0.4491	0.4491	-0.2931	0.4491	0.4491	-0.2931	0.4491
$\widehat{2}=(-1,1)$	0	0	0	0	0	0	0	0	0
$\widehat{3}=(0,1)$	0	0	0	0	0	0	0	0	0
$\widehat{4}=(1,1)$	0.844	0.4491	0.844	0.844	0.4491	0.844	0.844	0.4491	0.844

(4) The following are marginal properties:

$$\frac{1}{d^n} \sum_{\{\gamma_r\}} R_L(\{\gamma_r, \delta_r\}|\rho) = \frac{1}{d^n} \sum_{\widehat{\gamma}} R_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \mathcal{C}(X; \delta_0, ..., \delta_{n-1}) = \mathcal{C}(X; \widehat{\delta});$$

$$\frac{1}{d^n} \sum_{\{\delta_r\}} R_L(\{\gamma_r, \delta_r\}|\rho) = \mathcal{C}(P_L; \gamma_0, ..., \gamma_{n-1});$$

$$\frac{1}{d^n} \sum_{\widehat{\delta}} R_G(\widehat{\gamma}, \widehat{\delta}|\rho) = \mathcal{E}(P_G; \widehat{\gamma});$$

$$\frac{1}{d^n} \sum_{\{\gamma_r, \delta_r\}} R_L(\{\gamma_r, \delta_r\}|\rho) = \frac{1}{d^n} \sum_{\widehat{\gamma}, \widehat{\delta}} R_G(\widehat{\gamma}, \widehat{\delta}|\rho) = 0. \tag{136}$$

Proof (1) For factorisable density matrices $\mathfrak{R}(\rho) = \rho$ and then $R_L = \widetilde{R}_L = R_G = \widetilde{R}_G = 0$.
(2) We prove this using Eq. (31) with both ρ and $\mathfrak{R}(\rho)$.
(3) We prove this using Eq. (78) with both ρ and $\mathfrak{R}(\rho)$.
(4) We prove this using Eq. (107) with both ρ and $\mathfrak{R}(\rho)$.

$\square$

The matrices R_ρ and $\widetilde{R}_\rho$ indicate the existence of both classical and quantum correlations.

6 Examples

In the examples below we take $d = 3$ and $n = 2$. In this case the global Fourier transform is unitarily inequivalent to the local Fourier transform. We work in the 'periods' of Eq. (34).

We consider the density matrix

$$\rho = |s\rangle\langle s|; \quad |s\rangle = \frac{1}{\sqrt{3}}|X; 0, 1\rangle + \frac{1}{2}|X; 1, -1\rangle + \sqrt{\frac{5}{12}}|X; -1, 0\rangle. \tag{137}$$

The state described by ρ is entangled. In this case the reduced density matrices are

$$\check{\rho}_0 = \frac{1}{3}|X; 0\rangle\langle X; 0| + \frac{1}{4}|X; 1\rangle\langle X; 1| + \frac{5}{12}|X; -1\rangle\langle X; -1|;$$

$$\check{\rho}_1 = \frac{5}{12}|X; 0\rangle\langle X; 0| + \frac{1}{3}|X; 1\rangle\langle X; 1| + \frac{1}{4}|X; -1\rangle\langle X; -1|. \tag{138}$$

In Tables 3, 4, 5 and 6 we present the local Wigner function $W_L(\gamma_0, \gamma_1; \delta_0, \delta_1)$, the local Weyl function $\widetilde{W}_L(\alpha_0, \alpha_1; \beta_0; \beta_1)$, and the matrices $R_L(\gamma_0, \gamma_1; \delta_0, \delta_1)$ and $\widetilde{R}_L(\alpha_0, \alpha_1; \beta_0; \beta_1)$ for the density matrix ρ in Eq. (137).

Table 3 The local Wigner function $W_L(\{\gamma_0, \gamma_1; \delta_0, \delta_1\}|\rho)$ for the density matrix ρ in Eq. (137)

$\{\gamma_0,\gamma_1\}$	$\{\delta_0,\delta_1\}$								
	$\{-1,-1\}$	$\{0,-1\}$	$\{1,-1\}$	$\{-1,0\}$	$\{0,0\}$	$\{1,0\}$	$\{-1,1\}$	$\{0,1\}$	$\{1,1\}$
$\{-1,-1\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0
$\{0,-1\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0
$\{1,-1\}$	0	0	0.9954	0.994	0	0	0	0.9788	0
$\{-1,0\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0
$\{0,0\}$	0	0	0.9954	0.994	0	0	0	0.9788	0
$\{1,0\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0
$\{-1,1\}$	0	0	0.9954	0.994	0	0	0	0.9788	0
$\{0,1\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0
$\{1,1\}$	0	0	-0.1227	0.128	0	0	0	0.0106	0

