Pricing Basket of Credit Default Swaps and Collateralised Debt Obligation by Lévy Linearly Correlated, Stochastically Correlated, and Randomly Loaded Factor Copula Models and Evaluated by the Fast and Very Fast Fourier Transform

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Abstract

In the last decade, a considerable growth has been added to the volume of the credit risk derivatives market. This growth has been followed by the current financial market turbulence. These two periods have outlined how significant and important are the credit derivatives market and its products. Modelling-wise, this growth has parallelised by more complicated and assembled credit derivatives products such as m\(^{th}\) to default Credit Default Swaps (CDS), m out of n (CDS) and collateralised debt obligation (CDO).

In this thesis, the Lévy process has been proposed to generalise and overcome the Credit Risk derivatives standard pricing model’s limitations, i.e. Gaussian Factor Copula Model. One of the most important drawbacks is that it has a lack of tail dependence or, in other words, it needs more skewed correlation. However, by the Lévy Factor Copula Model, the microscopic approach of exploring this factor copula models has been developed and standardised to incorporate an endless number of distribution alternatives those admits the Lévy process. Since the Lévy process could include a variety of processes structural assumptions from pure jumps to continuous stochastic, then those distributions who admit this process could represent asymmetry and fat tails as they could characterise symmetry and normal tails. As a consequence they could capture both high and low events’ probabilities.

Subsequently, other techniques those could enhance the skewness of its correlation and be incorporated within the Lévy Factor Copula Model has been proposed, i.e. the “Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”. Then the Lévy process has been applied through a number of proposed
limiting and mixture cases of the Lévy Skew Alpha-Stable distribution and Generalized Hyperbolic distribution.

Numerically, the characteristic functions of the $m^{th}$ to default CDS’s and $\binom{n}{m}^{th}$ to default CDS’s number of defaults, the CDO’s cumulative loss, and loss given default are evaluated by semi-explicit techniques, i.e. via the DFT’s Fast form (FFT) and the proposed Very Fast form (VFFT). This technique through its fast and very fast forms reduce the computational complexity from $O(N^2)$ to, respectively, $O(N \log_2 N)$ and $O(N)$.

Keywords: Lévy Factor Copula, Stochastic Correlated Lévy Factor Copula, Lévy Random Factor Loading Copula, Lévy Skew Alpha-Stable, Generalized Hyperbolic, $m^{th}$ to default CDS, $\binom{n}{m}^{th}$ to default CDS, CDO, FFT, VFFT.
# Table of Contents

**ABSTRACT**

**TABLE OF CONTENTS**

**DEDICATION**

**ACKNOWLEDGEMENTS**

**LIST OF FIGURES**

**LIST OF TABLES**

**LIST OF ORIGINAL WORK**

**ABBREVIATIONS AND NOTATIONS**

## 1.0 INTRODUCTION  
1.1 **CHAPTER OUTLINE**  
1.2 **MARKET MOTIVATION**  
1.3 **MODELLING MOTIVATION**  
1.4 **AIMS AND OBJECTIVES**  
1.5 **OUTLINE**

## 2.0 OVERVIEW OF CREDIT RISK DERIVATIVES INSTRUMENTS  
2.1 **OUTLINE**  
2.2 **INTRODUCTION**  
2.3 **CREDIT RISK DERIVATIVES INSTRUMENTS**  
2.3.1 **THE CREDIT DEFAULT SWAP**  
2.3.2 **M**\textsuperscript{th} **TO DEFAULT BASKET CREDIT DEFAULT SWAPS**  
2.3.3 **RANKED M OUT OF N BASKET CREDIT DEFAULT SWAPS**  
2.3.4 **COLLATERALISED DEBT OBLIGATION**  
2.4 **MODELLING DEFAULTS**  
2.4.1 **DEFAULT CORRELATION**  
2.4.2 **THE GAUSSIAN COPULA MODEL FOR TIME TO DEFAULT**  
2.4.3 **THE GAUSSIAN FACTOR COPULA MODEL FOR TIME TO DEFAULT**  
2.4.4 **THE LEVY FACTOR COPULA MODEL FOR TIME TO DEFAULT**  
2.5 **COMPUTATION: THE FAST AND VERY FAST FORMS OF THE DISCRETE FOURIER TRANSFORM**  
2.6 **CONCLUSION**
8.3.3 Very Fast Fourier Transform 308
8.3.4 Comparing DFT, FFT, and VFFT 311
8.4 Characteristic Functions of Number of Defaults, Cumulative Loss, & Loss Given Default 312
  8.4.1 Number of Defaults’ Characteristic Function 312
  8.4.2 Cumulative Loss Characteristic Function 319
  8.4.3 Loss Given Default’s Characteristic Function 323
8.5 Numerical Evaluation via Discrete, Fast, Very Fast Fourier Transform 328
  8.5.1 Extracting the Number of Default’s by the Inverse Fourier Transform and the DFT 329
  8.5.2 Evaluating the Number of Defaults’ Characteristic Function 338
  8.5.3 Extracting the Cumulative Loss by the Inverse Fourier Transform and the DFT 344
  8.5.4 Evaluating the Cumulative Loss’s Characteristic Function 347
  8.5.5 Evaluating the Loss Given Default’s Characteristic Function 353
8.6 Mathematical Summary 356

9.0 Conclusion and Recommendations for Further Work 359
  9.1 Conclusion 359
  9.2 Recommendations for Further Work 360

10.0 References 362
Dedication

To Him,

My Parents,

My Wife and My Coming Baby

To all of Them I Dedicate This Thesis.
Acknowledgements

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Thank You All
List of Figures

Figure 1.1: Comparing CR, IR, and Equity growth acceleration. 3

Figure 1.2: Comparing the percentage growth between the CR total notional
outstandings’ peak in the second half in 2007 with the total notional outstanding
from the first half 2001 till its peak. 4

Figure 1.3: Comparing the percentage growth between the IR and Equity total
notional outstanding’s peak in the first half in 2008 with their total notional
outstandings from the first half 2001 till its peak. 4

Figure 1.4: Comparing CR, IR, and Equity growth deceleration 5

Figure 1.5: CR Total Notional Outstanding from the beginning of 2006 till the
end of the first half in year 2009 6

Figure 1.6: IR Total Notional Outstanding from the beginning of 2006 till the
end of the first half in year 2009 6

Figure 1.7: Equity Total Notional Outstanding from the beginning of 2006 till
the end of the first half in year 2009 7

Figure 2.1: CDS cash flow 15

Figure 2.2: m^(th-to-default) CDS contract cash flow 17

Figure 2.3: Ranked m out of n CDS contract cash flow 20

Figure 2.4: CDO illustration, i.e. Cash Flow, asset side, and liability side
representation 23

Figure 2.5: Liability side’s losses distribution at a specific time that corresponds 24

Figure 2.6: A comparison between the VFFT of 2003 and Gauss matrix
algorithm, Cooley-Tukey algorithm and all other algorithms and optimisations
various developed since 1965 37
List of Tables

Table 2.1: statistical summary of STCDO contracts from 14th Nov 2003 till 1st Feb 2007 29

Table 6.1: Proposed Models. Proposed (p), Not Proposed (NP), Mixture (M) 141

Table 8.1: Comparing the DFT, FFT, and VFFT complexity and accuracy 312
List of Original work

- Chapter 5
  - Proposing the Lévy process under the reduced form intensity model or the assets reduced form in the context of Factor Copula Model, i.e. “Lévy Factor Copula Model”.
  - Proposing the Lévy process into the Stochastic Correlated Factor Copula, i.e. “Lévy Stochastic Correlated Factor Copula Model”. This Model has two structures:
    - Binary Structure Case, i.e. “Lévy Binary Stochastic Correlated Factor Copula Model”
    - Symmetric Dependence Structure Case, i.e. “Lévy Symmetric Stochastic Correlated Factor Copula Model”
  - Random Factor Loading Copula, i.e. “Lévy Random Factor Loading Copula Model”.

- Chapter 6
  - Proposing a number of distributions those admits the Lévy process on the proposed modes in chapter 5. The subsequent table is a copy of Table 6.1, which summarises the proposed models in chapter 6. Proposed (p), Not Proposed (NP), Mixture (M)
<table>
<thead>
<tr>
<th>Limiting or Mixture Case</th>
<th>Linear</th>
<th>Stochastic Correlated</th>
<th>Factor Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Binary</td>
<td>Symmetric</td>
<td></td>
</tr>
<tr>
<td>Lévy Skew Alpha-Stable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Norm. M Gaussian</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Stand. Lévy Alpha Skewed</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Skewed t</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Fractional t</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>M (Stand. Gaussian, Fractional-t)</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Variance Gamma</td>
<td>NP</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>M (Gaussian, Variance Gamma)</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>M (Fractional-t, Variance Gamma)</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Normal Inverse Gaussian</td>
<td>NP</td>
<td>P</td>
<td>NP</td>
</tr>
<tr>
<td>M (Gaussian, Normal Inverse Gaussian)</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>M (Fractional-t, Normal Inverse Gaussian)</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
</tbody>
</table>

- Chapter 8
  - Proposing the Fast Fourier Transform to evaluate the number of defaults’, cumulative losses’, and loss given defaults’ characteristic functions explicitly. Some other authors have proposed it implicitly except in (Debuysscher, 2003, Debuysscher and Szegő, 2003a, Debuysscher and Szegő, 2003b, Debuysscher and Szegő, 2005) it was implemented in different way.
- Proposing the Very Fast Fourier Transform to evaluate the number of defaults’, cumulative losses’, and loss given defaults’ characteristic functions explicitly.

- Proposing the Lévy Skewed Alpha-Stable, Standard Lévy, Generalized Hyperbolic, Variance Gamma, and Normal Inverse Gaussian characteristic Function as a Loss Given Default’s structure.
Abbreviations and Notations

\[ \Delta_{y_1}^{Y_2} \] 2-order difference of \( y \)

\( \xi \) A uniform distribution

\( \rho \) Correlation parameter

\( \rho \) Spearman’s rho

\( \tau \) Kendall’s tau

\( \varphi \) Characteristic function

\( \psi \) Moment generating function

\( \phi \) Cumulant characteristic function or characteristic exponent function

\( \kappa \) Cumulant function

\( \sigma\text{-algebra} \) Sigma Algebra; see definition 4.1

\( \Phi \) The cumulative Gaussian Distribution function

\( \Phi_{\rho_{12}} \) The bivariate cumulative Gaussian Copula Function with correlation \( \rho_{12} \)

\( \Phi^{-1} \) The Inverse cumulative Gaussian Distribution function

\( \Omega \) Set of all possible outcomes

\( \Gamma_t \) The \( \mathbb{F}\)-hazard process of the time to default

\( \gamma_t \) The \( \mathbb{F}\)-intensity of the time to default

\( \delta_i \) The Recovery Rate
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{tr+di}^n$</td>
<td>The summation of the truncation and discretisation error</td>
</tr>
<tr>
<td>$\varepsilon_{trunc}^n$</td>
<td>Truncation error</td>
</tr>
<tr>
<td>$\tilde{\rho}$</td>
<td>Stochastic correlation coefficient</td>
</tr>
<tr>
<td>$\tau^{mth}$</td>
<td>$m$th time to default</td>
</tr>
<tr>
<td>$\varphi_{c_t}$</td>
<td>Cumulative loss’s characteristic function</td>
</tr>
<tr>
<td>$\varphi_{N_i}$</td>
<td>Number of default’s characteristic function</td>
</tr>
<tr>
<td>$\tau_i$</td>
<td>Random default times associated with the credit entity $i$.</td>
</tr>
<tr>
<td>$\mapsto$</td>
<td>Imply</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Infinity</td>
</tr>
<tr>
<td>$(\cdot,\cdot)$</td>
<td>Conditional probability</td>
</tr>
<tr>
<td>$(1 - \delta_i)$</td>
<td>The unrecovered rate</td>
</tr>
<tr>
<td>$(\Omega,\mathcal{G},\mathbb{Q}^*)$</td>
<td>Probability space used in chapter 4-8.</td>
</tr>
<tr>
<td>$(\mathcal{A}<em>{t_i})</em>{t\in[0,T],i\in\mathbb{K}}$</td>
<td>Firm value under geometric Brownian motion</td>
</tr>
<tr>
<td>$(n)_m^{th}$ to default CDS</td>
<td>$m$ out of $n$ Credit Default Swap</td>
</tr>
<tr>
<td>$(n)_{(R_m,R_M)}^{th}$ to default CDS</td>
<td>Ranked between $R_m$ and $R_M$ out of $n$ Credit Default Swap, i.e. $1 \leq R_m \leq m \leq R_M \leq n$</td>
</tr>
<tr>
<td>${X_t}_{t\in[0,T]}$</td>
<td>Stochastic Process</td>
</tr>
<tr>
<td>$\mathcal{AL}$</td>
<td>Accrued premium payments Leg</td>
</tr>
<tr>
<td>$B_i$</td>
<td>Bernoulli random variable.</td>
</tr>
<tr>
<td>$\mathcal{B}_{ti}$</td>
<td>The discount factor</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>( \mathcal{C}_t )</td>
<td>The cumulative loss</td>
</tr>
<tr>
<td>( \mathcal{C}_t^{[s,d,s_B]} )</td>
<td>The tranche’s cumulative loss</td>
</tr>
<tr>
<td>( \mathcal{C}_t^{(-i)th} )</td>
<td>The cumulative loss excluded ( i )</td>
</tr>
<tr>
<td>( \mathcal{C}_t^{[s,d,s_B]}_i^{(-i)th} )</td>
<td>The excluded ( i ) tranche’s cumulative loss</td>
</tr>
<tr>
<td>( C(x, y) )</td>
<td>Copulas’ function of ( x ) and ( y )</td>
</tr>
<tr>
<td>( C_{II} )</td>
<td>Product Copula function</td>
</tr>
<tr>
<td>( CBO )</td>
<td>Collateralised Bond Obligations</td>
</tr>
<tr>
<td>( CDX )</td>
<td>Credit Default Index that contains North American and Emerging Market companies</td>
</tr>
<tr>
<td>( CDO )</td>
<td>Collateralised Debt Obligation</td>
</tr>
<tr>
<td>( CDO^2 )</td>
<td>Collateralised Debt Obligations-squared structures</td>
</tr>
<tr>
<td>( CDS )</td>
<td>Credit Default Swap</td>
</tr>
<tr>
<td>( CLO )</td>
<td>Collateralised Loan Obligations</td>
</tr>
<tr>
<td>( CLT )</td>
<td>Central Limit Theorem</td>
</tr>
<tr>
<td>( CR )</td>
<td>Credit Risk</td>
</tr>
<tr>
<td>( D_i )</td>
<td>Loss given default</td>
</tr>
<tr>
<td>( DFT )</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>( DL )</td>
<td>Default payment Leg</td>
</tr>
<tr>
<td>( Dom(X) )</td>
<td>Domain of ( X )</td>
</tr>
<tr>
<td>( E )</td>
<td>Expectation</td>
</tr>
</tbody>
</table>
\( \mathbb{E}^{\mathbb{Q}^*} \)  
Expectation under the martingale probability measure

\( F \)  
Time to default filtration

\( F^\mathbb{Q} \)  
Cumulative distribution function under the martingale measure

\( F_X^{(-1)} \)  
Quasi-Inverse function of \( X \)

\( F_{X,Y} \)  
Bivariate function of \( X \) and \( Y \)

\( F_X^{-1} \)  
Inverse function of \( X \)

\( f^w_t \)  
The spot forward rate

\( \mathcal{FFT} \)  
Fast Fourier Transform

\( \mathcal{F} t(\nu) \)  
Fractional-\( t \) distribution

\( \mathcal{G} \)  
An enlarged filtration, i.e. \( \mathcal{G} = \mathcal{F} \vee \mathbb{H} \)

\( \mathcal{G} \)  
\( \sigma \)-algebra that includes all sets that the needed statements

\( G^k_t \)  
The probability of the \( k^{th} \)-to-default time survival function.

\( \mathcal{GCD} \)  
greatest common divisor function

\( GCLT \)  
Generalised Central Limit Theorem

\( GDP \)  
Gross Domestic Product

\( \mathcal{G}\mathcal{H}(\lambda, \alpha, \beta, \delta, \mu) \)  
generalised hyperbolic distribution

\( \mathbb{H} \)  
The natural filtration of the time to default

\( j() \)  
Indicator Default Process
\( \mathcal{J}_{t_i} \)  
Idiosyncratic Risk Factors

\( \mathbb{I} \)  
Numbers belongs to \([0, 1]\)

\( IR \)  
Interest Rate

\( ISDA \)  
International Swaps and Derivatives Association

\( \inf \)  
Infimum

\( iTraxx \)  
Credit Default Index that contains companies from around the world

\( K \)  
Any number between 1 and \( n \), where \( n \) is the number of referenced credit entities.

\( K^- \)  
Any number between 1 and \((n-1)\), where \( n \) is the number of referenced credit entities.

\( \mathcal{K} \)  
The nominal

\( K_{\lambda}(\cdot) \)  
the modified Bessel function of the third kind of order \( \lambda \)

\( \mathcal{L}(\alpha, \beta, \gamma, \delta; 1) \)  
Lévy Skewed Alpha-Stable distribution

\( \mathcal{M} \)  
Mixture of two distributions

\( \mathcal{M}(\ldots) \)  
Upper Frechet-Hoeffding bound

\( \mathcal{M}_t \)  
Systematic Market Risk Factor

\( \mathcal{M}BS \)  
Mortgage-Backed Securities

\( \mathcal{M}^N_\mathcal{G} \)  
Normalized Double Mixture Gaussian distribution

\( \mathcal{M}^{SG}_N(\nu, \rho) \)  
Mixture Standard Gaussian & Normalised Fractional-\( \mathcal{F} \) Distribution

\( \mathcal{M}^{N\mathcal{F}_t}(\nu, \alpha, \beta, \delta, \mu, \rho) \)  
Mixture Normalised Fractional-\( \mathcal{F} \) \& Normal Inverse
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

\[ \mathbb{M}^{SG}_{N \mathcal{G}}(\alpha, \beta, \delta, \mu, p) \]
Gaussian Distribution

\[ \mathbb{M}^{N \mathcal{F}t}_{\nu \mathcal{G}}(\nu, \lambda, \alpha, \beta, \mu, p) \]
Mixture Normalised Fractional-\( t \) & Variance Gamma Distribution

\[ \mathbb{M}^{SG}_{\nu \mathcal{G}}(\lambda, \alpha, \beta, \mu, p) \]
Mixture Standard Gaussian & Variance Gamma Distribution

\[ m^{th\text{-to-default}}\text{CDS} \]
mth-to-default Credit Default Swap

\[ \mathbb{N} \]
the set of non-negative integers

\[ \mathcal{N}_t \]
Counter Process

\[ \mathcal{N}_t^i \]
Counter Process associated with credit entity \( i \).

\[ \mathcal{N}_{(i \text{-th})\tau_t} \]
Counter Process, excluding \( i \)

\[ \mathcal{N}^\mathcal{Ft}(\nu) \]
Normalised Fractional-\( t \) distribution

\[ \mathcal{N}^\mathcal{IG}(\alpha, \beta, \delta, \mu) \]
Normal Inverse Gaussian Distribution

\[ O(.) \]
Computational Complexity

\[ \mathcal{O} \mathcal{TC} \]
Over-The-Counter

\[ \mathcal{P}_\Delta \]
The Periodic premium

\[ \mathcal{P} \mathcal{L} \]
premium payment leg

\[ p_{ti}^{\xi_i | \mathcal{M}_t} \]
The probability of default conditioned upon the Systematic Market Risk Factor \( \mathcal{M}_t \)

\[ q_{ti}^{\xi_i | \mathcal{M}_t} \]
The probability of survival time conditioned upon the Systematic Market Risk Factor \( \mathcal{M}_t \)
\( \mathbb{Q} \) or \( \mathbb{Q}^* \) Martingale Probability Measure

\( \mathbb{Q} \) Concordance Function

\( \mathbb{Q}_c |_{B,\infty[} \) The distribution of the function \( C_t \) at the period \( [B, \infty[ \).

\( \mathbb{R} \) Real Numbers

\( \mathcal{R}_M \) Upper Rank Barrier of mth-to-default

\( \mathcal{R}_m \) Lower Rank Barrier of mth-to-default

\( r_{\text{DFT}} \) The \( \text{DFT} \) resolution

\( r_{\text{Disc}} \) Discrete Density Resolution

\( r_{\text{FFT}} \) The \( \text{FFT} \) resolution

\( r_{\text{VFFT}} \) the \( \text{VFFT} \) resolution

\( \text{Ran}(F_X) \) Range of the function \( X \).

\( \mathcal{S}_B \) The tranche detachment point

\( \mathcal{S}_A \) The tranche attachment point

\( \text{Scaled } - t(\nu, \delta, \mu) \) Scaled-t distribution

\( SL(\alpha, \beta) \) Standard Lévy distribution

\( St (\nu, \beta, \delta, \mu) \) Skewed t distribution

\( \text{SCDO} \) Synthetic Collateralised Debt Obligations

\( \text{STCDO} \) Standardised Collateralised Debt Obligations Tranches

\( \text{sup} \) Supermum

\( t_{\Delta t} \) Payment length between \([t_{\ell-1}, t_{\ell}]\)
$t_L$  
Maturity date of the Contract.

$V_{F_{X,Y}}$  
The Volume of the bivariate function of X and Y

$\mathcal{VFFT}$  
Very Fast Fourier Transform

$\mathcal{VG}(\lambda, \alpha, \beta, \mu)$  
Variance Gamma Distribution

$W(., .)$  
Lower Frechet-Hoeffding bound

$y_{least}$  
The Least element in the Domain of Y
Chapter One

Introduction

1.1 Chapter Outline

- Market Motivation
- Modelling Motivation
- Aims and objectives
- Outline
The current financial market turbulence, which has followed a few years of considerable growth in the volume of the credit risk, which is denoted by $\mathcal{CR}$, derivatives market, has outlined how significant and important are these products. The development of theoretical and quantitative models to price $\mathcal{CR}$ derivatives has become a focal point of concentration for practitioners, regulators and researchers. The $\mathcal{CR}$ derivatives markets are in particular involved with $\mathcal{CR}$ transfer activities or redistributing them and the augmentation of more complex and model-driven trading strategies. The instruments of this market are fairly complex such as Synthetic Collateralised Debt Obligations, which is shorten as $\text{SCDO}$, which transfers the risk of a pool of single-name Credit Default Swaps, which is abbreviated by $\text{CDS}$, contracts. However, the complex nature and the dimension of the problems of the $\mathcal{CR}$ derivatives require advanced numerical methods.

