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SHAPE MORPHING USING PDE SURFACES

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ABSTRACT

A methodology for shape morphing using partial differential equation (PDE) surfaces is presented in this work. The use of the PDE formulation shows how shape morphing can be based on a boundary-value approach by which intermediate shapes can be created. Furthermore, the mathematical properties of the method give rise to several alternatives in which morphing one shape into another can be achieved. Three of these alternatives are presented here. The first one is based on the gradual variation of the weighted sum of the boundary conditions for each surface, the second one consists of varying the Fourier mode for which the PDE is solved whilst the third results from a combination of the first two. Examples showing the efficiency of these methodologies are presented. Thus, it is shown that the PDE based approach for morphing, when combined with a parametric variation of the boundary conditions, is capable of obtaining smooth intermediate surfaces automatically.

KEY WORDS

Geometric modelling, geometric algorithms, boundary representations, morphing.

1 Introduction

Shape morphing is defined in [1] as the gradual transformation of one shape into another. The areas in which this tool has become extremely powerful vary from applications in industrial design [2], geometric modelling and medicine [1] to the production of visual effects and computer animation [3–5]. The morphing of two specific shapes can be achieved via an unlimited number of sequences in which such a transformation can occur and therefore, many approaches have been proposed. Moreover, given the applications of this technique, sequences producing aesthetic effects are often pursued. In general such techniques are smooth and in some cases preserve shape.

The concept of morphing is not restricted to three-dimensional surfaces. As an example of this, [1] describes one of the methods employed to transform two-dimensional shapes by finding a correspondence rule between feature points. However, since the work presented here concerns three-dimensional shapes, some of the most relevant approaches to achieve morphing are outlined below. The reader is also referred to [4, 6] for further information.

According to [4] there are several approaches in which the techniques for achieving morphing are developed. These are:

Volume-based approaches: These approaches consider the entire surface representing the object, that is, they manipulate the object by modifying specific points known as control points and therefore, they provide excellent results when applied to objects that are represented as implicit surfaces. Therefore, they produce smooth transitions and some conservative properties can be achieved. Moreover, the implementation of these approaches is fairly straightforward.

Boundary-based approaches: These methods consist of varying specific values on the boundaries describing the objects (if the objects have been defined in such a way). However, a small perturbation of data defining these boundaries may lead to the production of an invalid object. Therefore, the process of morphing using this technique is not a trivial one. This problem is generally overcome by merging the two meshes associated with each of the objects into a third one, where a correspondence rule is then found.

A list of the most relevant work in developing techniques for each approach is also given in [4]. By contrast, the work presented in [6] states that the main problems in morphing are feature specification, warp generation and transition control. Moreover, [4] classifies the techniques available according to their role in addressing each of these problems and makes reference to work based on mesh warping [7], field morphing [8], energy minimisation [9] and free-form deformations [10] among others.

Works such as [1,3,5,11] are examples of some of the morphing techniques developed so far. For instance, [11] introduces a method that allows morphing between two objects using variational interpolation. In spite of the extensive work carried out in morphing, there are still a number of relevant areas in which little work has been done. For example, this is the case for multiple image morphing and the development of appropriate real-time interactive deformation tools.

Surface generation methods can prove useful for addressing some of the problems found in morphing. In particular, the PDE method, which is a fast boundary-value surface generation technique, may offer an excellent alternative for problems such as feature specification and transition control.

To the best of our knowledge, the PDE method proposed by Bloor and Wilson in [12] has not been used to carry out morphing and thus, our aim is to exploit its potential in this area. The PDE method discussed here offers numerous advantages. These include the speed at which individual morphs are generated and the intrinsic parametrisation of the surface shapes generated through the method. Such features may lead to smooth and controlled transitions between surfaces as required in morphing.

For the purpose of this work, three different methodologies are presented. The first one, is based on the change of the boundary conditions for the intermediate surfaces by using a weighted sum of the original boundary conditions of each surface. The second method is achieved by decreasing the Fourier mode for which the PDE is solved and the third one consists of a combination of the first two.

This paper is organised as follows: Section 2 outlines the mathematical basis of the PDE method in use, while Section 3 describes the methodology of the three proposed approaches for morphing using this method. The results obtained for each of the methods are discussed in Section 4 and conclusions together with future directions for this work are given in Section 5.