Table 4 The local Weyl function $\widetilde{W}_L((\alpha_0, \alpha_1 ; \beta_0, \beta_1)|\rho)$ for the density matrix ρ in Eq. (137)

$\{\alpha_0,\alpha_1\}$	$\{\beta_0,\beta_1\}$								
	$\{-1,-1\}$	$\{0,-1\}$	$\{1,-1\}$	$\{-1,0\}$	$\{0,0\}$	$\{1,0\}$	$\{-1,1\}$	$\{0,1\}$	$\{1,1\}$
$\{-1,-1\}$	$0.067 - 0.0295i$	0	0	0	$-0.125 + 0.0722i$	0	0	0	$0.067 - 0.0295i$
$\{0,-1\}$	$-0.059 + 0.0432i$	0	0	0	$0.125 - 0.0722i$	0	0	0	$-0.059 + 0.0432i$
$\{1,-1\}$	$-0.4921 - 0.8523i$	0	0	0	$-0.5 - 0.866i$	0	0	0	$-0.4921 - 0.8523i$
$\{-1,0\}$	$-0.0079 - 0.0727i$	0	0	0	$0.1443i$	0	0	0	$-0.0079 - 0.0727i$
$\{0,0\}$	0.9841	0	0	0	1	0	0	0	0.9841
$\{1,0\}$	$-0.0079 + 0.0727i$	0	0	0	$-0.1443i$	0	0	0	$-0.0079 + 0.0727i$
$\{-1,1\}$	$-0.4921 + 0.8523i$	0	0	0	$-0.5 + 0.866i$	0	0	0	$-0.4921 + 0.8523i$
$\{0,1\}$	$-0.059 - 0.0432i$	0	0	0	$0.125 + 0.0722i$	0	0	0	$-0.059 - 0.0432i$
$\{1,1\}$	$0.067 + 0.0295i$	0	0	0	$-0.125 - 0.0722i$	0	0	0	$0.067 + 0.0295i$

Table 5 The matrix $R_L(\{\gamma_0, \gamma_1; \delta_0, \delta_1\}|\rho)$ for the density matrix ρ in Eq. (137)

$\{\gamma_0,\gamma_1\}$		$\{\delta_0,\delta_1\}$								
		$\{-1,-1\}$	$\{0,-1\}$	$\{1,-1\}$	$\{-1,0\}$	$\{0,0\}$	$\{1,0\}$	$\{-1,1\}$	$\{0,1\}$	$\{1,1\}$
	$\{-1,-1\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833
	$\{0,-1\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833
	$\{1,-1\}$	-0.1042	-0.0833	0.9329	0.8204	-0.1389	-0.1042	-0.1389	0.8677	-0.0833
	$\{-1,0\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833
	$\{0,0\}$	-0.1042	-0.0833	0.9329	0.8204	-0.1389	-0.1042	-0.1389	0.8677	-0.0833
	$\{1,0\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833
	$\{-1,1\}$	-0.1042	-0.0833	0.9329	0.8204	-0.1389	-0.1042	-0.1389	0.8677	-0.0833
	$\{0,1\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833
	$\{1,1\}$	-0.1042	-0.0833	-0.1852	-0.0456	-0.1389	-0.1042	-0.1389	-0.1005	-0.0833

Table 6 The matrix $\widetilde{R}_L((\alpha_0, \alpha_1; \beta_0, \beta_1)|\rho)$ for the density matrix ρ in Eq. (137)