In this chapter, the $\mathcal{CR}$ derivatives’ market motivation will be discussed to cover the two contradictory periods, i.e. the booming period and the credit crunch period. This part will highlight how important are the $\mathcal{CR}$ derivatives and explain the influence that it has on the current global market. Since it is a modelling based thesis rather than market overview or regulator one, discussing how these two periods have influenced the $\mathcal{CR}$ derivatives modelling is an essential point. This will be introduced in the subsequent section as modelling motivation. Then the aim, objectives, and the thesis outline will be presented.

1.2 Market Motivation

Since the late of 1990s there has been a considerable growth in the volume of the $\mathcal{CR}$ derivatives market. Even though it was not the largest Over-The-Counter, which is denoted as $\text{OTC}$, derivative market from volume prospective, it was until the second half of year 2007 the most growthable derivative in this markets.
The CR derivatives total notional outstanding peak will be studied and compared percentagewise with the total notional outstanding peaks of the Interest Rate, which is shorten as IR, derivatives and the Equity derivatives as they are the main three OTC derivative markets in order to observe their volume growth. In order to specify the scope of this comparison, each derivative products will be specified. The CR derivatives consist of CDS referencing single credit entities, baskets, portfolios, and indices. Also, it comprises all types of collateralised debt obligation, which is expressed as CDO. The IR derivatives encompass the IR swaps, IR options, and cross currency swaps, while the Equity derivatives include the Equity swaps, Equity derivatives, and Equity forwards contracts.

If the CR, IR, and the Equity total notional outstandings’ growth acceleration are compared percentagewise between the first half of 2001 till each derivatives market peak, the CR will capture 90%, where the IR will be 6% and finally the Equity will be around 4%, as it is shown in Figure 1.1.

![Figure 1.1: Comparing CR, IR, and Equity growth acceleration.](image)

According to the International Swaps and Derivatives Association, which is abbreviated as ISDA, Market Survey presented in (ISDA, 1987-2009), the CR derivatives’ total notional outstanding has boomed from $0.63 trillion in the first half of year 2001 to
$62.17 trillion as its peak in the second half of year 2007. This is approximately equals to 9745% of growth; see Figure 1.2 for more details.

![Figure 1.2: Comparing the percentage growth between the CR total notional outstandings’ peak in the second half in 2007 with the total notional outstandings from the first half 2001 till its peak.](image1)

On the other hand, the IR derivatives’ total notional outstanding have grown by 710% from $57.31 trillion in the first half of year 2001 to $464.69 trillion as its peak point was in the first half of year 2008. In the same direction, the equity derivatives’ total notional outstanding by 414% from $2.31 trillion in the first half of year 2002 to $11.89 trillion in first half of 2008. This is exhibited in Figure 1.3.

![Figure 1.3: Comparing the percentage growth between the IR and Equity total notional outstanding’s peak in the first half in 2008 with their total notional outstandings from the first half 2001 till its peak.](image2)

Subsequently to this period, the market’s trend has been flipped, fluctuated, and entered what is called by the credit crunch period. In view of what happened to the CR derivatives market and the interconnectedness between the CR derivatives, other OTC
derivatives, credit entities, banks, financial markets, countries’ Gross Domestic Product, which is denoted by GDP, etc, the credit crunch period has emphasized how deeply the financial market is driven by the CR derivatives market.

To demonstrate this fact, the CR total notional outstanding will be studied and compared, in the same manner it was carried out previously, with the total notional outrstandings’ of the IR and the Equity derivatives’.

If the CR, IR, and the Equity total notional outrstandings’ growth deceleration are compared percentagewise between each derivatives market peak and their bottoms\textsuperscript{1}, the CR will capture -57\%, where the Equity will be -30\%, and finally the IR will be -13\%, as it is pied in Figure 1.4.

![Figure 1.4: Comparing CR, IR, and Equity growth deceleration.](image)

According to (ISDA, 1987-2009) the present CR derivatives’ total notional outstanding is estimated by $31.22 trillion in the first half of 2009. It has decreased since the second half of 2007 by around 50\%; from $62.17 trillion second half of 2007. Only since last year it has reduced by around 43\%, as it has been reduced from $54.61 trillion in the first mid of 2008. Still the turmoil is continuing as its total notional outstanding has been diminished by around 19\% from $38.56 trillion in second half of 2008, for more details see Figure 1.5.

\textsuperscript{1} When their bottom is not obviously appeared it will be compared against the first half of year 2009.
In contrary, as could be seen in Figure 1.6, the \( \mathcal{R} \) derivatives’ total notional outstanding have started recovering from last year partial climate. In the period between the second half of 2007 till the end of the first half of 2008 the \( \mathcal{R} \) derivatives’ total notional outstanding has increased by around 18% from $382.30 trillion to $464.69 trillion. The total notional outstanding of these derivatives has faced a slight decrease in first half of 2008, comparing to the percentage of the \( \mathcal{R} \) derivatives decrease in the following half by around 13%, i.e. 403.07 trillion, while its total notional outstanding has started increasing again in the first half of year 2009 by around 3%, i.e. 414.09 trillion.
In the same direction, the notional amount outstanding of the equity derivatives remained comparatively flat at $8.8 trillion compared to 27% reduction from $11.8 trillion in the second half of 2008.

![Equity Total Notional Outstanding from the beginning of 2006 till the end of the first half in year 2009](image)

It could be concluded that the CR derivatives market has a huge influence on the global market. This influence, in the booming period, was encouraging regulators, banks, and bankers behind them to insignificantly relaxed the regulations procedures, fast the risk transformation, and assuming the normality and liquidity of this market. This has led to many consequences one of the main obvious ones is the credit crunch period and its consequences. This problem could be explained in more details from a modelling point of view in the next subsection.

### 1.3 Modelling Motivation

Practitioners, regulators and researchers have concentrated developing theoretical and quantitative models those could price the CR derivatives. The complexity and the dimensionality\(^2\) nature of the CR derivatives require advanced numerical methods. However, the liquidity of the CR derivatives market promoted the practitioners and

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\(^2\) Since many CR derivatives products consist of a large number of underlying assets.
regulators to assume the normality of it. This assumption has relaxed the complexity of measuring and pricing in the contingent claims framework in the $\mathcal{CR}$ derivatives.

The “Gaussian Factor Copula Model” introduced in (Li, 2000), which is linearly correlated approach with deterministic parameters, has become the market standard model since it fulfils the normality assumption and could overcomes the dimensionality problem when integrated with some advanced numerical methods. Unappreciatively, with the chain of interconnectedness in this market, the normality assumption of the Gaussian Factor Copula has mispriced, hidden, and accumulated the actual risk by transferring and distributing the risk between the nodes of this chain in the booming period. Mispricing the $\mathcal{CR}$ derivatives and the accumulation of its hidden losses are two strong factors, on top of the fast and accelerated growth in its total notional amount, those have started and continued the credit crunch domino.

Conversely, researchers tried to enrich the literature by improving the standard model to capture the actual risk. They started since the beginning of the booming period, where practitioners and regulators have followed them after the credit crunch symptoms. These improvements have taken three directions. The first was by integrating skewness features within the standard model. The second was to replace the linear correlation by a stochastic correlation where the third one is by a stochastic risk exposure.

It could be concluded that this domino has increased significantly the need to build strong models those could capture the actual price of those products in order to rebalance the $\mathcal{CR}$ derivatives market and subsequently the whole financial markets domino. These models must simplify the complexity nature of the $\mathcal{CR}$ derivatives. Additionally, these models must incorporate some advanced numerical methods those could reduce the dimensionality characteristics of the $\mathcal{CR}$ derivatives.
1.4 Aims and objectives

The aim of this research is to introduce a new approach that could capture any \( \mathcal{CR} \) derivatives’ actual price, simplify its complexity nature, and reduce its dimensionality characteristics by generalising and standardising the existing Gaussian Factor Copula, Stochastic Correlated Gaussian Factor Copula, and Gaussian Random Factor Loading Copula Models through the Lévy process and by employing specific numerical techniques, i.e. the Fast Fourier Transform (\( \mathcal{FFT} \)) and the Very Fast Fourier Transform (\( V\mathcal{FFT} \)).

The objectives of this research are to:

1. Overview intuitively the structure and the cash flow of some important \( \mathcal{CR} \) derivatives instruments.
2. Review, mathematically, the concepts of the copula function and the time to default individually.
3. Build the copula function and the time to default within the Lévy Factor Copula context and its skewed versions, and sequentially, the later in the context of some important \( \mathcal{CR} \) derivatives products.
4. Extending, mathematically, the Lévy Factor Copula and its skewed versions from theory to application.
5. Apply \( \mathcal{FFT} \) and \( V\mathcal{FFT} \) algorithms to evaluate some important \( \mathcal{CR} \) derivatives instruments.

1.5 Outline

The next chapter; chapter two, is an introduction that allows the reader to gain an overview on the \( \mathcal{CR} \) derivatives products, i.e. basket credit derivatives and \( \mathcal{CDO} \), structure and cash flow without any heavy mathematics. Subsequently this chapter will focus on illustrating the blocks those build the propose \( \mathcal{CR} \) derivatives model.
Chapter three and four deals, mathematically, with the theories of copula and time to default. Those two chapters with the Lévy process and the Factor model are four major parts those will be implied, mathematically, to build the proposed models, i.e. the Lévy Factor Copula and its skewed versions, in chapter five. Chapter six moves the Lévy Factor Copula and its skewed versions from theory to application by introducing few new Factor Copula Models those admit the Lévy process, where chapter seven builds, mathematically, the pricing models within the contexts of the \( CR \) derivatives instruments, i.e. basket credit derivatives and \( CDO \).

Chapter eight describes and proves the characteristic functions of the number of defaults and the accumulated loss, which are essential to evaluate the basket \( CR \) derivatives instruments. Subsequently, this chapter proposes\(^3\) explicitly how to extract them by the Discrete Fourier Transform (\( DFT \)). Finally, this chapter consider two forms the \( DFT \). The first is the Fast Fourier Transform (\( FFT \)), where the second is the proposed and recommended form; the Very Fast Fourier Transform (\( VFFT \)).

Chapter nine concludes the current work and thesis and recommends some future work that could be carried out.

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\(^3\) Some others have proposed implicitly the use of FFT without explaining it.
Chapter Two
Overview of Credit Risk Derivatives Instruments

2.1 Outline

- Introduction

- Credit Risk Derivatives Instruments
  - The Credit Default Swap
  - $m^{th}$ to Default Basket Credit Default Swaps
  - Ranked $m$ out of $n$ Basket Credit Default Swaps
  - Collateralised Debt Obligation

- Modelling Defaults
  - Default Correlation
  - The Gaussian Copula Model for Time to Default
  - The Gaussian Factor Copula Model for Time to Default
  - The Lévy Factor Copula Model for Time to Default

- Computation: The Fast and Very Fast forms of the Discrete Fourier Transform

- Conclusion
2.2 Introduction

Recently, there has been a considerable growth in the volume of the Credit Risk (CR) derivatives market. The development of theoretical and quantitative models to price CR derivatives has become a focal point of concentration for practitioners, regulators and researchers. The CR derivatives markets are in particular involved with CR transfer activities or redistributing them and the augmentation of more complex and model-driven trading strategies. The instruments of this market are fairly complex such as Synthetic Collateralized Debt Obligations (SCDOs) which transfer the risk of a pool of single-name Credit Default Swaps (CDS) and the $n^{th}$ to default Credit Default Swaps $n^{th}$ to default CDS. In general, CR derivatives are contracts valued upon the creditworthiness of one or more credit entities, for example: companies, loans, or countries. Interested readers are referred to (Choudhry, 2004) and (Lehman-Brothers, 2003) for in-depth discussion and specification of these and some other CR derivatives products and the reasons they are traded for, while (Brown, 2005), (Steigman and Vassegh, 2005), (Bluhm et al., 2002), and (Bluhm and Overbeck, 2006) are referred for more structural information of their markets and their payoff and cash flow representation.

The purpose of this chapter is to explain intuitively and without any heavy mathematics the structure and the payoff of some CR derivatives instruments, i.e. the $n^{th}$ to default CDS, ranked m out of n CDS, and the CDOs. This will be achieved by explaining the standard CDS as a base model for the preceding ones. Subsequently, the dependencies between the underlying of the credit entities are clarified through the Gaussian copula model, the factor copula model, and its skewed versions. This chapter will be concluded by explaining briefly the purpose of proposing the Lévy process instead of the Gaussian when integrated in the factor copula model and its skewed versions and the purpose of
proposing the Fast \((FFT)\) and Very Fast \((VFFT)\) forms of the Discrete Fourier Transform.

2.3 Credit Risk Derivatives Instruments

2.3.1 The Credit Default Swap

Numerous exotic CR derivatives are structured upon the standard Credit Derivatives Swap \((CDS)\). A \(CDS\) is a contract that its payoff is valued upon the default losses of a credit entity in response for an agreed premium. Principally, the \(CDS\) is a bi-contract, i.e. the buyer and seller of the protection. A premium or spread is paid periodically by the protection buyer as an insurance payment until the maturity of the contract or default event occurs. In case of default, the protection seller compensates the fractional loss or called the unrecovered fraction of the credit entity’s value times the \(CDS\) notional amount. To clarify it more, the protection seller will not make any payment if there is no default. Payments flow made from the protection buyer to the protection seller is called the Protection Leg or “Premium Leg”. In contrary, the payments flow from the protection seller to the protection buyer is called the “Default Leg”.

In the case of default, the settlement of the \(CDS\) contract could be either a cash settlement or a physical settlement. In the former, the protection seller will make an immediate cash payment to the protection buyer at the time of default. In contrast, when a default event occurs in the case of the physical settlement, the protection buyer transfers the credit entity to the protection seller in return for the \(CDS\) National or equally for the notional of the credit entity. Subsequently, the protection seller can directly sell the credit entity in return for its value after default. The credit entity’s market value after default is equal to its recovered fraction’s value times the \(CDS\) notional amount. Again the protection seller’s loss is equal to the unrecovered fraction of the credit entity’s value times the \(CDS\) notional amount, which is equal to the cash
settlement. As a consequence of their resemblance, a cash settlement always will be assumed in the subsequent when modelling a CDS contract\textsuperscript{4}.

Oppositely to cash settlement promised by the protection seller, the protection buyer pays a periodic premium, called also “CDS Spread”. This periodic premium is fixed at the start of the CDS contract. Subsequent to the credit entity’s default, the protection buyer is obligate to pay the fractional amount of the premium payment that has accrued since the preceding periodic payment to the credit entity default, which is called the accrued premium or the “Accrued Leg”. When the credit entity default, the protection buyer payment stream “Premium Leg” will be terminated with the fractional payment “Accrued Leg”.

As declared previously, the periodic premiums are fixed at the start of the CDS contract, but it is done so that the expected value of the CDS contract is equal for both the protection seller and the protection buyer. This is achieved by equalising the expectation value of the premium leg to the expectation value of the default leg. The periodic premiums are characteristically made once, twice, or four time a year.

Figure 2.1 gives an example of a CDS cash flow streams over its contracted period. Assume that two parties have agreed to enter: (i) a 5 year CDS on the 1\textsuperscript{st} of January 2009, (ii) the notional principal is £100 Million, (iii) and the quarterly premium payment of 22.5 basis point for protection against the credit entity’s default.

Upon the previous assumption, two cases will be studied: the first is when there is no default until the maturity of the contract. The second is by assuming that the credit entity has defaulted in the 1\textsuperscript{st} of June 2010.

\textsuperscript{4} The same assumption will hold in the \(n^{th}\) to default CDS, \(m^{th}\) to default CDS, CDO.
In the case of no default, the protection buyer will pay periodic premium payments of £225,000 at the 1st of every quarter until the end of the CDS contract, i.e. 1st of January, 1st April, 1st July, and 1st of October of the years 2009 till 1st of January 2014, except for the year 2009 the 1st of January is excluded. At the end of the contract the protection buyer will pay £4.5 Million, where the protection seller did not pay anything in return.

In contrast, in the second case, where the credit entity has defaulted in the 1st of June 2010, the protection buyer will pay five periodic payment of £225,000 and then the regular payments will stop. Nevertheless, for the reason that these payments are made in predetermined dates, a closing payment has to be made to cover the accrued payment by the protection payer. The accrued premium payment is (approximately) equal to £150,000. Consequently, the protection seller has to pay the unrecovered fraction times the notional. By assuming that the credit entity’s recovered fraction is equal to 40%, the protection seller has to pay \((1 - 0.4) \times 100\) Million, which is equal to £60 Million.

2.3.2 mth to Default Basket Credit Default Swaps

Unlike the standard CDS contract, the Basket Credit Default Swaps are based on referencing a number of credit entities. Instead of valuing each CDS contract alone, they
will be considered as a one pool that is valued upon the number of defaults protected against. Since the proposed valuation is based on the event of a \( m^{th} \) to default credit entities, it is referred as the \( m^{th} \) to default CDS.

In the case of the \( 1^{st} \) to default CDS, the protection buyer will continue paying the periodic premium leg regularly until either the \( 1^{st} \) to default event occur for any credit entity out of the basket or the maturity of the contract arrives, where the protection seller is obligated in the case of \( 1^{st} \) to default event to pay the default leg. In general, the protection buyer of a \( m^{th} \) to default CDS pays the periodic premium leg regularly until either the \( m^{th} \) to default events occur for any \( m \) credit entities out of the basket or the maturity of the contract arrives, where the protection seller is obligated in the case of \( m^{th} \) to default events to pay the default leg, where it is valued by the same approach as the standard CDS. Subsequent to the occurrence of the default, there is a settlement and the contract is concluded with no more payments by either party are required.

Matching up the valuation of \( m^{th} \) to default CDS with the standard CDS is noted in the following to standardise the concept. There are predetermined periodic premiums at the beginning of the \( m^{th} \) to default CDS contract that are calculated so that the expected payments by the protection buyer and the expected payment by the protection seller in the \( m^{th} \) to default CDS contract are equivalent. Similarly to the standard CDS contract, the expectation value of the premium leg and the expectation value of the default leg are equalised in order to achieve this task. Likewise the standard CDS, the periodic premiums are routinely made every quarter of the year, half of the year, or annually.

To exemplify the \( m^{th} \) to default CDS contract cash flow streams over its contracted period, Figure 2.2 is illustrated. Suppose that two parties have agreed to enter: (i) default swap contract of a basket that contains 10 companies, (ii) 5 year \( 3^{rd} \) to default CDS on the \( 1^{st} \) of January 2009, (iii) the notional principal is equal for all credit entities...
and their total is equivalent to £1 Billion, (iv) and the quarterly premium payment of 70 basis point for protection against the 3rd default event.

Upon the previous supposition, two cases will be studied: the first is when there is no more than two defaults occur until the maturity of the contract. The second is by assuming that third credit entity has defaulted on the 1st of June 2010.

In the case of no more than two defaults, the protection buyer will pay periodic premium payments of £700,000 at the 1st of every quarter until the end of the 3rd default CDS contract, i.e. 1st of January, 1st April, 1st July, and 1st of October of the years 2009 till 1st of January 2014, except for the year 2009 the 1st of January is excluded. At the end of the contract the protection buyer will pay £14 Million where the protection seller did not pay anything in return.

In contrast, in the second case, where the third credit entity has defaulted in the 1st of June 2010, the protection buyer have paid five periodic payment of £700,000 and then the regular payments will stop. On the other hand, for the reason that these payments are made in predetermined dates, a concluding payment has to be made to cover the accrued
payment by the protection payer. The accrued premium payment is (approximately) equal to £466,667. Consequently, the protection seller has to pay the unrecovered fraction times their notional. By supposing that the three defaulted credit entities’ recovered fraction is equal to 40% of their notional, the protection seller has to pay $(1 - 0.4) \times 300$ Million, which is equal to £180 Million.

2.3.3 Ranked $m$ out of $n$ Basket Credit Default Swaps

$m$ out of $n$ Basket Credit Default Swaps, which is referred as the $(\binom{n}{m})^{th}$ to default $CDS$, is another type of the Basket Credit Default Swaps contracts, where the protection seller is obliged to pay the default leg to the protection buyer, when a $m^{th}$ to default event occur. The default leg cashflow and valuation method is proceeded in the same manner that the $m^{th}$ to default $CDS$ is carried out. Complementary to the default leg of $m^{th}$ to default $CDS$, the premium leg of the $(\binom{n}{m})^{th}$ to default $CDS$ is paid upon the un-defaulted credit entities of the whole basket until $m^{th}$ to default events occur.

In this subsection, particular product of the $(\binom{n}{m})^{th}$ to default $CDS$ will be considered called the ranked credit entities of the $(\binom{n}{m})^{th}$ to default $CDS$. This kind of contracts supplies protection for ranked defaults in the basket, where it covers the defaults of the credit entities ranked between $\mathcal{R}_m$ and $\mathcal{R}_M$ where the later is excluded, i.e. $1 \leq \mathcal{R}_m \leq m \leq \mathcal{R}_M \leq n$. The ranked $(\binom{n}{m})^{th}$ to default $CDS$ is referred as $(\binom{n}{[\mathcal{R}_m, \mathcal{R}_M])}^{th}$ to default $CDS$. Indeed, this means that instead of protecting against a particular number of defaults, the protection is covering a range of defaults.