2 The Bloor-Wilson PDE Method

The first direct application of the PDE method formulated by Bloor and Wilson in the area of computer aided design consisted of using it as a method for blend generation [12] immediately followed by applications to physical and biomedical systems. Afterwards, its use was extended to diverse areas such as automatic design optimisation and interactive design [13]. A brief description of the method is given below. However, if the reader is interested in further details of its mathematical formulation, [12] will be a good reference for that purpose.

This method produces a parametric surface $\mathbf{X}(u,\, v)$, which is defined as the solution to a PDE of the form,

$$\left(\frac{\partial^2}{\partial u^2} + a^2\frac{\partial^2}{\partial v^2}\right)^2 \mathbf{X}(u,\, v) = 0\,, \tag{1}$$

where u and v are the parametric surface coordinates, which are then mapped into the physical space; i. e., $(x(u,\, v),\, y(u,\, v),\, z(u,\, v))$ and a is a parameter inherent to the PDE. Equation (1) is solved subject to a specific set of four boundary conditions that define the value of $\mathbf{X}(u,\, v)$ and some of its derivatives at determined regions. It is worth mentioning that when $a = 1$, Equation (1) is known as the biharmonic equation, which models some phenomena occurring within areas such as fluid and solid mechanics and therefore, many alternatives for solving it have been developed. However, it is stressed that the use of this method is not restricted to Equation (1). For instance, this formulation was adapted in [14] where a sixth order PDE was considered in order to achieve fast surface modelling.

If the choice of boundary conditions is restricted to periodic ones, a closed form analytic solution to Equation (1) can be obtained. In particular, when the parametric region defined by u and v is restricted to $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$, the solution to Equation (1) is given by,

$$\mathbf{X}(u,\, v) = \mathbf{A_0}(u) + \sum_{n=1}^{\infty} [\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)]\,, \tag{2}$$

where,

$$\begin{aligned} \mathbf{A_0} &= \mathbf{a_{00}} + \mathbf{a_{01}}u + \mathbf{a_{02}}u^2 + \mathbf{a_{03}}u^3\,, &(3)\\ \mathbf{A_n} &= (\mathbf{a_{n1}} + \mathbf{a_{n3}u})\, e^{anu} + (\mathbf{a_{n2}} + \mathbf{a_{n4}}u)\, e^{-anu}\,, &(4)\\ \mathbf{B_n} &= (\mathbf{b_{n1}} + \mathbf{b_{n3}u})\, e^{anu} + (\mathbf{b_{n2}} + \mathbf{b_{n4}}u)\, e^{-anu}\,. &(5) \end{aligned}$$

The value of the constants $\mathbf{a_{ij}}$ and $\mathbf{b_{ij}}$ are determined by the specified boundary conditions, which for this purpose, have to be expressed in terms of a Fourier series. In general cases when all the boundary conditions can be exactly expressed in terms of finite Fourier series, Equation (2) will also be finite. However, when the solution is given in terms of an infinite series, it can be approximated by the sum of the first N Fourier modes and the so called remainder term; i. e.,

$$\begin{aligned} \mathbf{X}(u,\, v) &= \mathbf{A_0}(u) + \sum_{n=1}^{N} [\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)] \\ &+ \mathbf{R}(u,\, v)\,, \end{aligned} \tag{6}$$

where $\mathbf{R}(u,\, v)$ is a function defined as,

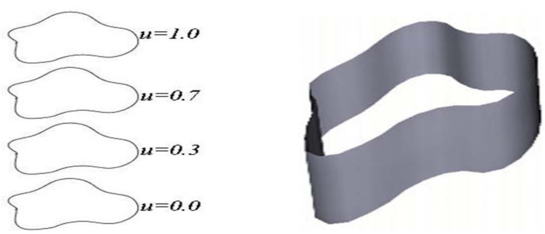
$$\begin{aligned} \mathbf{R}(u,\, v) &= \mathbf{r_1}(u)e^{wu} + \mathbf{r_2}(u)e^{-wu} \\ &+ \mathbf{r_3}(u)ue^{wu} + \mathbf{r_4}(u)ue^{-wu}\,, \end{aligned} \tag{7}$$

where w has been conveniently chosen as $w = a(N+1)$ and, $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$ and $\mathbf{r_4}$ are functions denoting the difference between the original boundary conditions and the ones satisfied by,

$$\mathbf{F}(u,\, v) = \mathbf{A_0}(u) + \sum_{n=1}^{N} [\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)]\,. \tag{8}$$