$\{\alpha_0,\alpha_1\}$	$\{\beta_0,\beta_1\}$								
	$\{-1,-1\}$	$\{0,-1\}$	$\{1,-1\}$	$\{-1,0\}$	$\{0,0\}$	$\{1,0\}$	$\{-1,1\}$	$\{0,1\}$	$\{1,1\}$
$\{-1,-1\}$	$0.067-0.0295i$	0	0	0	$-0.1354+0.0541i$	0	0	0	$0.067-0.0295i$
$\{0,-1\}$	$-0.059+0.0432i$	0	0	0	0	0	0	0	$-0.059+0.0432i$
$\{1,-1\}$	$-0.4921-0.8523i$	0	0	0	$-0.4896-0.848i$	0	0	0	$-0.4921-0.8523i$
$\{-1,0\}$	$-0.0079-0.0727i$	0	0	0	0	0	0	0	$-0.0079-0.0727i$
$\{0,0\}$	0.9841	0	0	0	0	0	0	0	0.9841
$\{1,0\}$	$-0.0079+0.0727i$	0	0	0	0	0	0	0	$-0.0079+0.0727i$
$\{-1,1\}$	$-0.4921+0.8523i$	0	0	0	$-0.4896+0.848i$	0	0	0	$-0.4921+0.8523i$
$\{0,1\}$	$-0.059-0.0432i$	0	0	0	0	0	0	0	$-0.059-0.0432i$
$\{1,1\}$	$0.067+0.0295i$	0	0	0	$-0.1354-0.0541i$	0	0	0	$0.067+0.0295i$

The correlations in Eqs. (22, 23) are

$$
\mathcal{C}_\rho(X; \delta_0, \delta_1) = \begin{bmatrix} -0.1042 & 0.2431 & -0.1389 \\ -0.0833 & -0.1389 & 0.2222 \\ 0.1875 & -0.1042 & -0.0833 \end{bmatrix};
$$

$$
\mathcal{C}_\rho(P_L; \gamma_0, \gamma_1) = \begin{bmatrix} -0.1093 & -0.1093 & 0.2187 \\ -0.1093 & 0.2187 & -0.1093 \\ 0.2187 & -0.1093 & -0.1093 \end{bmatrix}.
$$

(139)

We easily confirm that Eqs. (107) hold for the local Wigner and Weyl function.

For the global formalism in $\mathbb{Z}(9)$ we rewrite ρ as

$$
\rho = |s\rangle\langle s|; \quad |s\rangle = \frac{1}{\sqrt{3}}|X; \widehat{3}\rangle + \frac{1}{2}|X; \widehat{-2}\rangle + \sqrt{\frac{5}{12}}|X; \widehat{-1}\rangle. \tag{140}
$$

In Tables 7, 8, 9 and 10 we present the global Wigner function $W_G(\widehat{\gamma}, \widehat{\delta}|\rho)$, the global Weyl function $\widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho)$ and the matrices $R_G(\widehat{\gamma}, \widehat{\delta})$ and $\widetilde{R}_G(\widehat{\alpha}, \widehat{\beta})$ for the density matrix ρ in Eq. (137).

The correlations in Eq. (61) are

$$
\mathcal{C}_\rho(X; \widehat{\delta}) = \mathcal{C}_\rho(X; \delta_0, \delta_1);
$$
$$
\mathcal{E}_\rho(P_G; \widehat{\gamma})
$$
$$
= \begin{bmatrix} -0.0419 & -0.1093 & 0.1250 & -0.0832 & 0.2187 & -0.0832 & 0.1250 & -0.1093 & -0.0419 \end{bmatrix}^T.
$$

(141)

We easily confirm that Eq. (107) hold for the global Wigner and Weyl function.

In general there is no simple relation that links the local with the global quantities. We see this by comparing the expectation values of the local observables $\mathcal{X} \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{X}, \mathcal{X} \otimes \mathcal{X}, \mathcal{P} \otimes \mathbf{1}, \mathbf{1} \otimes \mathcal{P}, \mathcal{P} \otimes \mathcal{P}$ for the density matrix ρ in Eq. (137), with the expectation values of the global observables $\mathcal{X}_G, \mathcal{P}_G$ for the same density matrix (written in the 'global language' in Eq. (140)):

$$
\mathrm{Tr}[\rho(\mathcal{X} \otimes \mathbf{1})] = -0.1667; \quad \mathrm{Tr}[\rho(\mathbf{1} \otimes \mathcal{X})] = 0.0833; \quad \mathrm{Tr}[\rho(\mathcal{X} \otimes \mathcal{X})] = -0.25;
$$
$$
\mathrm{Tr}(\rho\mathcal{X}_G) = 0.0833
$$
$$
\mathrm{Tr}[\rho(\mathcal{P} \otimes \mathbf{1})] = 0; \quad \mathrm{Tr}[\rho(\mathbf{1} \otimes \mathcal{P})] = 0; \quad \mathrm{Tr}[\rho(\mathcal{P} \otimes \mathcal{P})] = -0.6561;
$$
$$
\mathrm{Tr}(\rho\mathcal{P}_G) = 0. \tag{142}
$$

The $\mathcal{X}, \mathcal{P}, \mathcal{X}_G, \mathcal{P}_G$, have been given in Eqs. (100, 101).