The default payments of the $(\binom{n}{[\mathcal{R}_m, \mathcal{R}_M])}^{th}$ to default $CDS$ could occur in one or more of the subsequent dates: $t^{(\mathcal{R}_m+1)}_{\mathcal{R}_m}, \ldots, t^{m}_{\mathcal{R}_m}, \ldots, t^{(\mathcal{R}_M)}_{\mathcal{R}_M}$ if the $m \in [\mathcal{R}_m, \mathcal{R}_M)$ and the $(m^{th \ to \ default}) < t_i$. Each default occur in between of the ranking barriers obligates
a payment equals to the unrecovered fraction of the defaulted credit entity times its nominal. However, looking at the default payment from mathematical point of view and trying to fulfil the equilibrium the expected value of the default leg with the expected value of the premium leg is essential in the \( \binom{n}{R_{m},R_{M}} \) to default CDS. Indeed, a basic algebra confirms that the default leg is equivalent to the sum of the default legs paying the credit entities’ unrecovered fraction over all possible \( m^{th} \) to default events, but this part is not covered in this subsection, where it will be detailed in Chapter 8.

Analogously to the \( m^{th} \) to default CDS, the premium payments dates of the \( \binom{n}{[R_{m},R_{M}]} \) to default CDS are predetermined. In contrary, the premium legs are not always equal, where it depends on the number of defaulted credit entities and their ranking at the prearranged payment dates. Thus, the premium payments amounts could be one of three cases. The first is when the number of defaults are less than the lower credit entities’ ranking \( R_{m} \), the premium is equivalent to entire protected credit entities between \( R_{m} \) and \( R_{M} \). The second is when the number of default has exceeded the lower credit entities’ ranking \( R_{m} \) but it did not rise above the higher credit entities’ ranking \( R_{M} \), the premium is equivalent to the remaining protected credit entities between \( R_{m} \) and \( R_{M} \). The last case is when the number of default has exceeded the higher credit entities’ ranking \( R_{M} \), the premium leg is terminated and it is said that the basket of credit default swap is exhausted. At this point it is worth mentions that the number of defaults are sorted in order, as it is in the case of all basket credit default products, so that the defaults are classified to be in three ranges: below, in between, or above the ranking barriers.
Figure 2.3 exemplify the \( \left( \frac{n}{|\mathcal{R}_m, \mathcal{R}_M|} \right)^{th} \) to default CDS contract cashflow streams over its agreed life. Presume that bi-contract has been agreed between two parties to enter: (i) 5 year \( \left( \frac{20}{|4, 14|} \right)^{th} \) to default CDS on the 1st of January 2009, (ii) the total notional principal of the whole basket is £2 Billion, i.e. £100 Million each, (iii) the notional principal of the protected credit entities is equal for and their sum is equal to £1 Billion, (iv) and the quarterly premium payment of 70 basis point for this protection contract.

Upon the previous supposition, three cases will be studied: the first is when the number of defaults until maturity do not exceed four default events. The second is by assuming that \((4 - 6)^{th}\) to default credit entity events have occurred, respectively, in the 1st of September 2011, 1st of January 2012, and 1st February 2013 with no more default events. The last case is when \((4 - 15)^{th}\) to default credit entity events have occurred, respectively, in the 1st of May 2009, 1 of August 2009, 1st of November 2009, 1st of February 2010, 1st of September 2010, 1st of March 2011, 1st of April 2011, 1st of

Since the default events under the lower protection barrier does not have effects on the cash flow of either the periodic premium payments’ legs or the default payments’ legs, these events are not going to be considered or mentioned in this example. For example, in the case of no more than three defaults, the protection buyer will pay periodic premium payments of £700,000 at the 1st of every quarter until the end of the \( \left( \frac{20}{(4,14)} \right)^{th} \) to default CDS contract, i.e. 1st of January, 1st April, 1st July, and 1st of October of the years 2009 till 1st of January 2014, except for the year 2009 the 1st of January is excluded. At the end of the contract the protection buyer has paid £14 Million where the protection seller did not pay anything in return.

In contrast, in the second case the first three default events do not affect any payment stream. As a consequence of the occurrence of 4th to default event in the 1st of September 2011, (approximately) £(466,667/10) is paid as an accrued leg after ten periodic premium payments each equal to £700,000 by the protection buyer. In return the protection seller pays as a default leg an amount equals to \((1 - 0.4) \times 100\) Million, which is equal to £60 Million. As stated before, the premium payments are not equal, since they are influenced by the number of defaults and the ranking range. The 11th periodic payment will be discounted to cover only the in-between un-defaulted ranked credit entities, which is equal to £630,000. Even though the 5th to default event have occurred in the same date as the 12th periodic payment is prearranged, this periodic payment is not affected and will equal to £630,000 with no accrued payment required. But its effect is transferred to the subsequent payments. In return to the 5th to default event, the protection seller is obliged to pay a default leg equals to £60 Million. On account of the 6th to default event in the 1st of February 2013, (approximately)
£(186,667/9) is paid as an accrued leg subsequent to four periodic premium payments, from 1st of April 20012 till 1st of January 2013, each equal to £560,000 by the protection buyer. In return the protection seller pays as a default leg an amount equals to £60 Million. The \( \left( \frac{20}{14} \right)^{th} \) to default CDS contract will be concluded with four periodic payments each equals to £490,000 with no further payments by either parity. The sum of the premium legs paid in this case was £12,460,000 and £67,407 as accrued legs. In return the sum of the default payment legs are equal to £180 Million.

In the last case with an extreme assumption of 15 defaults in the basket, a summarised explanation will be articulated. The protection buyer is obliged to pay the periodic premium payment legs and the accrued payment legs in-between the prearranged dates in case of any default event conditional on it occurrence is in between the ranking barrier. The periodic payment legs, which are made in the predetermined dates, are equal, respectively, to £700,000, £700,000, £700,000, £700,000, £630,000, £630,000, £560,000, £560,000, £490,000, £420,000, £350,000, £280,000, £210,000, £140,000, £140,000, £70,000, £0, £0, and finally £0. In the last three payments it could be observed that as the number of defaults exceeds the upper ranking barrier, the protection buyer pays nothing as the periodic premium payments are concluded. However, the protection buyer is obligated to pay the accrued payment legs, which are made in 1st of May 2009, 1 of August 2009, 1st of November 2009, 1st of February 2010, 1st of September 2010, 1st of March 2011, 1st of April 2011, 1st of September 2011, 1st of January 2012, 1st of May 2012, 1st of September 2012, 1st of February 2013, 1st of July 2013, 1st of September 2013, 1st of December 2013 and each of them, respectively, equals to £(233,333/10), £(420,000/9), £(373,333/8), £(0/7), £(280,000/6), £(0/5), £(93,333/4), £(140,000/3), £(46,667/2), £(23,333/1), £0, £0, and finally £0. As in the periodic premium legs, the last three payments are equal to zero, as a consequence of
exceeding the upper ranking barrier and thus the termination of the protection payments. In return, at the in-between ranking defaulted credit entities’ dates, default payment legs are executed. Each of these payments is equal to £60 Million.

2.3.4 Collateralised Debt Obligation

In this subsection, the collateralised debt obligations (CDO) will be explained analogously to the basket default swaps and in its context, i.e. \( m^{th} \) to default CDS and \( \left( \left( R_{m}, R_{M} \right) \right)^{th} \) to default CDS, which were explained, respectively, in Subsection 2.3.2 and Subsection 2.3.3.

The (CDO)s mainly consists of three major parts; explicitly the issuer, the asset side, and the liability side, see Figure 2.4.

The issuer is a virtual entity that is responsible for linking the asset side by the liability side by issuing notes for individual transaction type, i.e. CDOs. The CDOs assets side could be built upon an individual or many of one or more types of reference entities, i.e. CDO tranches, CDO of CDOs, i.e. CDO-squared structures (CDO\(^2\)) and CDO\(^n\), Credit Default Swaps (CDS), Synthetic Collateralised Debt Obligations (SCDO), Collateralised Bond Obligations (CBOs), Mortgage-Backed Securities (MBS), Collateralised Loan Obligations (CLOs), etc. In opposite, the issuer issues, generally,
the liability side by tranching a corresponding three positions of the capital structure of the CDO, i.e. the Equity Tranche, the Mezzanine Tranche, and finally the Senior Tranche.

The CDOs produce more adaptable and flexible product than the basket default swaps products if its structure is observed exteriorly, i.e. the liability side, and complex if its structure is viewed interiorly, i.e. the assets side that represents the reference portfolio. In contrast, the cash flows in the CDOs are almost identical to the basket default swaps in the assets side, where default payments are due to any default event in return of periodic premium payments. On the contrary, the cash flow in liability side is quite complex as a consequence of its tranche structure that waterfalls its interests and repayments cash flow as bottom-up, tranche by tranche, while, oppositely, the loss waterfalls are structured top-down, tranche by tranche, in case of losses. Figure 2.5 exemplifies the losses distribution at a specific time that corresponds to the liability side and to the statistic order of defaults of the underlying reference portfolio, i.e. the asset side.

![Diagram](image-url)

**Figure 2.5:** Liability side’s losses distribution at a specific time that corresponds.
In the case of interests and repayments the senior tranche notes’ holders receives their portion firstly then the mezzanine tranche notes’ holders receives their segment secondly, and finally the equity tranche notes’ holders receives their fraction. This sequence influences the amount of payments they are receiving at each time since the losses are affecting those amounts of payments in the opposite direction. The equity tranche notes’ holders suffer the initial losses until the end of equity tranche capacity, then the mezzanine tranche notes’ holders tolerate the subsequent losses until the end of the mezzanine volume, and finally if any losses have exceeded the equity and then the mezzanine tranches, the senior tranche notes’ holders have to carry them.

The $CDO$’s could be seen in the same manner of the $\left( \mathcal{R}_{m,M} \right)^{th}$ to default $CDS$, where the payments, in the later, depends on the number of defaulted credit entities that are in between a ranking level, and those ranking level in the $CDO$’s corresponds to the tranches points.

In order to understand the $CDO$ mechanism and how its losses and returns are allocated, in the following, a simple cash $CDO$ example will be exemplified analogously to Figure 2.4. Assume that the notional principal of the referenced portfolio is equal of £1 Billion. The percentages of losses are assumed to be calculated after detecting the unrecovered defaulted credit entities. The referenced portfolio is tranched in three tranches, i.e. the first is the equity tranche, which covers 10% of the $CDO$ and ranged 0%-10% with a 40% return on the remaining fraction of the first 10% of the referenced portfolio. The equity tranche notes’ are equal to £100 Million. The Second is the mezzanine tranche, which protect 20% of the $CDO$ and ranged 10%-30% with a 15% return on the remaining fraction of the ranged tranche of the referenced portfolio. The mezzanine tranche notes’ are equal to £200 Million. The Last is the senior tranche, which insure the remaining 70% of the $CDO$ and ranged 30%-100% with an 8% return on the
remaining fraction of ranged tranche of the referenced portfolio. The mezzanine tranche notes’ are equal to £700 Million.

Upon the previous assumptions, four cases will be studied: the first, which is an extreme case, is when there are no losses at all. The second is when the total losses are equal to 4.5%, while the third when the total losses are equal to 15%. The last case, which is an extreme case also, is when the total loss is equal to 50%.

In the first case all tranches notes’ holders will not lose any amount of their total capital and get 100% of their promised return, to be precise the senior tranche notes’ holders get their return of 8% on their 70% firstly, which is equal to £56 Million, then secondly the mezzanine tranche notes’ holders get their return of 15% on their 20% of investment, which is equal to £30 Million. Finally, the equity tranche notes’ holders get their return of 40% from their 10% of investment, which is equal to £40 Million.

Accordingly, if the total losses are equal to 4.5%, the senior and the mezzanine tranches notes’ holders are not affected by this loss and will not lose any amount of their capital and get 100% of their promised return, to be precise the first will get their return of 8% from their 70% firstly, which is equal to £56 Million, will the second will get their return of 15% from their 20% of investment, which is equal to £30 Million. In contrast to them, the equity tranche notes’ holders will loss 45% of their total capital and will get 55% of their promised return, which is going to be equal to 22% of return instead of 40%. The remaining capital is equal to £55 Million and the return will equal to £12.1 Million.

In the third case, where the total loss is equal to 15%, the senior tranches notes’ holders are not affected by this loss and will not loss any amount of their capital and get 100% of their promised return of 8% on their 70% firstly. The mezzanine tranche notes’ holders will loss 25% of their total capital, which is equal to 5% of the whole CDO’s
Capital amount, and will get 75% of their promised return, which is going to be equal to 11.3% of return instead of 15%. The remaining capital is equal to £150 Million and the return will equal to £16.9 Million. In this situation the equity tranche notes’ holders will lose their whole capital and in sequence no return is paid back.

In the case of 50% loss of the total principal amount, the senior tranches notes’ holders will loss 28.6% of their total capital, which is equal to 20% of the whole CDO’s Capital amount, and will get 71.4% of their promised return, which is going to be equal to 5.7% of return instead of 8%. The remaining capital is equal to £500 Million and the return will equal to £28.6 Million. In this condition the equity and the mezzanine tranches notes’ holders will lose their whole capital and consecutively no return is paid back.

Figure 2.4 could demonstrate another CDO structure called the Synthetic CDOs, which is symbolised as SCDO. The SCDO s referenced portfolio is purely created from a number of CDS contracts. The SCDO issuer sells those it to third parties. As a consequence of any default event in the referenced portfolio, this referenced credit entity will pass to the SCDO’s tranche holders. Analogously to the case CDO in the previous example, instead of the direct losses in the capital and interests’ repayments, the cash flows are structured similarly to the CDS contracts. In other words, the notes holders are the protection seller and the third parties are the protection buyer. The equity tranche notes’ holders are responsible for the default legs payoffs of the CDSs until the notional principal reaches the capacity of the equity tranche, then the mezzanine tranche notes’ holders are liable for the default legs payoffs of the CDSs until the notional principal reaches the volume of the mezzanine tranche, and finally the senior tranche notes’ holders are accountable for the default legs payoffs of the residual CDSs’ notional principal. In return, each tranche notes’ holders are getting periodic premium legs that reflect the amount of risk they are responsible for.
To demonstrate the $\textit{SCDO}$ contract cash flow streams over its contracted period, Figure 2.3, as the $\left( \frac{n}{R_M} \right)^{th}$ to default $\textit{CDS}$, and Figure 2.4, as a cash $\textit{CDO}$, could be used analogously to understand the mechanism of the $\textit{SCDO}$. It could be seen, in the case of equity, mezzanine, senior tranches, as three $\left( \frac{n}{R_M} \right)^{th}$ to default $\textit{CDS}$ contracts, where the $R_m$ is the attachment of the tranche and the $R_M$ is its detachment point. Alternatively, $\textit{SCDO}$’s could be seen analogously to the cash $\textit{CDO}$ in Figure 2.4, where the equity tranche notes’ holders are responsible for the default legs until total losses reaches the 10% in return to 2500 basis points as periodic premium legs. When 10% of total losses is reached, the mezzanine tranche notes’ holders are liable for any extra losses until it reaches 30% of the total notional principal in return to 900 basis points as periodic premium legs. Finally, when total losses exceed the 30%, the senior tranche notes holders are accountable for the residual losses in return to 250 basis points as periodic premium legs.

Similarly to the interest repayments in the cash $\textit{CDO}$ and the periodic premium legs in the $\left( \frac{n}{R_M} \right)^{th}$ to default $\textit{CDS}$, the periodic premium legs in the $\textit{SCDO}$ are decreasing along with the increase of the number of defaults.

Furthermore, many alternative $\textit{CDO}$ structures are available in the market, where CDX and iTraxx indices are examples. CDX and iTraxx are a standardised $\textit{CDO}$ tranches those have launched to the market generated by an underlying portfolio.

Trading these standardised $\textit{CDO}$ tranches are known as single tranche $\textit{CDO}$, which is signified as $\textit{STCDO}$. A $\textit{STCDO}$ contract is an agreement that two parties agrees to enter a protection contract that one of them represents the protection seller against losses that affects that tranche and the other party corresponds to the protection buyer. Contrast to the $\textit{SCDO}$ tranches, where the referenced credit entities portfolio is tranched by selling a
CDS contracts, the STCDO are not part of the SCDO, which means that its two parties are not trading the actual credit entities that build up these indices and their tranches but they are trading the movements and actions that those indices are facing. STCDO cash flows are calculated in the same manner as SCDO are carried out.

CDX.NA.IG index is an example of the CDX family indices and it presents default protection contract on 125 of equally weighted North American investment-grade rated issuers. Its equity, junior mezzanine, senior mezzanine, senior, super senior, and second super senior tranches protects the losses, respectively, between 0%-3%, 3%-7%, 7%-10%, 10%-15%, 15%-30%, and finally 30%-100%. Oppositely, iTRAXX Europe index is a member of the iTRAXX family indices and it provides default protection contract on 125 of equally weighted European investment-grade rated issuers. Its equity, junior mezzanine, senior mezzanine, senior, super senior tranches, and second super senior tranches protects the losses, respectively, between 0%-3%, 3%-6%, 6%-9%, 9%-12%, 12%-22% and finally 22%-100%. Table 2.1 is a statistical summary of STCDO contracts from 14th November 2003 till 1st of February 2007 (Chan-Lau and Ong, 2007).

<table>
<thead>
<tr>
<th>Index</th>
<th>Tranche</th>
<th>Attachment-Detachment Points</th>
<th>Tranche Prices (In Basis Points)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Average</td>
</tr>
<tr>
<td>iTRAXX Europe</td>
<td>Equity*</td>
<td>0% - 3%</td>
<td>2,688</td>
</tr>
<tr>
<td></td>
<td>J.Mezzanine</td>
<td>3% - 6%</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>S.Mezzanine</td>
<td>6% - 9%</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Senior</td>
<td>9% - 12%</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>S.Senior</td>
<td>12% - 22%</td>
<td>15</td>
</tr>
<tr>
<td>CDX.NA.IG</td>
<td>Equity*</td>
<td>0% - 3%</td>
<td>3,971</td>
</tr>
<tr>
<td></td>
<td>J.Mezzanine</td>
<td>3% - 7%</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>S.Mezzanine</td>
<td>7% - 10%</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>Senior</td>
<td>10% - 15%</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>Super Senior</td>
<td>15% - 30%</td>
<td>10</td>
</tr>
</tbody>
</table>

* The Equity Tranche quotes are representing the upfront premium and 500 bps as periodic premium payments.

Table 2.1: statistical summary of STCDO contracts from 14/11/2003 till 1/2/2007 (Chan-Lau and Ong, 2007)
2.4 Modelling Defaults

2.4.1 Default Correlation

Exteriorly observing the cash flow of the CR derivatives products with numerous referenced credit entities those were mentioned previously, i.e. $m^{th}$ to default $CDS$, $(\binom{n}{m})^{th}$ to default $CDS$, $CDO$s, $SCDO$, and $STCDO$, gives an erroneous indication regarding the complexity of those products, where they are seen as simple products. However, when a deep look at those products are made, a fundamental question that requires a sophisticated answer could raise regarding the default times, to be precise with a number of credit entities those build up a referenced portfolio, how its entities are dependent with each other and as a consequence how their default times are related?

Answering this question needs some basic clarifications. companies which share some common circumstances, such as country, type of industry, market, material used in manufacturing, transportation methods, etc, have more possibility to be affected by the same external events, which, in sequence, may lead to an analogous financial difficulties at similar time. These common conditions, which lead two companies to default at around the same time, are called the default correlation\(^5\). Acknowledging the existence of default correlation among the credit entities, extinctions the complete diversification concept in credit risk products. In spite of that, the economical situations influence the default rate regardless of its similarity.

Back to answer the main question, the default correlation is an essential tool that joins each and every credit entity to each other and, as a consequence, calculating the probability distribution for the whole referenced portfolio could be achieved. The

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\(^5\) The correlation is a measure that corresponds to the amount of association between two or more variables.
default correlation could be modelled by either the structural model or the reduced form models.

In the case of structural model, each company defaults when its assets value is under a pre-specified barrier. The Default correlation, in this model, is achieved by correlating the stochastic processes that each credit entity admits, i.e. the stochastic process admitted by CreditEntity1 is correlated to the stochastic process admitted by CreditEntity2. However, in the reduced form the default correlation is accomplished by correlating the stochastic processes by the microeconomic variables that each company is influenced by, i.e. when CreditEntity1 and CreditEntity2 are influenced by the same microeconomic variable, such as oil price, the default correlation is high and therefore the default intensities of both credit entities are similar.

The main advantage of the structural model that the default correlation could be increased as much as needed, while in the reduced form the scope of default correlations that could be reached is limited, i.e. the default correlation is low even if their default intensities are perfectly correlated. This disadvantage could be solved by extending the model’s default intensities to incorporate with large jumps. Since the default correlation, in the reduced form model, reflects the microeconomic variables then economical cycle could exhibits its correlation structure. Nonetheless, the structural model is computationally slow, which is a significant disadvantage.

2.4.2 The Gaussian Copula Model for Time to Default

Gaussian copula is a reduced form model that has become one of the most important practical mechanisms to model the time to default. This method quantifies the default times correlations for each and every pairs of credit entities. Gaussian copula model presumes that all credit entities will default eventually. However, the possibility of default, generally, is required over the following 1, 5, 10 years.
To understand the structure of the Gaussian copula and how it does joins the times’ of default probabilities together, in the subsequence, a brief introduction to the copula will be illustrated to discover its overall concepts without getting deeply in mathematics. Subsequently, a general view of how it is implemented in order to compute the correlated time to default. For detailed mathematical implementation of copula in the context of credit derivatives products the reader could jump to Chapter 3 to 7.

Copulas’ function, denoted by $C(x, y)$, parameters are constructed from the marginal function of the joint distribution function, therefore, it must fulfil the joint distribution function properties. The first requirement is that, the output must be between 0 and 1, i.e. $0 \leq C(x, y) \leq 1$. Secondly, if one of the events, i.e. $x$ or $y$, has probability equals to zero, then the output of $C(x, y)$ must equal zero. The third opposites the second requirement, where if one of the events is surely occurring, to be exact $x$ or $y$, has probability equals to one, then the output of $C(x, y)$ must be the probability of the other argument. The fourth and the last requirement is that if the probabilities of both arguments are increasing, then the output of $C(x, y)$ must be increasing. With these four requirements, $C(x, y)$ is called a copula function.

The copula function could be constructed from or destructed to the joint distribution; this is achieved by the mean of Sklar’s Theorem. This theorem articulate that any copula function acquiring univariate probability distributions as arguments produces a joint distribution and that, on the contrary, any joint probability distribution could be rephrased in terms of a copula function acquiring the marginal distributions as arguments.