Therefore, Equation (7) guarantees that the original boundary conditions are exactly satisfied in Equation (6)in spite of the truncation of the series. An example of a surface generated by the PDE method in use is outlined in Figure 1. The generating positional boundary conditions are presented in Figure 1.a, and the associated surface is sketched in Figure 1.b. In this example, the expansion has been truncated after 5 modes. Notice that the same boundary curve has been employed for different values of u so that a cylindrical surface has been obtained.

It is important to stress that the method employed in this work differs slightly from the standard one proposed in [12] where two positional boundary conditions and their respective derivative boundary conditions are required to solve the PDE, whereas the method described above is formulated so that the PDE is solved by using four positional boundary conditions. This technique is faster for solving this kind of PDE than methods such as the ones based on either finite element or finite differences. As far as surface generation is concerned, the speed with which the PDE is solved (results are obtained in virtually real time) makes this technique an excellent choice. Additionally, the mathematical properties of this solution can be exploited so that the PDE method can be adapted to some problems in different areas of computer-aided geometric design such as morphing.



(a) (b)

Figure 1. Example of a surface generated by the PDE method in use. The boundary curves are shown in (a) and their PDE surface is sketched in (b).

3 Methodology

A method for morphing using the proposed PDE method can be achieved. The aim is to create intermediate surfaces between two given surfaces; the source surface, denoted by $\mathbf{S_s}$, and the target one represented by $\mathbf{S_t}$. These intermediate surfaces can be generated by taking advantage of the mathematical features inherent to Equation (6).

Now, given the freedom with which the boundary conditions of any intermediate surface can be specified, the use of some mathematical properties of Equation (6) may be useful in the task of finding iterative boundary conditions for each of the intermediate surfaces so that a smooth transition from $\mathbf{S_s}$ to $\mathbf{S_t}$ can be achieved.

Firstly, the property of closure of Fourier series; i. e., the sum of any two Fourier series is equal to another Fourier series, permits a gradual variation of the sum of the boundary conditions associated with $\mathbf{S_s}$ and the ones specified for $\mathbf{S_t}$.

Secondly, given that the larger the number of Fourier modes considered in the Fourier series expansion, the better the boundary conditions are satisfied, intermediate surfaces can be obtained in two stages: The first one consists of decreasing the number of modes employed in the computation of $\mathbf{S_s}$ until the number of modes employed reaches one. The second stage calculates intermediate surfaces using the boundary conditions associated with $\mathbf{S_t}$ and gradually increasing the number of modes from one until it reaches the number of Fourier modes for which $\mathbf{S_t}$ was originally computed. Nevertheless, when using this feature to achieve morphing, it is mandatory to neglect the remainder term from Equation (6) since this term is responsible for the exact satisfaction of the boundary conditions regardless of the number of Fourier modes included in the expansion.

The two features described above can be exploited in the implementation of three different alternatives for morphing since they are not mutually exclusive and can be combined to increase the variety of choices available for morphing.

3.1 Variation of the boundary conditions

The first method consists of using linear combinations of the boundary conditions associated with the source and target surfaces. Thus, let $\mathbf{B_s} = \{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{S_4}\}$ be the set of boundary conditions specified for the $\mathbf{S_s}$ and $\mathbf{B_t} = \{\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}\}$ be the set of boundary conditions representing $\mathbf{S_t}$. Now, the i^{th} intermediate set of boundary conditions $\mathbf{B_i} = \{\mathbf{I_1}, \mathbf{I_2}, \mathbf{I_3}, \mathbf{I_4}\}$ can be achieved by,

$$\begin{aligned} \mathbf{I_1} &= (1-\epsilon)\mathbf{S_1} + \epsilon\mathbf{T_1}, \\ \mathbf{I_2} &= (1-\epsilon)\mathbf{S_2} + \epsilon\mathbf{T_2}, \\ \mathbf{I_3} &= (1-\epsilon)\mathbf{S_3} + \epsilon\mathbf{T_3}, \\ \mathbf{I_4} &= (1-\epsilon)\mathbf{S_4} + \epsilon\mathbf{T_4}, \end{aligned} \quad (9)$$

where,

$$\epsilon = \frac{\gamma i}{m},$$

with $\gamma \geq 0$ and m is the total of intermediate surfaces to be created.