The results for the local observables are different from the global observables. For strongly correlated systems global quantities might be physically more relevant.

We note that an observable can be written in both the local and global formalism (using the map in Eq. (35)). For example, for the above system we consider the

Table 7 The global Wigner function $W_G(\hat{\gamma}, \hat{\delta}|\rho)$ for the density matrix ρ in Eq. (137)

$\hat{\gamma} = (\gamma_0, \gamma_1)$		$\hat{\delta} = (\delta_0, \delta_1)$								
		$\widehat{-4} = (-1, -1)$	$\widehat{-3} = (0, -1)$	$\widehat{-2} = (1, -1)$	$\widehat{-1} = (1, 0)$	$\widehat{0} = (0, 0)$	$\widehat{1} = (1, 0)$	$\widehat{2} = (-1, 1)$	$\widehat{3} = (0, 1)$	$\widehat{4} = (1, 1)$
	$\widehat{-4} = (-1, -1)$	0.1003	0	0.25	0.4167	0	0.1294	0	-0.2732	0
	$\widehat{-3} = (0, -1)$	-0.2887	0	0.25	0.4167	0	-0.3727	0	0.0106	0
	$\widehat{-2} = (1, -1)$	0.4423	0	0.25	0.4167	0	0.571	0	0.4454	0
	$\widehat{-1} = (-1, 0)$	-0.5425	0	0.25	0.4167	0	-0.7004	0	0.8278	0
	$\widehat{0} = (0, 0)$	0.5774	0	0.25	0.4167	0	0.7454	0	0.9788	0
	$\widehat{1} = (1, 0)$	-0.5425	0	0.25	0.4167	0	-0.7004	0	0.8278	0
	$\widehat{2} = (-1, 1)$	0.4423	0	0.25	0.4167	0	0.571	0	0.4454	0
	$\widehat{3} = (0, 1)$	-0.2887	0	0.25	0.4167	0	-0.3727	0	0.0106	0
	$\widehat{4} = (1, 1)$	0.1003	0	0.25	0.4167	0	0.1294	0	-0.2732	0

Table 8 The global Weyl function $\widetilde{W}_G(\widehat{\alpha}, \widehat{\beta}|\rho)$ for the density matrix ρ in Eq. (137)

$\widehat{\alpha} = (\alpha_0, \alpha_1)$	$\widehat{\beta} = (\beta_0, \beta_1)$								
	$\widehat{-4} = (-1,-1)$	$\widehat{-3} = (0,-1)$	$\widehat{-2} = (1,-1)$	$\widehat{-1} = (1,0)$	$\widehat{0} = (0,0)$	$\widehat{1} = (1,0)0$	$\widehat{2} = (-1,1)$	$\widehat{3} = (0,1)$	$\widehat{4} = (1,1)$
$\widehat{-4} = (-1,-1)$	$-0.3001-0.4118i$	0	0	$-0.1614-0.2795i$	$-0.3667-0.3069i$	$-0.1614-0.2795i$	0	0	$-0.3001-0.4118i$
$\widehat{-3} = (0,-1)$	$-0.3307-0.0727i$	0	0	0.3227	0.1443i	0.3227	0	0	$-0.3307-0.0727i$
$\widehat{-2} = (1,-1)$	$0.2859-0.5526i$	0	0	$-0.1614+0.2795i$	$-0.3292+0.7845i$	$-0.1614+0.2795i$	0	0	$0.2859-0.5526i$
$\widehat{-1} = (-1,0)$	$0.0142-0.1408i$	0	0	$-0.1614-0.2795i$	$0.1959+0.2254i$	$-0.1614-0.2795i$	0	0	$0.0142-0.1408i$
$\widehat{0} = (0,0)$	0.6614	0	0	0.3227	1	0.3227	0	0	0.6614
$\widehat{1} = (1,0)$	$0.0142+0.1408i$	0	0	$-0.1614+0.2795i$	$0.1959-0.2254i$	$-0.1614+0.2795i$	0	0	$0.0142+0.1408i$
$\widehat{2} = (-1,1)$	$0.2859+0.5526i$	0	0	$-0.1614-0.2795i$	$-0.3292-0.7845i$	$-0.1614-0.2795i$	0	0	$0.2859+0.5526i$
$\widehat{3} = (0,1)$	$-0.3307+0.0727i$	0	0	0.3227	$-0.1443i$	0.3227	0	0	$-0.3307+0.0727i$
$\widehat{4} = (1,1)$	$-0.3001+0.4118i$	0	0	$-0.1614+0.2795i$	$-0.3667+0.3069i$	$-0.1614+0.2795i$	0	0	$-0.3001+0.4118i$