Accordingly, the copula function could join different types of marginal distributions, i.e. Gaussian, t-student, Gamma, etc., that shape each and every individual credit entity by a specified joint distribution, such as Gaussian, that structures the whole output...
appearance of this function. Structuring the marginal distributions could be achieved through transforming the time to default into new variables. This is an outstanding property that copula function provides. For example if the joint distribution of the $t_1$ and $t_2$ those represent the default time of CreditEntity1 and CreditEntity2 are normal, then its marginal distribution are normal. However, the opposite is not true, where if $t_1$ and $t_2$ are normally distributed, it does not imply that the joint distribution is normal. This drawback could be overcome by the copula function, transforming the marginal distribution and then joining them by the required joint distribution. For example, if the joint distribution of $t_1$ and $t_2$ is normally distributed, then to ensure the normality of $t_1$ and $t_2$ the Gaussian copula is introduced. Firstly, $t_1$ and $t_2$, respectively, are transformed to new normally distributed variables $x_1$ and $x_2$, i.e. $x_1 = \Phi^{-1}(F^Q(t_1))$ and $x_2 = \Phi^{-1}(F^Q(t_2))$, where $F^Q$ is the cumulative distribution function under the martingale measure. Note that the transformation is a percentile-to-percentile transformation, which means that the information needed to be captured is protected. After assembling $x_1$ and $x_2$, the joint probability distribution is assumed to be bivariate normal distribution. This method is called the Gaussian Copula and it has many significant advantages, beside what is mentioned previously, it improves the reduced form model in order to calculate the default time. One of them is that the Gaussian Copula could be extended to any number of credit entities and transformed and shaped in the same manner, and the other is that the joint distribution of default times are defined only by the individuals default time cumulative distribution functions and their correlation parameter.

With $\mathbb{Q}[\tau_1 < t_1, \tau_2 < t_2] = \Phi_{\rho_{12}}\left(\Phi^{-1}(F^Q(t_1)), \Phi^{-1}(F^Q(t_2))\right)$ as an example of the Gaussian Copula function, it could be summarised as following: with $\tau$ as any time
before the default time \( t \) or, in other words, what is the amount of time that each credit entity will take until default, \( Q[t_1 < \tau_1, \tau_2 < t_2] \) means what is the joint probability, or how likely, that both CreditEntity1 and CreditEntity2 will default. By means of the normally transformed default time distribution \( t \), i.e. \( \Phi^{-1}(F^Q(t)) \), and the default correlation \( \rho_{12} \), the Gaussian copula function, specifically \( \Phi_{\rho_{12}} \), associates the individual transformed default time probability distributions and return a single number that represents the probability that both CreditEntity1 and CreditEntity2 will default.

2.4.3 The Gaussian Factor Copula Model for Time to Default

A factor model is usually used in order to prevent the correlation ambiguity among the credit entities default times in the Gaussian Copula model. Furthermore, the factor model will establish a correlation structure between the universal risk factor, denoted by \( \mathcal{M} \), which represents the factors that may generate a credit default event across all referenced credit entities, and the idiosyncratic risk, denoted by \( \mathcal{J}_i \), which is a specific factor that affects a specific reference credit entity and may generate a credit default event. This factor model is mathematically represented as \( x_i = \rho_i \mathcal{M} + \sqrt{1 - \rho_i^2} \mathcal{J}_i \), and could be substituted conditionally on the universal factor, in the Gaussian Copula model, to give the probability of default time as \( p_{t|\mathcal{M}_t} = \Phi \left( \frac{\Phi^{-1}(F^Q(t)) - \rho_i \mathcal{M}}{\sqrt{1 - \rho_i^2}} \right) \) for each and every credit entity \( i \). The correlation between each pair \( x_i \) and \( x_j \) is equal to \( \rho_{ij} \).

This model is now the standard market model. It depends, as could be seen, on the linear correlation structure with deterministic parameters. However, this model has faced some problems, for instance in the case of \( CDO \) it cannot fit its tranches, where it underprices the equity and senior tranches and overprices the mezzanine tranche. As a consequence, this model has been extended to incorporate with skewed features by
modifying the distributions it admits. In parallel to this improvement, another route has been followed to improve the base model to incorporate a skewed correlation features instead of the linear ones have been lunched. This direction has been achieved by replacing the linear correlation by a stochastic correlation or stochastic risk exposure and, respectively, called “Stochastic Correlated Gaussian Factor Copula Model” and “Gaussian Random Factor Loading Copula Model”. Principally, the base model and its extended versions were based on the normality assumption, and if it went further, it will replace directly the normal distribution and therefore the Gaussian copula by an alternative distribution function.

Since the explanation of these skewed correlated models require some deep mathematical expression and illustration and to avoid the ambiguity while explaining these models, no more explanation is provided in this chapter. Interested reader can jump to chapter 5 for more modelling explanation.

2.4.4 The Lévy Factor Copula Model for Time to Default

Conversely to the normality assumption in the Gaussian process, the Lévy process explains and fit the financial market returns and its components, such as credit entities assets, and products, such as credit risk derivatives, in more accurate way; these components, assets’, and the products’ processes contain jumps or spikes and their empirical distributions contains fat tails and skewness those could not be captured by the Gaussian process, where the Lévy process could.

This thesis introduces the Lévy process, which supplies a suitable framework that sufficiently overcomes the Gaussian process drawbacks, in order to generalising the standard “Gaussian Factor Copula Model” through the Lévy processes, which is formally called “Lévy Factor Copula Model” and expand and homogenise the atomic approach of exploring this problem. The Lévy factor copula model presents an endless
number of alternatives distributions that admits the Lévy process definition and its properties and therefore could replace the Gaussian distribution, for instance Lévy skew alpha-stable distribution, normalized mixture Gaussian distribution, generalized hyperbolic distribution, a skewed t distribution, a variance gamma distribution, a normal inverse Gaussian distribution, etc. The only constrain is that the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ have to be infinitely divisible distributions with zero mean and equal finite variance.

Again, the Lévy Factor Copula Model will be extended to incorporate with the enhanced correlation skewed models mentioned previously, i.e. “Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”. For the same reasons mentioned in the previous subsection, this subsection will stop here.

2.5 Computation: The Fast and Very Fast forms of the Discrete Fourier Transform

Incorporating some advanced numerical methods those could reduce the dimensionality of the CR derivatives is the other aim CR derivatives modellers have in addition to Simplifying its nature. This former aim will be achieved by modelling the default characteristic function and then recovering its probability distribution by the Discrete Fourier Transform (DFT). Two forms of the DFT will be considered. The first is the Fast Fourier Transform (FFT), where the second is the proposed and recommended form; the Very Fast Fourier Transform (VFFT). If the loss distribution is needed, computing it will be considered in the same manner as the default distribution is carried out. For a comparison of this new technique with standard FFT techniques see Figure 2.6.
2.6 Conclusion

This chapter has overviewed the cash flow of some CR derivatives products, i.e. $m^{th}$ to default $CDS$, $(n \choose m)^{th}$ to default $CDS$, and $CDO$’s, intuitively and without any heavy mathematics. Subsequently, how the credit entities default times do depends and relate to each other. This part is explained through copula model, the factor copula model, and its skewed versions. The importance of rephrasing the Gaussian process by the Lévy process and the implementation of the $FFT$ and $VFFT$ has concluded this chapter.

After this overview, a reader with basic mathematical background, can easily dive through the rest of this thesis and comprehend the modelling of the credit derivatives and its requirements.

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"The solid black line is the $O(n^2)$ matrix algorithm known to Gauss in 1805. The dashed line underneath it is the famous Cooley-Tukey algorithm of 1965 with complexity $O(n \cdot \log(n))$. This algorithm is widely credited as "... the single breakthrough that made modern signal processing a practical proposition ...". All the other black lines are the results of various other algorithms and optimisations developed since 1965. The red line is the VFFT of 2003 with complexity $O(n)$. Remember the paradigm shift that Cooley-Tukey made possible. You are now looking at the next one" SIEFKE, S. J. 2003. The Very Fast Fourier Transform. Available: http://www.simonshepherd.supanet.com/vfft.htm [Accessed April 2007].
Chapter Four

Copula Function

3.1 Outline

- Introduction
- Basics and Joint Distribution
  - Basic Definitions
  - Joint and Marginal probabilities
- Copula Function
- Sklar’s Theorem
- Copula by Random Variables
- Frechet-Hoeffding Boundaries
- Copula’s Invariant Property
- Dependence Concept
  - Linear
  - Concordance
    - Concordance as Dependence Measure
    - Kendall’s tau
    - Spearman’s
  - Tail Dependence
- Summary
3.2 Introduction

Copulas’ Function parameters are constructed from the Marginal Functions of the Joint Distribution Function. As a consequence, its properties directly correspond to the properties of the Joint Distributions Function. The purpose of this chapter is to introduce an easier review of the theory of copula. The main part of this review will be utilised in Chapter 5 in the context of CDO and basket default swaps pricing Models.

Copula, which is a Latin noun that means “a link, tie, or bond” (Simpson, 1977), has first appear and introduced in (Sklar, 1959) and then translated in English by the same scholar in (Sklar, 1973). Significant developments were achieved, where copula was considered in the context of probabilistic metric spaces, in (Schweizer and Sklar, 1974), (Schweizer and Wolff, 1981), and (Schweizer and Sklar, 1983). An excellent mathematical survey of the copula’s function was presented in (Schweizer and Sklar, 1983). Nevertheless, in (Hoeffding, 1940) and (Hoeffding, 1941) Hoeffding has, independently, earlier pioneered the idea that copula depends on, by launching the best possible boundaries as well as investigating invariant dependence measures under strictly increasing transformations. Subsequently, considerable developments were carried in (Kimeldorf and Sampson, 1975), (Deheuvels, 1978), and (Deheuvels, 1979). For methodically mathematical modern literature of copula and its relationship to other works, (Joe, 1997) and (Nelsen, 2007) are excellent references; as the illustration of this chapter mainly depends on.

This chapter starts by outlining the joint distribution with some other basic Definitions those give an initial point to start building the first block “Copula Function and its properties”. Subsequently, The first block illustrates the forward-backward link between the joint and the copula functions in Section 3.4-3.6, the boundaries of copula in Section 3.7, the effect of copula being invariant in Section 3.8, and concluding with a number of
dependence concepts, i.e. linear correlation, Concordance: Kendall’s tau “τ” and Spearman’s rho “ρ”, and tail dependence in Section 3.9. In Section 3.10 a mathematical summary of the concepts needed in future chapters are noted.

3.3 Basics and Joint Distribution

In this subsection the basic and fundamental Definitions and terminologies needed to start building the blocks of this thesis, especially this chapter and more precisely Copulas’ Function, are represented. This section starts by outlining some basic and fundamental Definitions followed by some common Joint Distribution properties, which are utilised in the context of the Copula’s Function, in Section 3.3, where it is referenced to (Davenport, 1970), (Peyton and Peebles, 2001), and (Ross, 2007). Section 3.3 gives an entrance point to the first block “Copula Function” to be built.

3.3.1 Basic Definitions

Definition 3.1 (Random Variable)

A Real Random Variable or “Random Variable” is a function $X$ that is defined on a sample space $S = \text{Dom}(X)$, with an outcome probability element $x$ that corresponds to a sample range $s$, such that:

$$X(s) = x, \forall s \in S^7$$

Definition 3.2 (Random Vector)

A Random Vector is a finite-dimensional vector-valued function of random variables, where both, the random vector and its corresponding random variables, are defined on a sample space $S$.

$$X(s) = (x_1, x_2, \cdots, x_n)$$

$$= (X_1(s), X_2(s), \cdots, X_n(s))^8 \quad \forall s \in S$$

---

7 For short $X(s)$ will be represented as $X$.
8 For short $X_i(s)$ is represented as $X_i$
Definition 3.3 (Grounded Function)

A function \( F_{X,Y} \subseteq \text{Dom}(X \times Y) \) is Grounded if it has a least element \( x_{\text{least}} \in \text{Dom}(X) \) and \( y_{\text{least}} \in \text{Dom}(Y) \) such that

\[
F_{X,Y}(x_{\text{least}}, y) = F_{X,Y}(x, y_{\text{least}}) = 0
\]

Definition 3.4 (F-Volume Function)

A \( F_{X,Y} \) - Volume of \( Z \), where \( Z \) is any \([x_1, x_2] \times [y_1, y_2] \in \text{Dom}(X \times Y)\), denoted by \( V_{F_{X,Y}}(Z) \), is given by

\[
V_{F_{X,Y}}(Z) = F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)
\]

which will be denoted by the 2-order difference as

\[
V_{F_{X,Y}}(Z) = \Delta_{x_1}^{y_2} \Delta_{x_2}^{y_1} F_{X,Y}(x, y)
\]

Definition 3.5 (2-Increasing Function)

A function \( F_{X,Y} \) is said to be 2-Increasing if \( V_{F_{X,Y}}(Z) \geq 0 \), for all \( Z \in \text{Dom}(X \times Y) \).

Lemma 3.1 (Nondecreasing Function)

Let \( x_1, x_2 \in X, x_1 \leq x_2, y_1, y_2 \in Y \), and \( y_1 \leq y_2 \). Let \( F_{X,Y} \) be a 2-increasing function, where \( F_{X,Y} \in \text{Dom}(X \times Y) \). Then \( x \mapsto F_{X,Y}(x, y_2) - F_{X,Y}(x, y_1) \) is Nondecreasing on \( X \), and \( y \mapsto F_{X,Y}(x_2, y) - F_{X,Y}(x_1, y) \) is Nondecreasing on \( Y \).

Corollary 3.1 (\( F_{X,Y} \) as Nondecreasing Function in both Dimensions)

Let \( X \) and \( Y \) be nonempty spaces, and \( F_{X,Y} \) be a grounded 2-increasing function, where \( F_{X,Y} \in \text{Dom}(X \times Y) \). If \( x_1 = x_{\text{least}} \) and \( y_1 = y_{\text{least}} \) are substituted in Lemma 3.1, then \( F_{X,Y} \) become Nondecreasing function in both dimensions.

### 3.3.2 Joint and Marginal probabilities

This subsection describes the joint and marginal distributions and their densities, the continuity concept, right differentiable function, and finally by the Definitions of the quintile and inverse function.
Definition 3.6 (Distribution Function)

A Cumulative Probability Distribution Function or a “Distribution Function” of a random variable $X$, denoted by $F_X(x)$, is defined as $F_X(x) = P\{X \leq x\}, x \in \mathbb{R}$ such that

1. $F_X(-\infty) = 0$
2. $F_X(+\infty) = 1$
3. $0 \leq F_X(x) \leq 1$, $F_X$ is normalised.
4. $F_X$ is nondecreasing function.
5. $F_X(x)$ is continuous from the right.

Lemma 3.2 (Continuity on the Right)

A distribution function $F_X$ with a random variable $X$ is Continuous from the Right if

$$F_X(x + 0) = F_X(x), \forall x \in \mathbb{R}$$

Proof:

If $F_X(x + 0) - F_X(x) = 0$, then it is continuous on the right. Let $\varepsilon_i$ be a monotone-decreasing sequence, where $\varepsilon_{i+1} \leq \varepsilon_i$ and $\lim_{i \to +\infty} \varepsilon_i = 0$, then $F_X(x + 0) = \lim_{i \to +\infty} F_X(x + \varepsilon_i)$. It follows that

$$F_X(x + 0) - F_X(x) = \lim_{i \to +\infty} F_X(x + \varepsilon_i) - F_X(x)$$
$$= \lim_{i \to +\infty} [F_X(x + \varepsilon_i) - F_X(x)]$$
$$= \lim_{i \to +\infty} P\{X \in (x, x + \varepsilon_i]\}$$
$$= P\left[ \lim_{i \to +\infty} (X \in (x, x + \varepsilon_i]) \right]$$
$$= P\left( \bigcap_{i=1}^{\infty} (x, x + \varepsilon_i] \right)$$
$$= P[\emptyset]$$
$$= 0$$

In the literature you may find that 0 is denoted by $0^+$ and the same for the lift continues as $0^-$. 
**Lemma 3.3 (Continuity on the Left)**

A distribution function $F_X$ with a random variable $X$ is Continuous from the Left if

$$F_X(x - 0) + P[X = x] = F_X(x), \forall x \in \mathbb{R}$$

**Proof:**

Proof of Lemma 3.2 could be paraphrased to prove the fact that the $F_X$ is continuous from the left if $P[X = x] = 0$

**Theorem 3.1 (Continuity)**

A distribution $F_X$ with a random variable $X$ is Continuous if

$$F_X(x - 0) = F_X(x) = F_X(x + 0) \quad \forall x \in \mathbb{R}$$

**Definition 3.7 (Density Function)**

A probability density function or a “density function” of a random variable $X$, denoted by $f_X(x)$, is defined as $f_X(x) = \frac{dF_X(x)}{dx}$, $x \in \mathbb{R}$ such that

1. $f_X(x) \geq 0, \forall x$
2. $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(\xi) \, d\xi$
4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x) \, dx$

**Lemma 3.4 (Right Differentiable)**

Let $f_X$ be right continuous density function, then the distribution function $F_X$ is right differentiable.

**Proof:**

Using Lemma 3.2 and Theorem 3.1, then $\forall i > 0, \exists \varepsilon_i > 0$ with $u \in [x, x + \varepsilon_i]$ where $|f_X(u) - f_X(x)| < i$. 

10 In the points where it exist.
Let \( 0 < \varepsilon < \varepsilon_i \), \( \forall u \in [x, x + \varepsilon[ \), we have \( f_X(x) - i \leq f_X(u) \leq f_X(x) + i \), which could be written as

\[
f_X(x) - i \leq \frac{\int_x^{x+\varepsilon} f_X(u) \, du}{\varepsilon} \leq f_X(x) + i.
\]

The right equality states that

\[
\lim_{\varepsilon \to 0} \left( \frac{\int_x^{x+\varepsilon} f_X(u) \, du}{x + \varepsilon - x} \right) = f_X(x),
\]

and thus \( F_X(x) \) is right differentiable and the related derivative is \( f_X(x) \).

**Definition 3.8 (Joint Distribution)**

A Joint Probability Distribution Function or a “Joint Distribution Function” is a function of random variables \( X \) and \( Y \) given by \( F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\} \), where \( F_{X,Y} \in \mathbb{R}^2 \) such that

1. \( F_{X,Y} \) is grounded.
2. \( F_{X,Y}(\infty, \infty) = 1 \)
3. \( 0 \leq F_{X,Y}(x,y) \leq 1 \)
4. \( F_{X,Y}(x,y) \) is 2-increasing and then nondecreasing in both argument\(^{11}\).
5. \( F_{X,Y}(x, \infty) = F_X(x) \) and \( F_{X,Y}(\infty,y) = F_Y(y) \), called marginal distribution.

**Definition 3.9 (Marginal Distribution)**

Let \( F_{X,Y} \) be a joint distribution function that corresponds to the random variables \( X \) and \( Y \), then a Marginal Distribution Function of a random variable \( X \) could be obtained as in Definition 3.8, property 5. Or equivalently, it could be calculated by integrating the joint distribution over \( Y \), i.e. \( F_X(x) = \int_{-\infty}^{+\infty} F_{X,Y}(x,y) \, dy \).

\(^{11}\) See Lemma 3.1
Definition 3.10 (Joint Density)

A Joint Probability Density Function or a “Joint Density Function” is function of a random variables X and Y given by

\[ f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}, \]

such that

1. \( f_{X,Y}(x, y) \geq 0 \)
2. \( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \, dx \, dy = 1 \)
3. \( F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(\xi_1, \xi_2) \, d\xi_2 \, d\xi_1 \)
4. \( F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1, \xi_2) \, d\xi_2 \, d\xi_1 \) and \( F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 \)
5. \( P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{X,Y}(x, y) \, dx \, dy \)
6. \( f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \, dy \) and \( f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \, dx \)

Definition 3.11 (Marginal Density)

Let \( f_{X,Y}(x, y) \) be joint density function that corresponds to the random variables X and Y, then a Marginal Probability Density Function or a “Marginal Density Function” of a random variable X could be given from Definition 3.10 property 6, or equivalently by differentiating the X marginal distribution function over X, i.e \( f_X(x) = \frac{dF_X(x)}{dx} \).

Lemma 3.5 (Independent Random Variable)

X and Y are said to be continuous independent random variables with, respectively, marginal distribution functions \( F_X \) and \( F_Y \), if their joint distribution function \( F_{X,Y} \) is defined as

\[ F_{X,Y}(x,y) = F_X(x) \times F_Y(y) \]

Definition 3.12 (Percentile)

Let \( F_X \) be a distribution function of a random variable X, then \( u \) percentile is the smallest number range of that function, such that
u = \inf \{ u: F_X(x) \geq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}
= \sup \{ u: F_X(x) \leq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}\text{12}

**Definition 3.13 (Inverse Quasi)**

Let \( F_X \) be a distribution function of a random variable \( X \), then a quasi-inverse of the percentile \( u \), denoted by \( F_X^{-1}(u), u \in \mathbb{I} \), is given by

\[
F_X^{-1}(u) = \inf \{ x: F_X(x) \geq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}
= \sup \{ x: F_X(x) \leq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}
\]

**Lemma 3.6 (Inverse Function)**

Let \( F_X \) be strictly increasing distribution function of a random variable \( X \), then it has only a single quasi-inverse of the percentile \( u \), which is an ordinary inverse and denoted by \( F_X^{-1} \), such that

\[
F_X^{-1}(u) = \inf \{ x: F_X(x) \geq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}
= \sup \{ x: F_X(x) \leq u, x \in \mathbb{R}, \text{and } u \in \mathbb{I} \}
\]

### 3.4 Copula Function

Defining the Copula Function with a brief rationalisation by the mean of the basic Definitions is the core of this subsection. This subsection along with the “Sklar’s Theorem” and “Copula by Random variables” subsections are the core of first block.