The iterative formulation described in Equation (9) provides intermediate boundary conditions generating surfaces that are gradually blending from the original source surface and simultaneously creating the target surface with a constant rate. The exclusion of the remainder term in Equation (6) is not mandatory; however, for the purposes of speed, this term was omitted. Therefore, the intermediate surface is then given by,

$$\mathbf{S_i}(u, v) = \mathbf{A_0}(u) + \sum_{n=1}^{N}[\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)], \quad (10)$$

where $\mathbf{A_0}$, $\mathbf{A_n}$ and $\mathbf{B_n}$ are subject to the set of boundary conditions specified by Equation (9). This method is particularly useful when the morphing of two surfaces with perfect cylindrical symmetry is required.

3.2 Variation of the number of Fourier modes

The second method to be discussed throughout this work is based upon the iterative manipulation of the number of Fourier modes associated with the expansions of the source and target surfaces. Let N_s and N_t be the number of Fourier modes for which $\mathbf{S_s}$ and $\mathbf{S_t}$ have been respectively expanded. For the purposes of this work, let the number of intermediate surfaces be $m = N_s + N_t$ and thus the i^{th} intermediate surface is obtained according to the following formulation,

$$\mathbf{S_i}(u,\, v) = \mathbf{A_0}(u) + \sum_{n=1}^{N_s - i} \left[\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)\right],$$
$$\text{for} \quad i = 1, ..., N_s\,,$$
$$\mathbf{S_i}(u,\, v) = \mathbf{A_0}(u) + \sum_{n=1}^{i - N_s} \left[\mathbf{A_n}\cos(nv) + \mathbf{B_n}\sin(nv)\right],$$
$$\text{for} \quad i = N_s + 1, ..., m\,, \tag{11}$$

where the boundary conditions employed to solve Equation (11) for each of the i^{th} intermediate surfaces are determined by,

$$\mathbf{B_s}\,, \quad \text{for} \quad i = 1, ..., N_s\,,$$
$$\mathbf{B_t}\,, \quad \text{for} \quad i = N_s + 1, ..., m\,.$$

Notice that, as mentioned before, the remainder term is not included in Equation (11) for computing $\mathbf{S_i}$ since such a term would automatically satisfy the prescribed boundary conditions and therefore, little or no morphing is likely to be achieved between intermediate surfaces.

3.3 Combination of a gradual variation of the boundary conditions and the number of Fourier modes

This method results from the combination of the ones presented above. The intermediate PDE surfaces are found according to a corresponding rule similar to Equation (11). The respective boundary conditions are given by Equation (9).

4 Results

The efficiency of the methodologies explained in the previous section is shown through the use of particular examples, each of which is suitable for the requirements.

4.1 Variation in the boundary conditions

The method consisting of a gradual variation of the boundary conditions has been assessed by specifying the constant rate γ at which the boundary conditions are to be changed. Now, ϵ in Equation (9) is determined for each of the intermediate surfaces. Thus, each intermediate surface is then computed by solving Equation (6) and using the boundary conditions determined by Equation (9). Once again, it is stressed that Equation (2) could be used for computing the intermediate surfaces. However, since it is not essential to satisfy the boundary conditions for the intermediate surface, the remainder term can be neglected and thus unnecessary computations are avoided.

For the purposes of the particular example shown in this section, the value of γ has been chosen to be 0.1 and therefore, nine intermediate surfaces are found. Moreover, two surfaces with perfect cylindrical symmetry have been chosen as the source and target surface. Two different kind of glasses have been selected as the source and target surface, $\mathbf{S_s}$ is regarded as a water glass whereas $\mathbf{S_t}$ and is a representation of a wine glass.