Table 9 The matrix $R_G(\widehat{\gamma}, \widehat{\delta}|\rho)$ for the density matrix ρ in Eq. (137)

$\widehat{\gamma} = (\gamma_0, \gamma_1)$		$\widehat{\delta} = (\delta_0, \delta_1)$								
		$\widehat{-4} = (-1, -1)$	$\widehat{-3} = (0, -1)$	$\widehat{-2} = (1, -1)$	$\widehat{-1} = (1, 0)$	$\widehat{0} = (0, 0)$	$\widehat{1} = (1, 0)$	$\widehat{2} = (-1, 1)$	$\widehat{3} = (0, 1)$	$\widehat{4} = (1, 1)$
	$\widehat{-4} = (-1, -1)$	-0.0039	-0.0833	0.1875	0.2431	-0.1389	0.0253	-0.1389	-0.3843	-0.0833
	$\widehat{-3} = (0, -1)$	-0.3928	-0.0833	0.1875	0.2431	-0.1389	-0.4768	-0.1389	-0.1005	-0.0833
	$\widehat{-2} = (1, -1)$	0.3381	-0.0833	0.1875	0.2431	-0.1389	0.4668	-0.1389	0.3343	-0.0833
	$\widehat{-1} = (-1, 0)$	-0.6467	-0.0833	0.1875	0.2431	-0.1389	-0.8046	-0.1389	0.7167	-0.0833
	$\widehat{0} = (0, 0)$	0.4732	-0.0833	0.1875	0.2431	-0.1389	0.6412	-0.1389	0.8677	-0.0833
	$\widehat{1} = (1, 0)$	-0.6467	-0.0833	0.1875	0.2431	-0.1389	-0.8046	-0.1389	0.7167	-0.0833
	$\widehat{2} = (-1, 1)$	0.3381	-0.0833	0.1875	0.2431	-0.1389	0.4668	-0.1389	0.3343	-0.0833
	$\widehat{3} = (0, 1)$	-0.3928	-0.0833	0.1875	0.2431	-0.1389	-0.4768	-0.1389	-0.1005	-0.0833
	$\widehat{4} = (1, 1)$	-0.0039	-0.0833	0.1875	0.2431	-0.1389	0.0253	-0.1389	-0.3843	-0.0833

Table 10 The matrix $\widetilde{R}_G(\widehat{\alpha}, \widehat{\beta}|\rho)$ for the density matrix ρ in Eq. (137)

$\widehat{\alpha} = (\alpha_0, \alpha_1)$	$\widehat{\beta} = (\beta_0, \beta_1)$								
	$\widehat{-4} = (-1,-1)$	$\widehat{-3} = (0,-1)$	$\widehat{-2} = (1,-1)$	$\widehat{-1} = (1,0)$	$\widehat{0} = (0,0)$	$\widehat{1} = (1,0)$	$\widehat{2} = (-1,1)$	$\widehat{3} = (0,1)$	$\widehat{4} = (1,1)$
$\widehat{-4} = (-1,-1)$	$-0.3001-0.4118i$	0	0	$-0.1614-0.2795i$	$-0.3342-0.3351i$	$-0.1614-0.2795i$	0	0	$-0.3001-0.4118i$
$\widehat{-3} = (0,-1)$	$-0.3307-0.0727i$	0	0	0.3227	0	0.3227	0	0	$-0.3307-0.0727i$
$\widehat{-2} = (1,-1)$	$0.2859-0.5526i$	0	0	$-0.1614+0.2795i$	$-0.3735+0.7316i$	$-0.1614+0.2795i$	0	0	$0.2859-0.5526i$
$\widehat{-1} = (-1,0)$	$0.0142-0.1408i$	0	0	$-0.1614-0.2795i$	$0.0827+0.2729i$	$-0.1614-0.2795i$	0	0	$0.0142-0.1408i$
$\widehat{0} = (0,0)$	0.6614	0	0	0.3227	0	0.3227	0	0	0.6614
$\widehat{1} = (1,0)$	$0.0142+0.1408i$	0	0	$-0.1614+0.2795i$	$0.0827-0.2729i$	$-0.1614+0.2795i$	0	0	$0.0142+0.1408i$
$\widehat{2} = (-1,1)$	$0.2859+0.5526i$	0	0	$-0.1614-0.2795i$	$-0.3735-0.7316i$	$-0.1614-0.2795i$	0	0	$0.2859+0.5526i$
$\widehat{3} = (0,1)$	$-0.3307+0.0727i$	0	0	0.3227	0	0.3227	0	0	$-0.3307+0.0727i$
$\widehat{4} = (1,1)$	$-0.3001+0.4118i$	0	0	$-0.1614+0.2795i$	$-0.3342+0.3351i$	$-0.1614+0.2795i$	0	0	$-0.3001+0.4118i$