**Definition 3.14 (Copula Function)**

A two dimensional Copula is a real function of \( X \) and \( Y \) on \( \mathbb{R} \), denoted by \( C_{X,Y} \), defined from \( \mathbb{I}^2 \) to \( \mathbb{I} \), such that

1. \( C_{X,Y} \) is grounded: \( C_{X,Y}(u, 0) = C_{X,Y}(0, v) = 0 \) \( \forall u, v \in \mathbb{I} \)
2. \( C_{X,Y}(u, 1) = u \) and \( C_{X,Y}(1, v) = v \) \( \forall u, v \in \mathbb{I} \)
3. \( C_{X,Y} \) is 2-incresing.

---

12 \( x \) could belong to a subset \( \mathbb{R} \), i.e. \( x \in \text{Dom}(X) \subseteq \mathbb{R} \).
The first property of the Definition ensures that the copula function is grounded from $\mathbb{I}^2$ to $\mathbb{I}$. Where the second property gives the way to construct the margins as if $C_{X,Y}$ is a joint distribution; this property is leads to the most significant Theorem, sklar’s Theorem, in the copula theory, which will be discussed in the next section. The third property could be examined by ensuring that the $C_{X,Y}$-volume is not but a positive number, such that $V_{C_{X,Y}}(Z) \geq 0^{13}$, As a consequence of this property Lemma 3.1 and Corollary 3.1 has stated that the $C_{X,Y}$ has to be a nondecreasing in both argument. Since the copula $C_{X,Y} \in \mathbb{I}^2$, the $C_{X,Y}$-volume could be observed as a volume of $\mathbb{I}^2$. As a consequence the appearance of the copula $C_{X,Y}$ is the shape of a skewed continuous surface on $\mathbb{I}^2$, where its $Z \in \mathbb{I}^2$ also.

The vital Theorem in the copula theory is stated in the subsequent section.

### 3.5 Sklar’s Theorem

The Theorem of this subsection, which was firstly appeared in (Sklar, 1959) and called now the Sklar’s Theorem, is foremost to the theory of copulas as it could be seen as the foundation theory. It has several applications in statistics and mathematical sciences. The copula could explain the relationship between the multivariate distribution and its univariate margins through the Sklar’s Theorem.

**Theorem 3.2 (Sklar’s Theorem)**

Let $F_{X,Y}$ be a bivariate joint distribution of $X$ and $Y$ with, respectively, margins $F_X$ and $F_Y$ and $C_{X,Y}$ be a copula function that admits Definition 3.14. Then there exist a copula function, $C_{X,Y}$, such that

$$F_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}$$

---

13 See the Definition 3.4 (F-Volume).
Moreover, the uniqueness of the copula function \( C_{X,Y} \) is conditional on the continuity of the margins \( F_X \) and \( F_Y \). In contrast, the nonexistence of this condition reduces the uniqueness on the \( \text{Ran}(F_X) \times \text{Ran}(F_Y) \). On the contrary, if \( C_{X,Y} \) is a copula function and \( F_X \) and \( F_Y \) are distribution functions, then \( F_{X,Y} \), which is identified as in the preceding expression, qualified to be a joint distribution with the margins \( F_X \) and \( F_Y \).

This inventiveness of this Theorem appears from the statement that the copula associates the decomposition of the marginal distributions with their joint distribution.

Extending the former Theorem by Lemma 3.6 is an immediate application, as it could be seen in the subsequent Corollary.

**Corollary 3.2 (Inversion Method-Copula)**

Let \( F_{X,Y} \) be a bivariate joint distribution of \( X \) and \( Y \) with continuous margins \( F_X \) and \( F_Y \) and \( C_{X,Y} \) be a copula function that admits Definition 3.14. Then

\[
C_{X,Y}(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \quad \forall u, v \in \mathbb{I}
\]

The preceding consequence of the copula function \( C_{X,Y} \) presents a procedure to construct the copula functions from its joint distribution function. This Corollary will be useful to present many corollaries in the next section.

Developing the copula function \( C_{X,Y} \) by the mean \( X \) and \( Y \) as random variables will be obtained in the subsequent section.

**3.6 Copula by Random Variables**

Discussing the random variable in this subsection follows Definition 3.2 of the random vector on a common probability space, where their values are described by a joint distribution function defined in Definition 3.8, and linking it to the copula function is the subject of this subsection. The copula function, more specifically the copula function defined in Sklar’s Theorem, could be rephrased in terms of their joint distribution functions of random variables.
Theorem 3.3 (Copula via Random Variables)

Let $F_{XY}$ be a bivariate joint distribution of random variables $X$ and $Y$ with, respectively, marginal distribution functions $F_X$ and $F_Y$, and $C_{XY}$ be a copula function that admits Definition 3.14 and follows Theorem 3.2. Then there exist a copula function, $C_{XY}$, such that

$$P\{X \leq x, Y \leq y\} = C_{XY}(P\{X \leq x\}, P\{Y \leq y\})$$

$$F_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}$$

Additionally, the uniqueness of the copula function $C_{XY}$ is conditional on the continuity of the marginal distribution functions $F_X$ and $F_Y$. In contrast, the nonexistence of this condition reduces the uniqueness on the $\text{Ran}(F_X) \times \text{Ran}(F_Y)$. On the contrary, if $C_{XY}$ is a copula function and $F_X$ and $F_Y$ are distribution functions, then $F_{XY}$, which is identified as in the preceding expression, qualified to be a joint distribution with the margins $F_X$ and $F_Y$.

3.7 Frechet-Hoeffding Boundaries

It is significant now to remark the consequence of how $Y$ is related to $X$ as a random variables, i.e. if $Y$ is an independent, decreasing, or increasing function of $X$. This question introduces the current subsection through Frechet-Hoeffding Theorem, which has been developed independently by Hoeffding’s gathered work translated in (Fisher N. I. and Sen P. K., 1994) and (Frechet, 1951).

Theorem 3.4 Frechet-Hoeffding

Let $X$ and $Y$ be random variables with, respectively, marginal distributions $F_X$ and $F_Y$ and their bivariate joint distribution $F_{XY}$. Then $F_{XY}(x, y)$ is bounded by the subsequent inequality:

$$\max(F_X(x) + F_Y(y) - 1, 0) \geq F_{XY}(x, y) \geq \min(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}$$
It seems comprehensible that the Frechet-Hoeffding Theorem presents the joint distribution function’s upper and lowers bounds, which are defined in the subsequent Definitions.

**Definition 3.15 (Upper Bound)**

Let $X$ and $Y$ be random variables with, respectively, marginal distributions $F_X$ and $F_Y$ and their bivariate joint distribution $F_{X,Y}$. Then the upper bound of the Frechet-Hoeffding Theorem is defined as

$$M(F_X(x), F_Y(y)) = \min(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}$$

**Definition 3.16 (Lower Bound)**

Let $X$ and $Y$ be random variables with, respectively, marginal distributions $F_X$ and $F_Y$ and their bivariate joint distribution $F_{X,Y}$. Then the lower bound of the Frechet-Hoeffding Theorem is defined as

$$W(F_X(x), F_Y(y)) = \max(F_X(x) + F_Y(y) - 1, 0) \quad \forall x, y \in \mathbb{R}$$

Subsequent to this brief introduction of the Frechet-Hoeffding Theorem, the question that has been remarked in the introduction of this section regarding how $Y$ is related to $X$ as a random variables, could be explained in the three subsequent Lemmas.

In the case of independence of $X$ and $Y$ the subsequent Lemma could be stated.

**Lemma 3.7 (Independent X, Y)**

Let $X$ and $Y$ are said to be continuous independent random variables with, respectively, marginal distribution functions $F_X$ and $F_Y$, then their joint distribution function $F_{X,Y}$ is defined as

$$F_{X,Y}(x,y) = F_X(x) \times F_Y(y) \quad \forall x, y \in \mathbb{R}$$

An immediate consequence could be acknowledged on the copula function, when $X$ and $Y$ are independent, in the subsequent Corollary.
Corollary 3.3 (Product Copula)

Let \( C_{X,Y} \) be a copula function that follows Theorem 3.3 and \( X \) and \( Y \) be independent random variables, where their joint distribution function \( F_{X,Y} \) follows Lemma 3.7, then \( C_{X,Y} \) is called the product copula, denoted by \( C_\Pi \), and given by

\[
C_\Pi(u, v) = C_{X,Y}(F_X(x), F_Y(y)) = uv \quad \forall x, y \in \mathbb{R}, \forall u, v \in \mathbb{I}
\]

The second case is when the \( Y \) is Monotone-Increasing of \( X \). To introduce this Lemma, it is important to define this terminology first.

Definition 3.17 (Monotone-Increasing)

Let \( x_1 \) and \( x_2 \), where \( x_1 > x_2 \), be any two elements of the random variable \( X \) with a corresponding Function \( F_X \). Then \( F_X \) is said to be Monotone-Increasing if and only if

\[
F_X(x_1) > F_X(x_2).
\]

Lemma 3.8 (Monotone-Increasing \( X, Y \))

Let \( X \) and \( Y \) be random variables with, respectively, marginal distributions \( F_X \) and \( F_Y \) and their bivariate joint distribution \( F_{X,Y} \). If \( Y \) is an monotone-increasing function of \( X \) that admits Definition 3.17, then \( F_{X,Y} \) is equal to its Frechet-Hoeffding upper bound, such that

\[
F_{X,Y}(x, y) = M(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}
\]

A direct result could be accepted on the copula function, when \( X \) and \( Y \) are monotone-increasing, in the next Corollary.

Corollary 3.4 (Copula’s Upper Bound)

Let \( C_{X,Y} \) be a copula function that follows Theorem 3.3 and \( X \) and \( Y \) be random variables, where \( Y \) is an monotone-increasing function of \( X \), with, respectively, marginal distributions \( F_X \) and \( F_Y \) and their bivariate joint distribution \( F_{X,Y} \) admits Lemma 3.8, then the Frechet-Hoeffding upper bound is given by:
The third and the last case that is required to be explained in this section is when $Y$ is Monotone-Decreasing of $X$. To initiate this Lemma, it is significant to define this term first.

**Definition 3.18 (Monotone-Decreasing)**

Let $x_1$ and $x_2$, where $x_1 < x_2$, be any two elements of the random variable $X$ with a corresponding Function $F_X$. Then $F_X$ is said to be Monotone-Decreasing if and only if $F_X(x_1) < F_X(x_2)$.

With Definition 3.18 in attention the Lemma of $Y$ being Monotone-Decreasing of $X$ could be declared.

**Lemma 3.9 (Monotone-Decreasing $X, Y$)**

Let $X$ and $Y$ be random variables with, respectively, marginal distributions $F_X$ and $F_Y$ and their bivariate joint distribution $F_{X,Y}$. If $Y$ is a monotone-decreasing function of $X$ and admits Definition 3.18, then $F_{X,Y}$ is equal to its Frechet-Hoeffding Lower bound, such that

$$F_{X,Y}(x, y) = W(F_X(x), F_Y(y)) \quad \forall x, y \in \mathbb{R}$$

Lemma 3.9 could be applied on the copula function to obtain an immediate outcome, which is stated in the subsequent Corollary.

**Corollary 3.5 (Copula’s Lower Bound)**

Let $C_{X,Y}$ be a copula function that follows Theorem 3.3 and $X$ and $Y$ be random variables, where $Y$ is a monotone-decreasing function of $X$, with, respectively, marginal distributions $F_X$ and $F_Y$ and their bivariate joint distribution $F_{X,Y}$ admits Lemma 3.9, then the Frechet-Hoeffding lower bound is given by the next equality:

$$W(u, v) = W(F_X(x), F_Y(y)) = \max(u + v - 1, 0) \quad \forall x, y \in \mathbb{R}, \forall u, v \in [0,1]$$
Now the Frechet-Hoeffding Theorem could be rephrased in term of copula function, and be proven by mixing the Frechet-Hoeffding Theorem in term of the joint probability distribution function and the copula’s Definition.

**Theorem 3.5 (Copula’s Frechet-Hoeffding Theorem)**

Let $C_{X,Y}$ be a copula function of a uniform random variables $u$ and $v$ that admits Theorem 3.3, $M$ be copula’s lower upper bonds that admits Corollary 3.4, and $W$ be copula’s lower and upper bonds that admits Corollary 3.5. Then $C_{X,Y}$ is bounded by $M$ and $W$, such that:

$$W(u + v - 1, 0) \geq C(u, v) \geq M(u, v) \quad \forall u, v \in I$$

**Proof:**

This Theorem will be proved in two components

1. **Right hand side:** Since

   $$C(u, v) \leq C(1, v) = v \quad \forall u, v \in I$$

   $$C(u, v) \leq C(u, 1) = u \quad \forall u, v \in I$$

   then $C(u, v) \geq \min(u, v) = M(u, v)$ is an immediate result.

2. **Left hand side:** Since

   $$V_C([u,1] \times [v,1]) \geq 0$$

   $$C(u, v) = V_C([u,1] \times [v,1]) + u + v - 1 \quad \forall u, v \in I$$

   then it implies that $C(u, v) \geq \max(u + v - 1, 0)$.

**3.8 Copula’s Invariant Property**

The significance of copula function rises from the fact that it encapsulates Sklar’s Theorem and copula’s invariant property$^{14}$; that the copula is invariant, while its margins could be modified if required, under strictly monotone-increasing transformation of $X$ and $Y$ (Schweizer and Wolff, 1981).

---

In this subsection, this property and some other monotone transformations will be introduced.

**Lemma 3.10 (Strictly Monotone-Increasing)**

Let $F_X$ and $F_{\alpha(X)}$ be, respectively, distribution functions of $X$ and $\alpha(X)$, and $\alpha$ be a strictly monotone-increasing function. Then $F_{\alpha(X)}(x) = F_X(\alpha^{-1}(x))$.

**Proof:**

Since $\alpha$ is a strictly monotone-increasing function, then

$$F_{\alpha(X)}(x) = P\{\alpha(X) \leq x\}$$

$$= P\{X \leq \alpha^{-1}(x)\}$$

$$= F_X(\alpha^{-1}(x))$$

The previous Lemma has given the required background to imply monotone-increasing transformation on the copula function, which is expressed in the next Theorem.

**Theorem 3.6 (Copula’s Strictly Monotone-Increasing Transformation)**

Let $C_{XY}$ be a copula function of continuous random variable $X$ and $Y$ that admits Theorem 3.3, $\alpha$ and $\beta$ be strictly monotone-increasing functions on $\text{Ran}(F_X) \times \text{Ran}(F_Y)$. Then $C_{\alpha(X),\beta(Y)} = C_{XY}$. Consequently, $C_{XY}$ is invariant under strictly monotone-increasing transformation of $X$ and $Y$.

**Proof:**

Assuming $F_X, F_Y, F_{\alpha(X)}$, and $F_{\beta(Y)}$ are distribution functions of, respectively, $X, Y, \alpha(X)$ and $\beta(Y)$. And using the result in Lemma 3.10, the transformation of the copula is given by

$$C_{\alpha(X),\beta(Y)}(F_{\alpha(X)}(x), F_{\beta(Y)}(y)) = C_{XY}(F_X(\alpha^{-1}(x)), F_Y(\beta^{-1}(y)))$$

$$= C_{XY}(F_{\alpha(X)}(x), F_{\beta(Y)}(y)) \quad \forall x, y \in \mathbb{R}$$

In view of the fact that $X$ and $Y$ are continuous, $\text{Ran}(F_X) = \text{Ran}(F_Y) = I$, combing it consequences $C_{\alpha(X),\beta(Y)} = C_{XY}$ on $\mathbb{R}^2$. 
The strictly monotone-increasing transformation condition of the previous Theorem could be relaxed to cover other monotone transformations. The subsequent three Lemmas could be clarified regarding the behaviour of the copula function.

**Definition 3.19 (Relative Complement)**

Let $X$ and $Y$ be continuous random variables, then the subsequent equality hold:

$$P\{X \leq x, Y \geq y\} = P\{X \leq x\} - P\{X \leq x, Y \leq y\}$$

**Lemma 3.11 (Strictly Monotone-Decreasing)**

Let $F_X$ and $F_{\alpha(X)}$ be, respectively, distribution functions of $X$ and $\alpha(X)$, and $\alpha$ be a strictly monotone-decreasing function. Then $F_X(\alpha^{-1}(x)) = 1 - F_{\alpha(X)}(x)$.

**Proof**

Similar to proof Lemma 3.10

**Lemma 3.12 ($\alpha$ strictly Monotone-Increasing, $\beta$ strictly Monotone-Decreasing)**

Let $C_{XY}$ be a copula function of continuous random variable $X$ and $Y$ that admits Theorem 3.3, $\alpha$ be strictly monotone-increasing function, and $\beta$ be strictly monotone-decreasing function on $\text{Ran}(F_X) \times \text{Ran}(F_Y)$, then the subsequent equality is valid:

$$C_{\alpha(X),\beta(Y)}(u,v) = u - C_{XY}(u,1-v)$$

**Proof:**

Assuming $F_X, F_Y, F_{\alpha(X)}$, and $F_{\beta(Y)}$ are distribution functions of $X,Y,\alpha(X)$ and $\beta(Y)$.

Using the fact $\alpha$ is strictly monotone-increasing function, $\beta$ is strictly monotone-decreasing function, and utilising Lemma 3.8, Lemma 3.9, Lemma 3.10, Lemma 3.11, Theorem 3.6, and Definition 3.19, the transformation of the copula is given by the subseqant chain of equality:

$$C_{\alpha(X),\beta(Y)}(F_{\alpha(X)}(x),F_{\beta(Y)}(y)) = C_{\alpha(X)}(F_X(\alpha^{-1}(x))) - C_{XY}(F_X(\alpha^{-1}(x)),F_Y(\beta^{-1}(y)))$$

$$= C_{\alpha(X)}(F_{\alpha(X)}(x)) - C_{XY}(F_{\alpha(X)}(x),1 - F_{\beta(Y)}(y))$$

$$= u - C_{XY}(u,1-v) \quad \forall x,y \in \mathbb{R}$$
Given that $X$ and $Y$ are continuous, $\text{Ran}(F_X) = \text{Ran}(F_Y) = \mathbb{I}$, combing it consequent the last result. on $\mathbb{I}^2$.

**Lemma 3.13 (α strictly Monotone-Decreasing, β strictly Monotone-Increasing)**

Let $C_{XY}$ be a copula function of continuous random variable $X$ and $Y$ that admits Theorem 3.3, $α$ be a strictly monotone-decreasing function, and $β$ be a strictly monotone-increasing function on $\text{Ran}(F_X) \times \text{Ran}(F_Y)$, then the subsequent equality hold:

$$C_{α(X),β(Y)}(u,v) = v - C_{XY}(1-u,v)$$

**Proof**

Proof of Lemma 3.12, could be rephrased to obtain the proof of this Lemma.

**Definition 3.20**

Let $X$ and $Y$ be continuous random variables, then the subsequent equality hold:

$$P\{X \geq x, Y \geq y\} = 1 - P\{X \leq x, Y \leq y\}$$

$$= 1 - P\{X \leq x\} - P\{Y \leq y\} + P\{X \leq x, Y \leq y\}$$

**Lemma 3.14 (α, β strictly Monotone-Decreasing)**

Let $C_{XY}$ be a copula function of continuous random variable $X$ and $Y$ that admits Theorem 3.3, and $α$ and $β$ be strictly monotone-decreasing function on $\text{Ran}(F_X) \times \text{Ran}(F_Y)$ then the subsequent equality hold:

$$C_{α(X),β(Y)}(u,v) = u + v - 1 + C_{XY}(1-u,1-v)$$

**Proof:**

Assuming $F_X,F_Y,F_{α(X)},$ and $F_{β(Y)}$ are distribution functions of $X,Y,α(X)$ and $β(Y)$.

And using the results in Lemma 3.11, Lemma 3.12, Lemma 3.13, and Theorem 3.6 and utilising Definition 3.19, the transformation of the copula is given by
\[ C_{\alpha(x), \beta(y)} \left( F_{\alpha(x)}(x), F_{\beta(y)}(y) \right) = 1 - C_{\alpha(x)} \left( F_X(\alpha^{-1}(x)) \right) - C_{\beta(y)} \left( F_Y(\beta^{-1}(y)) \right) \\
= 1 - (1 - \alpha) - (1 - \beta) + C_{X,Y} (1 - \alpha, 1 - \beta) \\
= \alpha + \beta - 1 + C_{X,Y} (1 - \alpha, 1 - \beta) \]

### 3.9 Dependence Concept

Dependence measure between random variables is the concept that fills the gap that copula invariant property under strictly monotone function omitted; in view of the fact that the later property does not apply multivariate elliptical distribution for instance the Gaussian. Furthermore, (Embrechts et al., 2003) had remark that linear correlation has been proven to be misleading measure of dependence in some cases. For additional argument regarding the dependence measure between random variables in copula function (Schweizer and Wolff, 1981) is an appropriate reference.

The more interesting measures will be the ones which can be solely defined in term of copula.

This subsection is introduced by the linear correlation, since it is the base and the one that will be employed to model the credit derivatives. Subsequently, the Definition of the concordance will be noted; in order to introduce the Kendall’s tau, Spearman’s rho, and the relationship between them. This subsection will be concluded with the tail dependence as measure of association.

#### 3.9.1 Linear

**Definition 3.21 (Linear Correlation Coefficient)**

Let \( F_X \) and \( F_Y \) be, respectively, distribution functions of the random variables \( X \) and \( Y \) and jointly follow the bivariate distribution function \( F_{X,Y} \), then the Linear Correlation Coefficient or “Linear Dependence”, denoted by \( \rho \), between \( X \) and \( Y \) is given by

\[
\rho(X,Y) = \frac{1}{\sqrt{\text{Var}(X)\sqrt{\text{Var}(Y)}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( F_{X,Y}(x,y) - F_X(x)F_Y(y) \right) \, dx \, dy
\]
Linear correlation coefficient could be rephrased by the copula function in preference to the joint distribution function. The subsequent Corollary states this result.