Figure 2 shows the source and target surfaces, which have been obtained with the aid of the Bloor-Wilson PDE method by using specific sets of boundary conditions for each respective case. Moreover, each of these surfaces is composed of two patches; that is, one patch corresponds to the base of the glass and another to its body.

(a) (b)

Figure 2. Source (a) and target (b) surfaces employed to assess the efficiency of the methodology based on the gradual variation of the boundary conditions.

Now, the i^{th} intermediate surface for this example is computed as follows: The boundary conditions associated with this intermediate surface are found by using Equation (3). A sequence varying from the source surface to the target one passing through the nine intermediate surfaces is shown in Figure 3. This sequence is outlined as follows: Figure 3.a corresponds to the water glass, which is transformed into Figure 3.b by changing its boundary conditions according to Equation 3 when $i = 1$. Then, Figure 3.b evolves into Figure 3.c and successively until Figure 3.k, which represents the wine glass, is achieved. The sequence qualitatively shows how the water glass is smoothly morphed into the wine one. For instance, it can be noticed that the base of the water glass, which is fairly small, starts to elongate progressively until it reaches the length of the base of the given wine glass. These results suggest that this method can be used for morphing surfaces without losing its advantages of speed and accuracy.

4.2 Results obtained by varying the number of Fourier modes in the series

The methodology concerning the variation of the Fourier modes taken for computing the original source and target surfaces cannot be employed in surfaces with perfect cylindrical symmetry. Therefore, a different set of source and target surface is required to assess the efficiency of this methodology. Thus, $\mathbf{S_s}$ and $\mathbf{S_t}$ have been specified as the

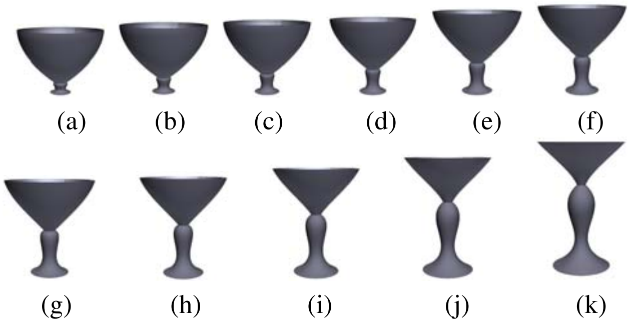


Figure 3. Sequence showing how the source surface $\mathbf{S_s}$ has been morphed into the target one $\mathbf{S_t}$ by using a gradual change in the boundary conditions and finding each of the intermediate surfaces via the proposed PDE method.

surface resulting from a set of boundary conditions so that the Fourier series associated with each of these curves can be expanded to 5 modes and with a non-zero remainder term. Figure 4 presents the source and target surfaces to be used. Figure 4.a sketches the source surface $\mathbf{S_s}$, whereas Figure 4.b is a graphical representation of the target one $\mathbf{S_t}$. Now, according to the methodology proposed in Section 3.2, it is possible to find 10 intermediate surfaces. The first five of them are computed by decreasing the Fourier mode for which Equation (6) and using the boundary conditions associated with $\mathbf{S_s}$. The other five are computed by increasing from 1 to 5, the mode for which the same equation is expanded and using the boundary conditions corresponding to $\mathbf{S_t}$. Figure 5 shows the sequence in which $\mathbf{S_s}$ is transformed into $\mathbf{S_t}$ through ten intermediate surfaces. The first (Figure 5.a) and last (Figure 5.l) figures correspond to the source and target surface respectively, whereas the remaining figures outline the transition between them. Again, a smooth transition is achieved; However it is not as smooth as the one obtained by varying the boundary conditions.

Figure 4. Source and target surfaces employed to assess the efficiency of the methodology consisting of varying the Fourier mode.

4.3 Results obtained by combining the two methodologies

For the purposes of assessing the results obtained by combining both methodologies, the gradual variation rate by which the boundary conditions of the intermediate surfaces are determined is again determined by Equation (9) for $\gamma = 1$. However, i in Equation (10) is allowed to vary from 0 to 10. Therefore, eleven intermediate surfaces can be potentially found.