observable (Hermitian operator)

$$\mathcal{O} = a|X; 0, 1\rangle\langle X; 0, 1| + b|X; 1, -1\rangle\langle X; 1, -1| + c|X; -1, 0\rangle\langle X; -1, 0|$$
$$+ d|X; 0, 1\rangle\langle X; 1, -1| + d^*|X; 1, -1\rangle\langle X; 0, 1|; \quad a, b, c \in \mathbb{R}; \quad d \in \mathbb{C},$$
(143)

which can also be written as

$$\mathcal{O} = a|X; \widehat{3}\rangle\langle X; \widehat{3}| + b|X; \widehat{-2}\rangle\langle X; \widehat{-2}| + c|X; \widehat{-1}\rangle\langle X; \widehat{-1}|$$
$$+ d|X; \widehat{3}\rangle\langle X; \widehat{-2}| + d^*|X; \widehat{-2}\rangle\langle X; \widehat{3}|.$$
(144)

Important physical quantities like the position can be defined locally like $\mathcal{X} \otimes \mathbf{1}$, $\mathbf{1} \otimes \mathcal{X}$, $\mathcal{X} \otimes \mathcal{X}$, or globally as $\mathcal{X}_G$ (defined in Eq. (97)). The same is true for local and global momenta. For strongly correlated systems the identity of each component becomes weak, and global quantities might be physically more appropriate for the description of these systems.

7 Discussion

In this paper we introduced local and global Fourier transforms and related phase space methods for multipartite systems. The multipartite system consists of n components, each of which is described with variables in $\mathbb{Z}(d)$ and with a d-dimensional Hilbert space $H(d)$. In the global formalism we take a holistic view of the system and describe it with variables in $[\mathbb{Z}(d^n)]$ and the d^n-dimensional Hilbert space $\mathfrak{H}$. Even if the various components of the system are located far from each other, in the case of strong interactions and strong correlations between them they might loose their individual identity.

In this case a holistic approach that uses global quantities, might be more appropriate.

In the local formalism the phase space is $[\mathbb{Z}(d) \times \mathbb{Z}(d)]^n$, and in the global formalism $[\mathbb{Z}(d^n)] \times [\mathbb{Z}(d^n)]$. We have explained that although the map in Eq. (35) is bijective, the ring $[\mathbb{Z}(d)]^n$ is not isomorphic to the ring $[\mathbb{Z}(d^n)]$ (because of Eq. (36)). The heart of the formalism is the local and global Fourier transforms. We have shown that for some values of d, n they are unitarily inequivalent to each other (Proposition 4.4).

We have compared and contrasted the local phase space formalism with the global phase space formalism. Examples of this are:

- Some of the local momentum states are the same as the global momentum states (Proposition 4.3).
- Density matrices which have only diagonal elements with respect to the position basis, have the same local and global Wigner function (Proposition 5.3). The difference between local and global Wigner functions, is contained entirely in the off-diagonal elements.
- We have calculated the time evolution in terms of both local variables and also global variables (Sect. 5.2)

- Classical and quantum correlations have been described in the local formalism with the matrices R_L, $\widetilde{R}_L$ and in the global formalism with the Wigner matrices R_G, $\widetilde{R}_G$.

The formalism could be used in the general area of Fast Fourier transforms (in a quantum or even classical context). For example, a link between the present formalism (in some special cases) and the Cooley–Tukey formalism has been discussed in Sect. 4.5.

The work is a contribution to the various approaches for multipartite systems. Unitary equivalence between the local and global Fourier transform (Eq. (64)), implies that the distinction between the concept of a multipartite system and that of a single system is weak. Unitary inequivalence (Eq. (65)) implies that the concept of a multipartite system is fundamentally different from that of a single quantum system.

Data availability No data were used in this paper, and therefore data availability is not applicable.

Declarations

Conflict of interest We have no conflicts of interest to disclose.

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