**Corollary 3.6 (Copula’s Linear Correlation Coefficient)**

Let $C_{X,Y}$ be a copula function, that admits Theorem 3.3, of the random variables $X$ and $Y$. Then the linear correlation coefficient $\rho$ between $X$ and $Y$ by the mean of the copula function is given by

$$
\rho(X,Y) = \frac{1}{\sqrt{\text{Var}(X)\sqrt{\text{Var}(Y)}}} \int_0^1 \int_0^1 \left( C_{X,Y}(u,v) - uv \right) dF_X^{-1}(u)dF_Y^{-1}(v)
$$

**Proof:**

Using Corollary 3.2 and the fact that $x = F_X^{-1}(u)$ and $y = F_Y^{-1}(v)$ authorise restructuring Definition 3.21 to give Corollary 3.6 as an immediate consequence.

The previous corollary explains the limitation of the linear correlation coefficient, since it depends on the inversion marginal distribution function, where these marginal distributions could be possibly not invariant under monotone transformations. Consequently, studying the dependence association in copula function by other types of dependence measure could be more appropriate.

**3.9.2 Concordance**

To facilitate other types of dependence measurement between random variables, concordance should be defined. For more details on the concordance functions and copula (Nelsen, 2002) is an appropriate reference.

**Definition 3.22 (Concordance)**

Let $(X,Y)$ be two-dimensional vector-valued of a continuous finite-dimensional random variables $X$ and $Y$ with $(x_1,y_1)$ and $(x_2,y_2)$ as two observations. Then $(x_1,y_1)$ and $(x_2,y_2)$ are

- **Concordant** if $((x_1,y_1)(x_2,y_2) > 0$
- **Discordant** if $((x_1,y_1)(x_2,y_2) < 0$
The previous Definition of concordance requests further facilitation in order to collaborate the copula’s function with concordance and the dependence measure, for instance Kendall’s tau.

**Definition 3.23 (Concordance Function $Q$)**

Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be two independent two-dimensional vector-valued those admits Definition 3.22, with common margins $F_X$ and $F_Y^{15}$, but (possibly) different joint distribution functions $F_{X_1,Y_1}$ and $F_{X_2,Y_2}$. Then the difference of the probability of concordance and discordance between them is called the Concordance Function, denoted by $Q$, and given by

$$Q = P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - P\{(X_1 - X_2)(Y_1 - Y_2) < 0\}$$

As a consequence of this Definition, copula function could be shown as the primary request for the $Q$ “concordance function” to be constructed.

**Lemma 3.15 (Concordance Function $Q$)**

Let $Q$ be the concordance function that follow Definition 3.23. Then $Q$ could be rephrased and is given by the subsequent equality:

$$Q = 2P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1$$

**Proof:**

By rephrasing Definition 3.23 equality, i.e. $P_G = P\{(X_1 - X_2)(Y_1 - Y_2) > 0\}$ and $P_L = P\{(X_1 - X_2)(Y_1 - Y_2) < 0\} = 1 - P_G$, the concordance function $Q$ could be given by the subsequent chains of equalities:

$$Q = P_G - (1 - P_G)$$
$$= 2P_G - 1$$
$$= 2P\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1$$

This lemma provides the foundation that allow building the concordance function of two copula functions.

$^{15} F_X = F_{X_1} = F_{X_2}, F_Y = F_{Y_1} = F_{Y_2}$
Theorem 3.7 (Q of the Copula Function)

Let $C_{X_1,Y_1}(F_X(x), F_Y(y))$ and $C_{X_2,Y_2}(F_X(x), F_Y(y))$ be copula functions those follow Theorem 3.3, of two independent two-dimensional vector-valued $(X_1, Y_1)$ and $(X_2, Y_2)$, and $Q$ be concordance function that follow Lemma 3.15. Then $Q(C_{X_1,Y_1}, C_{X_2,Y_2})$ is given by the subsequent equality:

$$Q(C_{X_1,Y_1}, C_{X_2,Y_2}) = 4 \int_0^1 \int_0^1 C_{X_2,Y_2}(u, v) dC_{X_1,Y_1}(u, v) - 1$$

Proof

Using the Definition 3.23, the probabilities in Lemma 3.15 could be evaluated by integrating over one of the two-dimensional vector-valued distribution; such that:

$$Q(F_{X_1,Y_1}, F_{X_2,Y_2}) = 2\mathbb{P}\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - 1$$

$$= 2\mathbb{E}[\mathbb{P}\{(X_1 - x)(Y_1 - y) \geq 0| X_1 = x_1, Y_1 = y_1\}] - 1$$

$$= 2\mathbb{E}[F_{X_1,Y_1}(x,y) - F_X(x) - F_Y(y) + 1] - 1$$

$$= 4[F_{X_2,Y_2}(x,y)] - 1$$

$$= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{X_1,Y_1}(x,y) dF_{X_2,Y_2}(x,y) - 1$$

furthermore, since $F_X(x)$ and $F_Y(y)$ are independent, it could be rephrased over $\mathbb{II}^2$ under Theorem 3.2 as follow:

$$Q(C_{X_1,Y_1}, C_{X_2,Y_2}) = 4 \int_0^1 \int_0^1 C_{X_2,Y_2}(u, v) dC_{X_1,Y_1}(u, v) - 1$$

3.9.2.1 Concordance as Dependence Measure

The dependence measure between random variables in this section is frequently entitled as “measure of concordance”, since it fulfils a set of axioms noted in (Scarsini, 1984).

Definition 3.24 (Measure of Concordance Axioms)

A numeric measure $\kappa_{X,Y}$ of dependence between two continuous random variables $X$ and $Y$, which $C_{X,Y}$ is their corresponding copula, is a measure of concordance if it satisfies the subsequent properties:
1. $\kappa_{X,Y}$ is defined for every pairs $X, Y$

2. $-1 \geq \kappa_{X,Y} \geq 1$, and $\kappa_{X,-X} = 1$ and $\kappa_{X,-X} = -1$

3. $\kappa_{X,Y} = \kappa_{Y,X}$.

4. $\kappa_{X,Y} = \kappa_{\Pi} = 0$, when $X$ and $Y$ are independent.

5. $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$

6. If $C_1$ and $C_2$ are copulas such that $V(C_1) < V(C_2)$ then $\kappa_{C_1} < \kappa_{C_2}$

7. If $\{(X_n,Y_n)\}$ is a sequence of continuous random variables with copula function $C_n$ and if $\{C_n\}$ converge pointwise to $C$, then $\lim_{n \to +\infty} \kappa_{C_n} = \kappa_C$

Significance result could be illustrated from the last Definition, i.e. the upper and lower bounds of Frechet-Hoeffding Theorem and Sklar’s Theorem, are articulated in the subsequent Theorem.

**Theorem 3.8 (Monotone Measure of Concordance)**

Let $\kappa_{X,Y}$ of continuous random variables $X$ and $Y$ be a measure of concordance, then

1. If $Y$ is almost surely an monotone-increasing function of $X$ then $\kappa_{X,Y} = \kappa_M = 1$.

2. If $Y$ is almost surely a monotone-decreasing function of $X$ then $\kappa_{X,Y} = \kappa_W = -1$.

3. If $\alpha$ and $\beta$ are almost surely strictly monotone functions, respectively, on $\text{Ran}(X)$ and $\text{Ran}(Y)$, then $\kappa_{\alpha(X),\beta(Y)} = \kappa_{X,Y}$.

**3.9.2.2 Kendall’s tau**

The population form of Kendall’s tau “$\tau$” is an immediate consequence of Theorem 3.7, which is expressed by the $Q$ “concordance function” as a measure of dependence.

Kendall’s tau “$\tau$” was introduced by Fechner (1900), as noted in (Nelsen, 1991), and re-introduced by (Kendall, 1938) and (Kendall, 1970).

**Lemma 3.16 (“$\tau$” Kendall’s tau)**

Let $C_{X_1,Y_1}(F_X(x),F_Y(y))$ and $C_{X_2,Y_2}(F_X(x),F_Y(y))$ be copula functions those follow
Theorem 3.3, of two independent two-dimensional vector-valued \((X_1, Y_1)\) and \((X_2, Y_2)\), and \(Q\) be concordance function that follows Theorem 3.7. Then the population form of Kendall’s tau of \(X\) and \(Y\), denoted by \(\tau_{X,Y}\) or \(\tau_{C_{XY}}\), is given by the subsequent equality:

\[
\tau_{X,Y} = \tau_{C_{XY}} = Q(C_{X,Y}, C_{X,Y}) = 4 \int_0^1 \int_0^1 C_{XY}(u, v) dC_{XY}(u, v) - 1
\]

3.9.2.3 Spearman’s

Practically the same as Kendall’s tau “\(\tau\)”, the population form of dependence measure identified as Spearman’s rho “\(\rho\)” is based on three observed pairs in preference of two in Definition 3.21 and its consequences, i.e. Lemma 3.16 and Theorem 3.7, (Kruskal, 1958) and (Lehmann, 1966).

Lemma 3.17 (“\(\rho\)” Spearman’s rho)

Let \((X_1, Y_1)\) and \((X_2, Y_2)\) and \((X_3, Y_3)\) be three independent two-dimensional vector-valued those admits Definition 3.23, \(F_{XY}\) be the common joint distribution function of margins \(F_X\) and \(F_Y\), and \(C_{XY}\) be their copula function. Then the population form of Spearman’s rho, denoted by \(\rho_{X,Y}\) or \(\rho_{C_{XY}}\) is identified as the proportional to the probability of concordance minus discordance for both vectors \((X_2, Y_2)\) and \((X_2, Y_3)\), such that:

\[
\rho_{C_{XY}} = 12 \int_0^1 \int_0^1 C_{XY}(u, v) dudv - 3
\]

Proof:

\[
\rho_{C_{XY}} = 3(\mathbb{P}((X_1 - X_2)(Y_1 - Y_3) > 0) - \mathbb{P}((X_1 - X_2)(Y_1 - Y_3) < 0))
\]

\[
= 3Q(C_{XY}, \Pi)
\]

\[
= 12 \int_0^1 \int_0^1 uv \ dC_{XY}(u, v) - 3
\]

\[
= 12 \int_0^1 \int_0^1 C_{XY}(u, v) dudv - 3
\]

Following the last two Definitions about the Spearman’s \(\rho\) and Kendall’s \(\tau\) we have the
Theorem 3.9 (\( \rho \) and \( \tau \) are measure of concordance)

Let \( \tau_{XY} \) be the Kendall’s tau that admits Lemma 3.16 and \( \rho_{XY} \) be the Spearman’s rho that admits Lemma 3.17, both of continuous random variables \( X \) and \( Y \). Then they said to be measures of concordance if and only if they satisfy the measure of concordance axioms articulated in Definition 3.24 and Theorem 3.8.

Proof

The proof of this Theorem is just a hint proof rather than a mathematical proof. From the Definition of \( \rho \) and \( \tau \) the axioms of Definition 3.24 could be satisfied as follow:

1. An immediate satisfaction according to the probability Definition.
2. By applying the Frechet-Hoeffding Theorem.
3. Since \( Q \) is symmetric in its argument, i.e. \( Q(C_1, C_2) = Q(C_2, C_1) \)
4. By the Definition the \( C_\Pi \)
5. By combining 2 and 3.
6. By the concordance order (Nelsen, 2007)
7. By the Lipchitz condition implies that any family of copulas’ function is equicontinuous, as a consequence the convergence of \( \{C_n\} \) to \( C \) is uniform, (Nelsen, 1998), page 137.

And for the Theorem 3.8:

1. If \( Y \) is fixed to be \( X \), their copulas’ function will be the upper Frechet-Hoeffding bound.
2. If \( -Y \) is fixed to be \( X \), their copulas’ function will be the lower Frechet-Hoeffding bound.
3. It is an immediate consequence of applying the copula’s invariant property on \( \rho \) and \( \tau \).
3.9.3 Tail Dependence

Tail dependence of random variables is a measure which observes the relationship between different randomly extreme events constructed by different marginal distribution functions those jointly occurring, i.e. the probability that two companies default together.

**Definition 3.25 (Tail Dependence)**

Let $X$ and $Y$, respectively, be continuous random variables with distribution functions $F_X$ and $F_Y$. Then the upper and lower tail dependence coefficients, respectively, denoted $\lambda_U$ and $\lambda_L$ (if each or both of them exists) are given by

$$
\lambda_U = \lim_{p \to 1^-} P\{Y > F_Y^{-1}(p)|X > F_X^{-1}(p)\}^{16}
$$

$$
\lambda_L = \lim_{p \to 0^+} P\{Y \leq F_Y^{-1}(p)|X \leq F_X^{-1}(p)\}
$$

$\lambda_U, \lambda_L \in (0, 1]$

The preceding Definition could be rephrased by mean of copula and Bays Theorem and Definition 3.20 in copula invariant.

**Corollary 3.7**

Let $C$ be a copula function that admits Theorem 3.3, $p$ be the percentile that admits Definition 3.12 and $p \in \mathbb{I}$, and $\lambda_U$ and $\lambda_L$ be, respectively, the upper and lower tail dependence coefficients those admit Definition 3.25. Then $\lambda_U$ and $\lambda_L$ could be rearticulated (if each or both of them exists) as

$$
\lambda_U = \lim_{p \to 1^-} \frac{C(p,p) - 2p + 1}{1 - p} \quad \lambda_U, \lambda_L \in (0, 1]
$$

$$
\lambda_L = \lim_{p \to 0^+} \frac{C(p,p)}{p}
$$

In view of the fact that the upper and lower tail dependence coefficients are element of the unit interval, $\lambda_U, \lambda_L \in \mathbb{I}$, and could be written in term of the copula function, then the numeric measure of association as described by (Scarsini, 1984) are satisfied.

---

16 $p$ is the percentile as defined in Lemma 3.12 and used in Sklar's Theorem, i.e. Theorem 3.2.
3.10 Mathematical Summary

Copula Function

- $C_{X,Y} : \mathbb{I}^2 \rightarrow \mathbb{I}$, $X \& Y \in \mathbb{R}$ such that $C_{X,Y}$ is grounded, 2-increasing, and the margins could be constructed: $C_{X,Y}(u, 1) = u$.

- $F_{X,Y}(x, y) = C_{X,Y}(F_X(x), F_Y(y))$.

- $C_{X,Y}(u, v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v))$.

- $P\{X \leq x, Y \leq y\} = C_{X,Y}(P\{X \leq x\}, P\{Y \leq y\})$

- $F_{X,Y}(x, y) = C_{X,Y}(F_X(x), F_Y(y))$

- $C_{\Pi}(u, v) = uv$.

- $\max(u + v - 1, 0) = W(u + v - 1, 0) \geq C(u, v) \geq M(u, v) = \min(u, v)$

Copula’s Invariant Property

- $\alpha \uparrow^{\uparrow} \& \beta \uparrow^{\uparrow}: C_{\alpha(X), \beta(Y)} = C_{X,Y}$.

- $\alpha \uparrow \& \beta \downarrow: C_{\alpha(X), \beta(Y)}(u, v) = u - C_{X,Y}(u, 1 - v)$.

- $\alpha \downarrow \& \beta \uparrow: C_{\alpha(X), \beta(Y)}(u, v) = v - C_{X,Y}(1 - u, v)$.

- $\alpha \downarrow \& \beta \downarrow: C_{\alpha(X), \beta(Y)}(u, v) = u + v - 1 + C_{X,Y}(1 - u, 1 - v)$

Dependence Concept

- Linear: $\rho(X, Y) = \frac{1}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \int_0^1 \int_0^1 (C_{X,Y}(u, v) - uv) dF_X^{-1}(u) dF_Y^{-1}(v)$

- Concordance: $Q(C_{X_1, Y_1}, C_{X_2, Y_2}) = 4 \int_0^1 \int_0^1 C_{X_2, Y_2}(u, v) dC_{X_1, Y_1}(u, v) - 1$
  
  - Kendall’s tau: $\tau_{X,Y} = 4 \int_0^1 \int_0^1 C_{X,Y}(u, v) dC_{X,Y}(u, v) - 1$
  
  - Spearman’s rho: $\rho_{C_{X,Y}} = 12 \int_0^1 \int_0^1 C_{X,Y}(u, v) du dv - 3$

- Tail Dependence: $\lambda_U, \lambda_L \in (0, 1]$
  
  - Upper: $\lambda_U = \lim_{p \rightarrow 1^-} \frac{C(p, p) - 2p + 1}{1 - p}$, Lower: $\lambda_L = \lim_{p \rightarrow 0^+} \frac{C(p, p)}{p}$.

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37 $\uparrow$: Strictly Monotone-Increasing and $\downarrow$: Strictly Monotone-Decreasing
Chapter Four

Time To Default Modelling

4.1 Outline

- Introduction
- Basic Definitions of Measure, Stochastic Process, and Martingale
- Set up and notation
- Conditionally Independent Default Times
  - Canonical construction of conditionally independent default times
  - Conditional independent default times properties
  - Dynamically Conditionally Independent
  - Dynamically Conditional independent default times properties
  - Minimum of Default Times
- Mathematical Summary
4.2 Introduction

Modelling default times and the hazard rate of associated random times by the mean of a simple multivariate Cox process framework will be recalled in this section. This concept has first appeared as a purely mathematical methodical concept as stopping times associated with enlargement filtrations in the 1970’s by a number of French scholars as in: (Dellacherie, 1970), (Bremaud and Yor, 1978), (Dellacherie and Meyer, 1978), (Jeulin and Yor, 1978), and (Jeulin, 1980). Subsequently, financial modellers, like (Kusuoka, 1999) and (Elliott et al., 2000), have comprehensively analysed the random times properties. Modelling credit derivatives and their events via the classic concept of the hazard rate of associated random times has attract a significant number of practitioners and researchers in the current years as in (Jarrow and Turnbull, 1995), (Duffie et al., 1997), (Jarrow et al., 1997), (Duffie, 1998a), (Lando, 1998), (Madan and Unal, 1998), (Jarrow and Yu, 2000), and (Duffie and Lando, 2001). It has been studied in relation to Cox processes in (Last and Brandt, 1995) and to the theory of martingales in (Bremaud, 1981), (Jeanblanc and Rutkowski, 2000a), (Jeanblanc and Rutkowski, 2000b), and (Bielecki and Rutkowski, 2002a). In (Bielecki and Rutkowski, 2002a) the default time processes, hazard and martingale hazard process and their intensities are comprehensively studied, where in that context the arbitrage pricing theory was studied in (Musiela and Rutkowski, 1997). Finally, the conditionally independent defaults was studied in (Bielecki and Rutkowski, 2002b), where this chapter stands on.

The stated results will propose some building blocks for the pricing of basket credit derivatives as well as synthetic CDO’s.
4.3 Basic Definitions of Measure, Stochastic Process, and Martingale

In order to study the time to default, hazard process, and the martingale hazard process and their intensities, some basic definitions will be recalled. This section will start by some fundamentals on probability space, random variables, measures, stochastic process, and the definition of martingale.

In the subsequent the expectation, $E$, will be usually on a $d$-dimensional real space, $\mathbb{R}^d$.

**Definition 4.1 (Sigma Algebra)**

Let $\mathbb{D}$ be a set. Then a $\sigma$-algebra is a nonempty collection $\mathcal{D}$ of subsets of $\mathbb{D}$ such that the subsequent conditions hold:

i. $\mathcal{D}$ is in $\mathbb{D}$, i.e. $\mathcal{D} \in \mathbb{D}$

ii. $\mathcal{D}$ contains the empty set, i.e. $\emptyset \in \mathcal{D}$

iii. If $X_n$ is a disjoint sequence of $\mathcal{D}$, then its union is in $\mathcal{D}$, i.e. If $\{X_n: X_n \in \mathcal{D}, n \geq 1\}$, then $\cup_{n \geq 1} X_n \in \mathcal{D}$.

iv. If $X$ is in $\mathcal{D}$, then its complement is in $\mathcal{D}$, i.e. $\forall X \in \mathcal{D}, X^c \in \mathcal{D}$.

The sigma algebra is interpreted as a collection of events those could be assigned a probability, where it is mainly used to define the measure concept.

**Definition 4.2 (Measure Space)**

Let $\mathbb{D}$ be a $\sigma$-algebra of subsets $\mathcal{D}$ that admits Definition 4.1. Then its measurable space is denoted as $(\mathbb{D}, \mathcal{D})$.

**Definition 4.3 (Measure)**

Let $\mathbb{D}$ be a $\sigma$-algebra of subsets $\mathcal{D}$ that admits Definition 4.1, $(\mathbb{D}, \mathcal{D})$ be its measurable space that admits Definition 4.2. Then a measure on $(\mathbb{D}, \mathcal{D})$ is nonnegative real function $\alpha: \mathcal{D} \rightarrow [0, \infty)$ such that:

i. $\alpha(\emptyset) = 0$

ii. $\forall\{X_n, n \geq 1\}$ of disjoint elements of $\mathcal{D}$, $\alpha(\cup_{n \geq 1} X_n) = \sum_{n \geq 1} \alpha(X_n)$.  

Chapter Four: Time To Default Modelling
With the measure definition in hand, its space and mass are needed in order to define the probability measure.

**Definition 4.4 (Finite Measure)**

Let $\alpha$ be a measure that admits Definition 4.3 on a measurable space $(\mathcal{D}, \mathcal{D})$ that admits Definition 4.2. Then $\alpha$ is said to be finite if $\alpha(\mathcal{D}) < \infty$.

**Definition 4.5 (Mass Measure)**

Let $\alpha$ be a measure that admits Definition 4.3 on a measurable space $(\mathcal{D}, \mathcal{D})$ that admits Definition 4.2. Then $\alpha$’s mass is equal to the quantity $\alpha(\mathcal{D})$.

**Definition 4.6 (Probability Measure)**

Let $\alpha$ be a measure that admits Definition 4.3 on a measurable space $(\mathcal{D}, \mathcal{D})$ that admits Definition 4.2, where $\alpha(\mathcal{D})$ is its mass. Then when $\alpha(\mathcal{D}) = 1$, $\alpha$ is called the probability measure.

**Definition 4.7 (Probability Space Measure)**

Let $\alpha$ be a measure that admits Definition 4.3 on a measurable space $(\mathcal{D}, \mathcal{D})$ that admits Definition 4.2, where $\alpha(\mathcal{D})$ is its probably measure, i.e. $\alpha(\mathcal{D}) = 1$. Then $(\mathcal{D}, \mathcal{D}, \alpha)$ is its probability space.