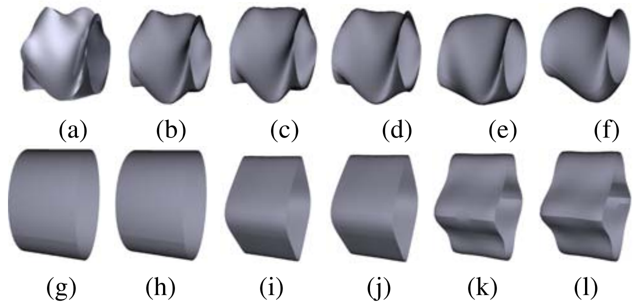


Figure 5. Morphing sequence obtained by varying the Fourier mode of the expansion.

In order to take all possible advantage of this combination, the intermediate surfaces will be computed as follows: The first five are calculated by decreasing the Fourier mode from 5 to 1 along with the first five sets of boundary conditions. The sixth intermediate surface is estimated by using one Fourier mode together with the sixth set of boundary conditions. This one corresponds to the one with the central morph; i.e., it resembles both source and target surfaces equally. The remaining five surfaces are determined by an increment in the Fourier mode and the rest of the pre-determined sets of boundary conditions. The same example used in the previous section is used to assess this case. Figure 6 shows the sequence in which $\mathbf{S_s}$ is transformed into $\mathbf{S_t}$ by sketching the original surfaces and their corresponding 11 intermediate surfaces. A smooth transition similar to the one outlined in Figure 3 is achieved showing that a simple manipulation of some of the parameters inherent to the PDE method in use leads to smooth transition in morphing, making this method a very powerful one.

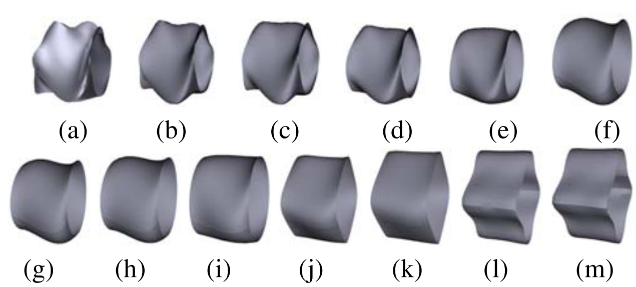


Figure 6. Morphing sequence obtained by combining both methodologies.

5 Conclusions

A method for shape morphing using PDE surfaces has been described. The mathematical properties inherent to the proposed PDE method enable the use of different methodologies in which morphing can be achieved. Three alternatives are described in this work: the first consists of a linear combination of the prescribed boundary conditions of the source and target surfaces, the second one achieves morphing by varying the number of Fourier modes for which the source and target surfaces are expanded and the third is a combination of the previous two. A particular example for each of these methodologies has been chosen and excellent results were obtained.

The third method is likely to be more successful the other methodologies proposed in the majority of cases; however, there are cases in which this method will not offer any additional advantage. For instance, the first of these methodologies is particularly useful when surfaces with perfect cylindrical symmetry are to be morphed since their Fourier expansion only requires one mode to exactly satisfy the prescribed boundary condition. Moreover, if such surfaces are to be transformed with the aid of the second methodology, no morphing can be achieved. Notice that the examples used in this work are somehow simple. However, they fulfil satisfactorily the task of showing how morphing is obtained when using the PDE method.

The major advantage of using the PDE method as a morphing tool relies on the fact that the intermediate surfaces can be obtained in virtually real time and the intermediate surfaces can be as smooth as required. Moreover, the morphing methodology presented in this work modifies the boundary curves exclusively, whereas other methodologies such as the ones based on subdivision NURBS modify each of the control points describing the surface, which is not a trivial task. Thus, the alternative avoids such complications.

Furthermore, the reader should be reminded that the use of this PDE method for this work considers objects with periodic boundary conditions and therefore, it restricts morphing to cases where the two objects are topologically equivalent. However it is not a restriction to the chosen PDE method in itself since the PDE can be fully solved numerically when the boundary conditions are not confined to periodic functions. Thus, the proposed PDE method represents an excellent choice for developing an interactive morphing tool and further studies concerning its potential may encourage a formulation of such a tool.

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