To understand $(\mathcal{D}, \mathcal{D}, \alpha)$, let as CR market probability space measure: $\mathcal{D}$ represent all possible outcomes that could happen in the CR market, $\mathcal{D}$ characterise the $\sigma$-algebra that contains all sets to build on the needed statements, i.e. the physical probability measure observed on CR market, and finally $\alpha$ represents the probability that $\mathcal{D}$ will happen.

**Definition 4.8 (Measureable Function)**

Let $(\mathcal{D}, \mathcal{D})$ and $(\mathcal{G}, \mathcal{G})$ be two measurable spaces those admits Definition 4.2. Then $f: \mathcal{D} \to \mathcal{G}$ is a measurable function if for any measurable set $X$ in $\mathcal{D}$ there exists an inverse function which is a measurable subset of $\mathcal{G}$, i.e.

$$\forall X \in \mathcal{D}, \quad \exists f^{-1}(X) = \{x \in \mathcal{D}: f(x) \in X\} \in \mathcal{G}$$
Definition 4.9 (Random Variable)

Let $(\mathcal{D}, \mathcal{D}, \alpha)$ be a probability space that admits Definition 4.7, $(\mathbb{G}, \mathcal{G})$ be a measurable space that admits Definition 4.2. Then the measurable function $\mathcal{X}: \mathcal{D} \rightarrow \mathbb{G}$ that admits Definition 4.8 is called a random variable. This random variable is $(\mathcal{D}, \mathcal{G})$-measurable, i.e.

$$\forall y \in \mathcal{G}, \quad \{s: \mathcal{X}(s) \in y\} \in \mathcal{D}$$

Relying on the scenario $s \in \mathcal{D}$, the random variable could have different values; $\mathcal{X}(s)$ indicates the realisation of the random variable $\mathcal{X}$, if the scenario $s$ occurs. The subsequent definition defined the random variable when associated with time.

Definition 4.10 (Stochastic Process)

Let $(\mathcal{D}, \mathcal{D}, \alpha)$ be a probability space that admits Definition 4.7, $\mathcal{X}$ be a random variable that admits Definition 4.9. Then the collection of $\mathcal{X}$ when associated with the time parameter, i.e. $\mathcal{X} = \{\mathcal{X}_t, 0 \leq t \leq T\}$, is called a stochastic process.

It is important to know that the stochastic process $\mathcal{X}$ is said to be adapted to the filtration $\mathbb{F}$ when, for each time $t$, $\mathcal{X}_t$ is known and $\mathcal{X}_t \in \mathcal{F}_t$. With the previous definitions in hand, the martingale could be defined.

Definition 4.11 (Martingale)

Let $\mathcal{X} = \{\mathcal{X}_t\}_{t \in [0,T]}$ be a stochastic process that admits Definition 4.10. Then $\mathcal{X}$ is said to be $\mathcal{F}_t$ martingale if the subsequent conditions hold:

i. $\mathcal{X}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}^+}$.

ii. $\mathbb{E}[|\mathcal{X}_t|] < \infty$ for each $t$.

iii. $\forall s, t: \quad s \leq t, \quad \exists \mathbb{E}[\mathcal{X}_t | \mathcal{F}_s] = \mathcal{X}_s$

The first condition declares that the value of $\mathcal{X}$ could be observed at each and every time $t$, where the third condition articulates that given the information available until
time $s$, the expectation of a future value of $\mathcal{X}$ equals the present value of $\mathcal{X}$ at time $s$, i.e. a martingale has no systematic drift.

### 4.4 Set Up and Notation

In view of the definitions stated previously, the standard time of default settings will be articulated. To intuitively explain this section setups, the probability space considered is $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, where the set of all possible outcomes is expressed by $\Omega$, the $\sigma$-algebra that includes all sets that the needed statements is built on are expressed by $\mathcal{G}$, and finally $\mathbb{Q}^*$ is the martingale probabilities measure that an event $\mathcal{G}$ will occur.

Furthermore, this subsection will articulate the main default time, space measure, and filtrations assumptions and definitions needed to build the rest of this chapter.

**Assumption 4.1 (Number of Considered Credit Entities: $n$)**

*The number of considered credit entities is $n$. Also it is assumed that the subsequent notations are valid, unless explicitly stated otherwise:*

- $i. \quad \mathbb{K} = 1, \cdots, n.$
- $ii. \quad \mathbb{K}^- = 1, \cdots, n - 1$

**Assumption 4.2 (Default Times $\tau_i$)**

*The random default times, denoted by $\tau_i$, where $i \in \mathbb{K}$, is presupposed to model the underlying credit entity $i$.*

In other words, the previous assumption means that for each and every credit entity $i$ its default time is denoted by $\tau_i$.

From this point when anything is associated with $i$ then it follows Assumption 4.2, i.e. $i \in \mathbb{K}$, where $\mathbb{K}$ follows Assumption 4.1, unless there is an association with $\mathbb{K}^-$, then both of them are going to be explicitly mentioned in that statement.

**Assumption 4.3 (Default Times Measure)**

*The default times, $\tau_i$, are defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$.*
It is sufficient to choose \( \mathbb{Q}^* \) as martingale probabilities measure; as a consequence to the scope, which is the valuation of credit risk derivatives, of this research.

**Assumption 4.4 (Default Times under the Measure \( \mathbb{Q}^* \))**

The subsequent assumptions of the default times under the measure \( \mathbb{Q}^* \) are valid, unless explicitly stated otherwise:

- \( \mathbb{Q}^*\{ \tau_i = 0 \} = 0, \forall t \in \mathbb{R}^+ \).
- \( \mathbb{Q}^*\{ \tau_i > t \} > 0, \forall t \in \mathbb{R}^+ \).
- \( \mathbb{Q}^*\{ \tau_i = \tau_k \} = 0 \) for arbitrary \( i, k \in \mathbb{K} \) with \( i \neq k \).

This assumption express in the first condition that it is impossible for a credit entity \( i \) to default at the initial time, where the second states that there is a probability that it will default after that. The third means that there are no simultaneous defaults at the same time, i.e. it is impossible for two credit entities to default at the same time. By looking again at the second condition more details could be observed, where \( \tau \) is assumed to be unbounded, i.e. it is not dominated with probability 1 by a constant. Therefore, to bound the \( \tau \), another filtration should be introduced.

**Assumption 4.5 (\( i^{th} \) Random Default Time: \( \tau^{i^{th}} \))**

The \( i^{th} \) random default time, denoted by \( \tau^{i^{th}} \), is the coupled of the ordered sequence \( \tau^{1st} \leq \tau^{2nd} \leq \cdots \leq \tau^{nth} \) with the pool \( \tau_1, \cdots, \tau_n \) of default times, such that

\[
\tau^{1st} = \min(\tau_1, \cdots, \tau_n) = \tau_1 \wedge \cdots \wedge \tau_n \\
\tau^{(i+1)th} = \min(\tau^{kth} : k \in \mathbb{K}, \tau_k > \tau^{ith}), i \in \mathbb{K}^- \\
\tau^{nth} = \max(\tau_1, \cdots, \tau_n)
\]

To intuitively express the previous assumption, the statements could be read as: the first condition means that by taking all the credit entities \( \tau^{1st} \) is equal to the first defaulted credit entity. The second articulates that if \( i \) credit entities have defaulted, the next random default time, \( \tau^{(i+1)th} \), is going to be the first credit defaulted provided that this
credit entity is not one of those defaulted earlier. The last condition expresses that the
nth default time will be associated with the last credit entity that default. In general
Assumption 4.4 and Assumption 4.5 formulate the time to default conditions.

Assumption 4.6 (Random Default Time, Excluded i: \( \tau^{(-i)th} \))

Let \( \tau^{i_{th}} \) be the ith random default time that admits Assumption 4.5, then it is assumed
that the first to default time for a set of credit references, where i is excluded, is denoted
by \( \tau^{(-i)th} \) and given by \( \tau^{(-i)th} = \min_{j \neq i} \tau_j \).

This assumption is usually used in order to find the random default time provided that it
is not associated with a specified credit entity. Moreover, it worth to mention that the
ith random default time is the 1st to default name if and only if \( \tau^{(-i)th} > \tau_i \) or
equivalently if and only if \( \tau^{1st} > \tau_i \).

Assumption 4.7 (Filtration \( \mathbb{F} \))

It is assumed that the reference filtration \( \mathbb{F} \) of the default times \( \tau_i \) is specified on the
probability space \( (\Omega, \mathcal{G}, \mathbb{Q}^*) \).

The complete information needed to construct the default time could be partially viewed
through the reference filtration \( \mathbb{F} \), where in the sequel an enlarge filtration will be
provided. In order to introduce this enlarged filtration, the indicator default process, the
default process, default counter process, and the natural filtration \( \mathbb{H} \) assumptions and
definitions are articulated.

Assumption 4.8 (Default Process Indicator: \( \mathcal{I} \))

A default process indicator, denoted by \( \mathcal{I} \), is supposed as a Boolean value driven upon a
default time \( t \), where \( t \in \mathbb{R}^+ \) coupled with the random default time \( \tau_i \) of the credit entity
i. It is represented by \( \mathcal{I}_{\{\tau_i \leq t\}} \).
In other words, \( J \) is the time to default jump process. This assumption is formalised in the subsequent definition.

**Definition 4.12 (Default Process of Credit Entity \( i: N^i_t \))**

A default process associated with the default credit entity \( i \), denoted by \( N^i_t \) where \( t \in \mathbb{R}^+ \), is given by the indicator default process \( J \) that follows Assumption 4.8, where it could be represented by the subsequent equality:

\[
N^i_t = J_{(\tau_{i} \leq t)}
\]

However, if the number of traceable default credit entities until time \( t \) is required, then this could be achieved by summing over all credit entities’ default processes. This could be accomplished as represented in the subsequent definition.

**Definition 4.13 (Default Counter Process: \( N_t \))**

Let \( N^i_t \) be a default process associated with the default credit entity \( i \) that admits Definition 4.12, then \( N_t \) is assumed to be a default counter process at time \( t \), where it could be represented by the subsequent equality:

\[
N_t = \sum_{i=1}^{n} N^i_t = \sum_{i=1}^{n} J_{(\tau_{i} \leq t)}
\]

With Assumption 4.6 and Definition 4.13 in hand, the next definition represent the number of defaults excluding a specified credit entity \( i \) at time \( t \).

**Definition 4.14 (Default Counter Process, of Excluded \( i: N^{(-i)^{th}}_t \))**

A counter process associated with the default time \( \tau^{(-i)^{th}} \) that admits Assumption 4.6 and \( i \) is excluded, for a set of credit references, denoted by \( N^{(-i)^{th}}_t \), where \( t \in \mathbb{R}^+ \), is given by the indicator default process \( J \) that follows Assumption 4.8, by the subsequent equality:

\[
N^{(-i)^{th}}_t = J_{\{\tau^{(-i)^{th}} \leq t\}} = \sum_{j \neq i} N^j_t
\]
The previous assumptions and definitions depend on the natural filtration $\mathbb{H}$, however, they were presented in order to give an indication of the meaning and use of the natural filtration $\mathbb{H}$ in this chapter.

**Assumption 4.9 (Filtration $\mathbb{H}$)**

The filtration $\mathbb{H}_t$ is assumed to represent the natural filtration of default time $\tau_i$, with $\mathbb{H} = \bigvee_{t=1}^{\infty} \mathbb{H}_t$, where $\mathbb{H}_t^i = \sigma(N_s^i, s \leq t)$ represent the family of jump processes, i.e. the indicator default process $\mathbb{I}$.

Now, after articulating the reference filtration $\mathbb{F}$ in Assumption 4.6 and the natural filtration $\mathbb{H}$ in Assumption 4.9, the enlarged filtration $\mathbb{G}$ could be represented.

**Assumption 4.10 (Filtration $\mathbb{G}$)**

It is postulated that $\mathbb{G}$ is an enlarged filtration, which is found by setting $\mathbb{G} = \mathbb{F} \lor \mathbb{H}$. Which could also hold upon Assumption 4.9 that $\mathbb{G}_t^i := \mathbb{F} \lor \mathbb{H}_t^i$.

**4.5 Conditionally Independent Default Times**

The valuations of basket credit derivatives and CDO under the underlying filtration are assumed to enclose conditional independence between default times. This assumption supports most of the composition of the basket credit derivatives; see for example (Kijima, 2000) and (Kijima and Muromachi, 2000).

**Assumption 4.11 (Conditionally independent Random Times)**

For every $T \in \mathbb{R}^+$ and arbitrary $t_i \in [0, T]$, the random times $\tau_i$ are assumed to be conditionally independent with respect to the filtration $\mathbb{F}$ under $\mathbb{Q}^*$, such that

$$\mathbb{Q}^*(\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_T) = \prod_{i=1}^{n} \mathbb{Q}^*(\tau_i > t_i | \mathcal{F}_T)$$

To introduce an informal factor model meaning for Assumption 4.11, which will be introduced in Chapter 5, the random times is generated by a universal risk factors and an idiosyncratic risk. The universal risk factors are the factors that may generate a credit
default event across all reference credit names. In contrast, the idiosyncratic risk is specific factor for a specific reference credit name which may generates a credit default event. Assumption 4.11 intuitively means that the idiosyncratic risk factors become independent of each other when the universal risk factors are predetermined.

It is important to note that the conditional independence of random default times $\tau_i$ does not imply their independence or their contradictory. Furthermore, the conditional independence may not be invariant under equivalent change of probability measure, i.e.

$$Q^*(\tau_i > t_i | \mathcal{F}) \neq Q(\tau_i > t_i | \mathcal{F}).$$

**Stress 4.1**

\forall t_i \in [0, T] \text{ and } u \in [0, T) \text{ the subsequent equality does not necessarily hold:}

$$Q^*(\tau_i > t_i | \mathcal{F}_T) = Q^*(\tau_i > t_i | \mathcal{F}_u)$$

This property will be utilised frequently in the rest of this section, thus the subsequent assumption will hold unless explicitly stated.

**Assumption 4.12 ($\mathbb{F}$-Default Process of $\tau$)**

For every $T > 0, u \in [0, T]$ it is assumed that:

$$Q^*(\tau_i > u | \mathcal{F}_T) = Q^*(\tau_i > u | \mathcal{F}_u)$$

and for any $t \in \mathbb{R}^+$, the $\mathbb{F}$-default process, denoted by $F$, and $\mathbb{F}$-survival process, denoted by $G$, of $\tau$ with respect to the filtration $\mathcal{F}$, are given as

$$F_t = Q^*\{\tau \leq t | \mathcal{F}_T\}$$

$$G_t := 1 - F_t = Q^*\{\tau > t | \mathcal{F}_T\}$$

Therefore, the process $F$ and $G$ under assumption 4.12, follows a bounded, non-negative $\mathbb{F}$-submartingale under $Q^*$ and as a consequence of having $\{\tau \leq t\} \subseteq \{\tau \leq s\}, \forall 0 \leq t \leq s$, $F_t$ could be found as:

$$E^{Q^*}(\mathcal{F}_s | \mathcal{F}_T) = Q^*\{\tau \leq s | \mathcal{F}_T\}$$

$$\geq Q^*\{\tau \leq t | \mathcal{F}_T\}$$

$$= F_t$$
After Definition 4.12 and Assumption 4.12 the $\mathbb{F}$-hazard process could be defined as follow.

**Definition 4.15 ($\mathbb{F}$-Hazard Process of $\tau$: $\Gamma$)**

The $\mathbb{F}$-hazard process of $\tau$ with respect to the filtration $\mathbb{F}$ under $\mathbb{Q}^*$, which is denoted by $\Gamma$, with the assumption of $F_t < 1$, $\forall t \in \mathbb{R}^+$, is given as:

$$
\Gamma_t = -\ln G_t = -\ln(1 - F_t), \forall t \in \mathbb{R}^+
$$

This definition gives more restriction on the time to default assumption, i.e. Assumptions 4.4, where its probability will not exceeds 1.

Subsequently to this definition, the $\mathbb{F}$-hazard process intensity of $\tau$ is usually assumed, in the recent reduced-form credit risk derivatives context, to have absolutely continuous sample paths. The next definition summarise the components of the hazard rate intensity function.

**Definition 4.16 ($\mathbb{F}$-Intensity of $\tau$: $\gamma$)**

The $\mathbb{F}$-intensity of $\tau$, denoted by $\gamma$, with $\Gamma$ as the hazard rate that admits Definition 4.15, is an $\mathbb{F}$-progressively measurable non-negative process that holds the subsequent equality:

$$
\Gamma_t = \int_0^t \gamma_u du
$$

$\mathbb{F}$-hazard process intensity of $\tau$, $\gamma$, is referenced sometimes as $\mathbb{F}$-hazard rate of $\tau$ or stochastic intensity of $\tau$ in literature, in particular when the reference filtration $\mathbb{F}$ is clear in the context.

At this point, the foundation to define an interrelated concept to the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of a random time $\tau$ is completed. This is needed to examine if the $\mathbb{F}$-hazard process of $\tau$, $\Gamma$, coincides with the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of $\tau$, $\Lambda$. 

Chapter Four: Time To Default Modelling
**Definition 4.17 ((\(\mathbb{F}, \mathbb{G}\))-Martingale hazard process: \(\Lambda\))**

A process is called \((\mathbb{F}, \mathbb{G})\)-martingale hazard process of \(\tau\), denoted by \(\Lambda\), if and only if:

1. It is an \(\mathbb{F}\)-predictable, right-continuous, increasing process under \(\mathbb{Q}^*\), where \(\Lambda_0 = 0\).
2. The process \(\tilde{M}_t = H_t - \Lambda_{t\wedge t}\) follows a \(\mathbb{G}\)-martingale under \(\mathbb{Q}^*\).

The use of this definition is going to appear in many lemmas’ and theorems’ statements; particularly it will be used clearly in Lemma 4.4.

After representing the definition of \((\mathbb{F}, \mathbb{G})\)-Martingale hazard process of \(\tau\), it worth following it with the definition of its Intensity.

**Definition 4.18 ((\(\mathbb{F}, \mathbb{G}\))-Martingale Intensity Process of \(\tau\): \(\lambda\))**

The \((\mathbb{F}, \mathbb{G})\)-martingale intensity process of \(\tau\) under \(\mathbb{Q}^*\), with \(\Lambda\) as defined in Definition 4.17, is \(\mathbb{F}\)-progressively measurable, nonnegative process \(\lambda\), that holds the subsequent equality:

\[
\Lambda_t = \int_0^t \lambda_u \, du
\]

Again, the \((\mathbb{F}, \mathbb{G})\)-martingale hazard process of \(\tau\), \(\Lambda\), is referenced sometimes as \((\mathbb{F}, \mathbb{G})\)-martingale hazard rate of \(\tau\) or \((\mathbb{F}, \mathbb{G})\)-martingale stochastic intensity of \(\tau\) in literature.

**4.5.1 Canonical Construction of Conditionally Independent Default Times**

In this subsection, an explicit construction of a conditionally independent family of random times with pre-specified \(\mathbb{F}\)-hazard processes will be provided. This methodology is the most commonly used to show how to construct the time to default when associated with a pre-specified hazard process \(\Gamma\). In order to examine the random default times constructed through this approach, i.e. the canonical construction, the random default time’s properties will be presented and proved, if needed, in the next subsection.
Assumption 4.13 (F-Adapted Processes: \( \Gamma \))

A given collection of \( \mathbb{F} \)-adapted increasing continuous stochastic processes, denoted by \( \Gamma^i \) where \( \Gamma \) admits Definition 4.15, defined on a common filtered probability space \( (\hat{\Omega}, \mathbb{F}, \mathbb{P}^\star) \) are assumed to hold the following statements:

\[
\begin{align*}
\Gamma^i_0 &= 0 \\
\Gamma^i_\infty &= \infty \\
\Gamma^i_t &< \infty, \quad \forall t \in \mathbb{R}^+
\end{align*}
\]

To intuitively explain the meaning of the previous assumption, \( \Gamma \) could represent the default-free securities market’s uncertainty, which is modelled through a reference filtration \( \mathbb{F} \) on the underlying probability space \( (\hat{\Omega}, \mathbb{F}, \mathbb{P}^\star) \).

Assumption 4.14 (Independent Uniform Random Variables \( \xi_i \))

It is assumed that a sequence of mutually independent random variables uniformly distributed on the interval \([0,1] \), denoted by \( \xi_i \) is defined on an auxiliary probability space \( (\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}}) \).

This assumption expresses \( \xi \) as a representation of the default-free, i.e. arbitrage-free, securities market. In other words, \( \xi \) could express a unique spot martingale measure \( \hat{\mathbb{P}} \), equivalent to \( \mathbb{P}^\star \), on the martingale probability space \( (\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}}) \). Connecting this assumption with Assumption 4.13 could introduce a space, which is large enough, to construct the random default time.

Definition 4.19 (Enlarge space \( (\Omega, \mathcal{G}, Q^\star) \))

\( (\Omega, \mathcal{G}, Q^\star) \) is defined as an enlarge space produced by a product of \( \Omega = \hat{\Omega} \times \hat{\Omega} \), \( \mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathbb{F}} \), and \( Q^\star = \mathbb{P}^\star \otimes \hat{\mathbb{P}} \).

After Assumption 4.13, Assumption 4.14 and Definition 4.19, conditional independent default times could be constructed by the canonical construction method as in Definition 4.20 and satisfies Assumption 4.3.
Definition 4.20 (Conditional Independent Default Times $\tau$)

Let $\Gamma$ be an $\mathbb{F}$-hazard process that admits Definition 4.15 and follows Assumption 4.13, and $\xi$ be a uniform distributed random variable that admits Assumption 4.14. Then the conditional independent default times, denoted by $\tau_i$ defined on the enlarged space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, which admits Definition 4.19, is given by the subsequent equality:

$$\tau_i = \inf\{t \in \mathbb{R}^+: I_i^t \geq -\ln \xi_i\}$$

With filtration $\mathbb{H} = \bigvee_{i=1}^n \mathbb{H}_t^i$, the enlarge filtration $\mathcal{G}$, and the enlarge space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, respectively, as in Assumption 4.9, Assumption 4.10 and Definition 4.19, the subsequent is an immediate corollary.

Corollary 4.1 (Complete Information by $\mathcal{G}_t$)

All information available to an agent at time $t$, including the observations of all random times $\tau_i$ is represented by $\sigma$-field $\mathcal{G}_t$, $\forall t \in \mathbb{R}^+$, Formally,

$$\mathcal{G}_t = \mathcal{F}_t \lor \sigma(\{\tau_1 < t_1\}, \cdots, \{\tau_n < t_n\}: t_1 \leq t, \cdots, t_n \leq t)$$

Finally it could be observed that the sequence of random times $\tau_i$ constructed above satisfies the required assumption, i.e. Assumption 4.3 and Assumption 4.10. The subsequent subsection will encapsulate random default times $\tau_i$ properties.

4.5.2 Conditional independent default times properties

Now, the properties of the random default times constructed above are presented.

Lemma 4.1 (Conditional Joint Probability on $\mathcal{F}_\infty$)

Let $\tau_i$ be the random default times that admits Definition 4.20, $\Gamma^i$ be a given family that follows Assumption 4.13, and $t_i \in \mathbb{R}^+$. Then the conditional joint probability of survival satisfies the subsequent equality:

$$\mathbb{Q}^*(\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_\infty) = \prod_{i=1}^n e^{-\int_0^t \Gamma_i^t}$$

$$= e^{-\sum_{i=1}^n t_i \Gamma_i^t}$$
Proof.

First, by assuming $t_i \in \mathbb{R}^+$ as an arbitrary numbers and with the random default times $\tau_i$ as in Definition 4.20, the following statement is valid:

$$\{\tau_i > t_i\} = \{\Gamma_i^t < -\ln \xi_i\} = \left\{e^{-\Gamma_i^t} > \xi_i\right\}$$

Secondly, as $\Gamma_i^t$ are noticeably $\mathcal{F}_\infty$-measurable, it is acceptable to say that:

$$Q^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_\infty\} = Q^*\left\{e^{-\Gamma_1^t} > \xi_1, \ldots, e^{-\Gamma_n^t} > \xi_n | \mathcal{F}_\infty\right\}$$

$$= Q^*\{e^{-x_1} > \xi_1, \ldots, e^{-x_n} > \xi_n | \mathcal{F}_\infty\}_{x_1=\Gamma_1^t, \ldots, x_n=\Gamma_n^t}$$

$$= \prod_{i=1}^{n} Q^*\{e^{-x_i} > \xi_i | \mathcal{F}_\infty\}_{x_i=\Gamma_i^t}$$

$$= \prod_{i=1}^{n} \bar{P}\{e^{-x_i} > \xi_i\}_{x_i=\Gamma_i^t}$$

$$= \prod_{i=1}^{n} e^{-\Gamma_i^t}$$

$$= e^{-\sum_{i=1}^{n} \Gamma_i^t}$$

Lemma 4.2 (Conditional Joint Probability on $\mathcal{F}_T$)

For an arbitrary numbers $t_i \in \mathbb{R}^+$, $\Gamma_i^t$ as a given family that follows Assumption 4.13, the random times $\tau_i$ as defined in Definition 4.20, and any $T \geq \max(t_1, \ldots, t_n)$ the subsequent equality hold:

$$Q^*(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T) = \prod_{i=1}^{n} e^{-\Gamma_i^t}$$

$$= e^{-\sum_{i=1}^{n} \Gamma_i^t}$$

Proof:

In view of the fact that the random variable $\Gamma_i^t$ is $\mathcal{F}_T$-measurable for any $T \geq t_i$, and using Lemma 4.1 and its proof, Lemma 4.2 is an immediate result, as it is shown in the subsequent equalities:
\[ \mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T\} = \mathbb{E}^{\mathbb{Q}^*}(\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_\infty\} | \mathcal{F}_T) \]
\[ = \mathbb{E}^{\mathbb{Q}^*}\left(\prod_{i=1}^{n} e^{-r_i^i} \bigg| \mathcal{F}_T\right) \]
\[ = \mathbb{E}^{\mathbb{Q}^*}\left(e^{-\sum_{i=1}^{n} r_i^i} \bigg| \mathcal{F}_T\right) \]
\[ = e^{-\sum_{i=1}^{n} r_i^i} \]

With Lemma 4.1 and Lemma 4.2 in hand the following corollary is an immediate result.

**Corollary 4.2 (Equality Joint Distribution Conditionally on \( \mathcal{F}_\infty \) and \( \mathcal{F}_T \))**

For an arbitrary numbers \( t_i \in \mathbb{R}^+, \Gamma^i \) as a given family that follows Assumption 4.13, and the random times \( \tau_i \) as defined in Definition 4.20. For every \( t_i \leq T \), the subsequent equalities hold:

\[ \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T\} = \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_\infty\} \]
\[ = e^{-r_i^i} \]

By analysing the previous corollary, a number of important consequences could be extracted. First, the equality mentioned in the corollary could be stretched to include two arbitrary dates, i.e. \( 0 \leq t \leq u \). Hence, \( \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_\infty\} = \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_u\} = \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T\} = e^{-r_i^i} \). It could be observed that only the last equality is essentially satisfied by the \( \mathbb{F} \)-hazard process \( \Gamma \) of \( \tau \), where the first two equalities could be seen as an extra features of the canonical construction of \( \tau \). In other words, the first two equalities are not essentially valid in a general set-up. This causes the conditional independence under \( \mathbb{Q}^* \) of the \( \sigma \)-fields \( \mathcal{H}_t \) and \( \mathcal{F}_t \) when \( \sigma \)-field \( \mathcal{F}_\infty \) is given.

The next lemma explains another property of the random default times when it is conditionally independent.

**Lemma 4.3 (Conditionally Independent of \( \tau_i \))**

For an arbitrary numbers \( t_i \in \mathbb{R}^+, \Gamma^i \) as a given family that follows Assumption 4.13, and \( \tau_i \) as the random times that admits Definition 4.20. Then \( \tau_i \) are conditionally independent with respect to the filtration \( \mathbb{F} \) under \( \mathbb{Q}^* \).
Proof:

Using Lemma 4.2 and Corollary 4.2, and for an arbitrary $T \geq t_i$'s, any fixed $T \in \mathbb{R}^+$, and $\tau_i$ with respect to the filtration $\mathbb{F}$, the following equalities hold:

$$
\mathbb{Q}^*\{\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_T\} = \prod_{i=1}^{n} e^{-\tau_i}
= \prod_{i=1}^{n} \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T\}
$$

where these equalities proves the random times are conditionally independent.

Lemma 4.4 ($\Lambda^i$ Coincides $\Gamma^i$)

If the $\mathbb{F}$-hazard process $\Gamma^i$ that admits Definition 4.15 and follows Assumption 4.13 hold, then the $(\mathbb{F}, \mathcal{G})$-martingale hazard process $\Lambda^i$ of the random default time $\tau_i$ that admits Definition 4.17 coincides with the $\mathbb{F}$-hazard process $\Gamma^i$, i.e.

$$
\Gamma^i = \Lambda^i, \quad \forall i \in \mathbb{K}
$$

Proof:

Using $\tilde{\Gamma}^i_t = \mathcal{H}^i_t - \Gamma^i_t$ is a $\mathcal{G}^i$-martingale, stated in (Bielecki and Rutkowski, 2002a), where $\Gamma^i$ is a $\mathbb{F}$-hazard process and a $(\mathbb{F}, \mathcal{G}^i)$-martingale hazard process of $\tau_i$. It is sufficient to confirm that $\tilde{\Gamma}^i$ is also a $\mathcal{G}$-martingale.

Since the process $\tilde{\Gamma}^i$ is $\mathcal{G}$-adapted, and that for any $t \leq s$ the $\sigma$-fields $\mathcal{G}^i_s$ and $\tilde{\Gamma}_t = \mathcal{H}^i_t \vee \cdots \vee \mathcal{H}^i_{t-1} \vee \mathcal{H}^i_{t+1} \vee \cdots \vee \mathcal{H}^i_T$ are conditionally independent given $\mathcal{G}^i_t$, the subsequent sequence of equalities conclude that $\Gamma^i$ is the $(\mathbb{F}, \mathcal{G})$-martingale hazard process of $\tau_i$:

$$
\mathbb{E}^{Q^*}(H^i_s - \Gamma^i_{s \wedge T} | \mathcal{G}^i_t) = \mathbb{E}^{Q^*}(H^i_s - \Gamma^i_{s \wedge T} | \mathcal{G}^i_t) = \mathbb{E}^{Q^*}(H^i_s - \Gamma^i_{s \wedge T} | \mathcal{G}^i_t \vee \tilde{\Gamma}_t)
= \mathbb{E}^{Q^*}(H^i_s - \Gamma^i_{s \wedge T} | \mathcal{G}^i_t)
$$

This lemma conclude an important property, i.e. the process $\Gamma^i$ represents, both, the $\mathbb{F}$-hazard and the $(\mathbb{F}, \mathcal{G})$-martingale hazard processes of the random time $\tau_i$. 
4.5.3 Dynamically Conditionally Independent Default Times

In this subsection the dynamical conditional independence of random default times with respect to a given filtration $\mathcal{F}$ is introduced. Furthermore, it will be shown in the following that to have a conditional independent random default time, it is essential for the random default time to be dynamically conditionally independent.

**Definition 4.21 (Dynamically Conditionally Independent Default time)**

For any $t \in [0,T]$, arbitrary $t_i \in [0,T]$, random default times $\tau_i$, and since $\mathcal{F}_T \vee \mathcal{H}_t = \mathcal{F}_T \vee \mathcal{G}_t$ for any $t \leq T$. Then $\tau_i$ are dynamically conditionally independent with respect to $\mathcal{F}$ under $\mathbb{Q}^*$, if and only if the subsequent equalities hold:

\[
\mathbb{Q}^*\{\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_T \vee \mathcal{H}_t\} = \prod_{i=1}^{n} \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T \vee \mathcal{H}_t\} = \prod_{i=1}^{n} \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T \vee \mathcal{G}_t\} = \mathbb{Q}^*\{\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_T \vee \mathcal{G}_t\}
\]

The second and third equality hold for the condition of having any $t \leq T$. It is noticeable that the previous property is much stronger than the conditional independence of the time to default.

In the sequel some important blocks will be built in order to examine the implication of having the random default time as dynamical conditional independent.

The following Lemma is a property that will be used commonly when constructing the expectation of a dynamical conditional independent time to default.

**Lemma 4.5 (Exception of a Dynamical Conditional Independent $\tau_i$)**

Let $Y$ be a $\mathcal{G}$-measurable random variable and any sub-$\sigma$-field $\mathcal{F}$ of $\mathcal{G}$ and $\tau_i$ be dynamically conditionally independent random default times, which admits Definition 4.21, be defined on the probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$. Then the subsequent equalities hold:
\[ \mathbb{E}^{Q^*}\{ \mathcal{G}_t \} = \mathbb{E}^{Q}\{ \mathcal{G}_t \} \]

where \( \mathcal{G}_t = \bigwedge_{i=1}^{n} \mathcal{H}^i_t \).

**Proof:**

**Part 1**

With Assumption 4.10 and Corollary 4.1, let

i. \( A \in \mathcal{G}_t \)

ii. \( B \in \mathcal{F} \)

iii. \( C \) denote \( \{ \tau_1 > t, \ldots, \tau_n > t \} \)

Then \( A \cap C = B \cap C \), when either:

i. \( B = \emptyset \), when \( A = \{ \tau_1 \leq u, \ldots, \tau_n \leq u \} \) for some \( u \leq t \)

ii. \( B = A \), when \( A \in \mathcal{F} \)

**Part 2**

By recalling that \( \mathcal{F} \subseteq \mathcal{G}_t \), the following equality could be proved, consecutively to prove the Lemma.

\[ \mathbb{E}^{Q^*}\{ \mathcal{C} \} = \mathbb{E}^{Q}\{ \mathcal{C} \} \frac{\mathbb{E}^{Q^*}\{ \mathcal{C} \} \mathbb{E}^{Q}\{ \mathcal{C} \}}{\mathbb{E}^{Q}\{ \mathcal{C} \} \mathbb{E}^{Q^*}\{ \mathcal{C} \}} \]

To prove this part, it is sufficient to prove that:

\[ \mathbb{E}^{Q^*}\{ \mathcal{C} \mathbb{Q}^*\{ \mathcal{C} \} \} = \mathbb{E}^{Q^*}\{ \mathcal{C} \mathbb{E}^{Q^*}\{ \mathcal{C} \} \} \mathbb{Q}^*\{ \mathcal{C} \} \]

Or equivalently, by taking into consideration part 1, it is sufficient to prove that for any \( A \in \mathcal{G}_t \) we have

\[ \int_A \mathcal{C} \mathbb{Q}^*\{ \mathcal{C} \} d\mathbb{Q}^* = \int_A \mathcal{C} \mathbb{E}^{Q^*}\{ \mathcal{C} \} d\mathbb{Q}^* \]

**Proof of part 2 Lemma 5:**
\[ \int_A J_C Y Q^*(C|F) dQ^* = \int_{A \cap C} Y Q^*(C|F) dQ^* \]
\[ = \int_{B \cap C} Y Q^*(C|F) dQ^* \]
\[ = \int_B J_C Y Q^*(C|F) dQ^* \]
\[ = \int_B E^Q'(J_C Y|F) Q^*(C|F) dQ^* \]
\[ = \int_{B \cap C} E^Q'(J_C Y|F) dQ^* \]
\[ = \int_{A \cap C} E^Q'(J_C Y|F) dQ^* \]
\[ = \int_A J_C E^Q'(J_C Y|F) dQ^* \]

Part 3

As a consequence of part 2, and by \( \mathcal{H}_t \subseteq \mathcal{G}_t \) being hold, the sequence of equalities hold and the prove of the Lemma is completed:

\[ E^Q'(J_C Y|G_t) = J_C E^Q'(Y|G_t) = J_C \frac{E^Q'(J_C Y|F)}{Q^*(C|F)} \]

A direct implication of Lemma 4.5, when having \( t_i \geq t \), is given in the subsequent corollary. The next corollary could be seen as a generalisation of the previous lemma.

Corollary 4.3 (Exception of a Dynamical Conditional Independent \( \tau_i \))

Let \( Y \) be a \( \mathcal{G}_t \) -measurable random variable and any sub-\( \sigma \)-field \( F \) of \( \mathcal{G}_t \) and \( \tau_i \) be dynamically conditionally independent random times, which admits Definition 4.21, be defined on the probability space \( (\Omega, \mathcal{G}, Q^*) \), and \( t_i \geq t \). Then the subsequent equality hold:

\[ Q^*\{\tau_1 > t_1, \ldots, \tau_n > t_n|F \vee \mathcal{H}_t\} = J_{\{\tau_1 > t_1, \ldots, \tau_n > t_n|F\}}^\mathcal{H}_t \]

where \( \mathcal{H}_t = \bigvee_{i=1}^n \mathcal{H}_t^i \).
The above statements have introduced to one of the most important properties the random default time when constricted by the conical approach.

**Theorem 4.1 (Conditional Independent iff Dynamically Independent)**

With respect to filtration $\mathbb{F}$ under $\mathbb{Q}^*$, the random default times $\tau_i$ are conditionally independent if and only if they are dynamically conditionally independent.

**Proof:**

Stating that the random times $\tau_i$ with respect to $\mathbb{F}$ are conditionally independent is equivalent to state that: for an arbitrary subsets $A_i \in [0, T]$ and $\forall T \in \mathbb{R}^+$ the following equality is valid:

$$
\mathbb{Q}^*\{\tau_1 \in A_1, \cdots, \tau_n \in A_n | \mathcal{F}_T\} = \prod_{i=1}^{n} \mathbb{Q}^*\{\tau_i \in A_i | \mathcal{F}_T\}
$$

and as a consequence, when $\mathcal{F}_T$ is given:

i. It implies that the $\sigma$-fields $\mathcal{H}_t^i$ are mutually conditionally independent for any $t \leq T$.

ii. $\mathcal{H}_t^i \subseteq \mathcal{H}_t^1$ and the $\sigma$-fields $\mathcal{H}_t^i$ and $\mathcal{H}_t^i$ are conditionally independent for $t \leq t_i \leq T$ and $\mathcal{H}_t^i := \mathcal{H}_t^1 \vee \cdots \mathcal{H}_t^{i-1} \vee \mathcal{H}_t^{i+1} \vee \cdots \mathcal{H}_t^n$ by using these implications and Corollary 4.3, the following bi-equalities are accepted:

$$
\mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T \vee \mathcal{H}_t^i\} = \mathbb{Q}^*\{\tau_i > t_i | \mathcal{F}_T \vee \mathcal{H}_t^i\}
$$

again by using Lemma 4.5, setting $\mathcal{F} = \mathcal{F}_T$, and using the conditional independence of the random times $\tau_i$, the preceding equality is equal to Corollary 4.3.

The subsequent chain of equality concludes that the conditional independence implies the dynamical conditional independence, which is enough to prove the articulated theorem:
\[ \mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T \vee \mathcal{H}_t \} = \prod_{i=1}^{n} \mathcal{J}_{(\tau_i > t)} \mathbb{Q}^*\{\tau_i > t | \mathcal{F}_T \} \]

4.5.4 Dynamically Conditional Independent Default Times Properties

In this section the properties of the conditional independent default times in Subsection 4.4.2 could be re-expressed and generalised to be valid for the scope of having a dynamically conditional independent default times. The proof is just an immediate result of the Lemma’s in Subsection 4.5.2 by the mean of Theorem 4.1, and thus their proofs are left to the reader.

**Lemma 4.6 (Dynamically Conditionally Independent Default time)**

For \( \Gamma^i \) as a given family that follows Assumption 4.13 and the random default times \( \tau_i \) as defined in Definition 4.20 with the respect to the filtration \( \mathcal{F} \) under \( \mathbb{Q}^* \), \( \tau_i \) are dynamically conditionally independent.

This property is, again, another angle that represents the required condition of having conditional independent random default time if and only if the random default time is a dynamical independent.

**Lemma 4.7 (Joint Dynamical Conditional Probability on \( \mathcal{F}_\omega \))**

Let \( \tau_i \) be the random default times that admits Definition 4.21, \( \Gamma^i \) be a given family that follows Assumption 4.13, then for every \( t_i \in \mathbb{R}^+ \) and \( t_i \in [t, \infty) \) the joint conditional probability of survival satisfies the subsequent equality:

\[ \mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_\omega \vee \mathcal{G}_t \} = \mathcal{J}_{(\tau_i > t)} e^{-\sum_{i=1}^{n}(t_i - \tau_i)} \]

The property is an enlargement representation of Lemma 4.1, i.e. it includes the default process indicator.
Lemma 4.8 (Joint Dynamical Conditional Probability on $\mathcal{F}_T$)

For arbitrary numbers $t \in [0,T)$, any $t_i \in [t,T]$, $\Gamma^i$ as a given family that follows Assumption 4.13, and the random default times $\tau_i$ as defined in Definition 4.21, the joint conditional probability of survival satisfies the subsequent equality:

$$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T \mathcal{\lor} \mathcal{G}_t\} = \mathcal{I}_{\{\tau_1 > t_1, \ldots, \tau_n > t_n\}} e^{-\sum_{i=1}^{n}(\Gamma^i - \Gamma^i_t)}$$

Again, this property coincides Lemma 4.2 with an enlarged view the incorporate the default process indicator.

4.5.5 Minimum of Default Times

In this subsection, the minimum of default times $\tau_i$ and their relations with processes as $\Gamma^i$ will be discussed. However, this problem could not be solved in a general form, where the knowledge of the time to default joint probability law role. Indeed, specifying the time to default joint probability assumptions and the choice of the filtration solves this problem.

In order to examine the minimum of default times $\tau_i$ and their hazard processes $\Gamma^i$ under different filtration, the $\mathbb{F}$-hazard process $\Gamma^i$ and $\Gamma^{1st}$ are going to be represented.

In view of Lemma 4.2 and Definition 4.16, the subsequent corollary is an immediate result.

Corollary 4.4 ($\mathbb{F}$-hazard process $\Gamma^i$)

If each hazard process $\Gamma^i$ that follows Assumption 4.13 admits the $\mathbb{F}$-intensity $\gamma^i$ that follows Definition 4.16, then for any arbitrary numbers $t_i \in \mathbb{R}^+$ and $T \geq \max(t_1, \ldots, t_n)$ the random default times $\tau_i$ that admits Definition 4.20 could be given by the subsequent equality:

$$\mathbb{Q}^*\{\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T\} = \prod_{i=1}^{n} e^{-\int_{0}^{t_i} \gamma^i_u du}$$
Lemma 4.9 (\( \mathbb{F} \)-hazard process \( \Gamma_{1^{st}} \))

The \( \mathbb{F} \)-hazard process \( \Gamma_{1^{st}} \) of the \( \tau_{1^{st}} \) random default time that follows Assumption 4.5 and under \( \sigma \)-field \( \mathcal{G}_t \) stated in Corollary 4.1 satisfies the subsequent equalities:

\[
\Gamma_{1^{st}} = \sum_{i=1}^{n} \Gamma_{i}^{t} = \sum_{i=1}^{n} \left( t_{i} \int_{0}^{\tau_{i}} \gamma_{i}^{t} du \right)
\]

Proof:

With Definition 4.16, Assumption 4.5, Corollary 4.1, Lemma 4.4, and Lemma 4.2 implies that:

\[
e^{-\Gamma_{1^{st}}} = \mathbb{Q}^{*}\{\tau_{1^{st}} > t | \mathcal{F}_T\} = \mathbb{Q}^{*}\{\tau_{1} > t, \ldots, \tau_{n} > t | \mathcal{F}_T\} = e^{-\sum_{i=1}^{n} \Gamma_{i}^{t}}
\]

After these two results, the \( 1^{st} \) time to default will be examined as \( \mathcal{G} \)-measurable random variable.

Lemma 4.10 (\( \tau_{1^{st}} \) to Default under \( \mathcal{G}_t \))

For any \( \mathcal{G} \)-measurable random variable \( Y \), any \( t \leq s \), and under Assumption 4.10, the following equality holds:

\[
\mathbb{E}^{\mathbb{Q}^{*}}(\mathbb{I}_{\{\tau_{1^{st}} > s\}} Y | \mathcal{G}_t) = \mathbb{E}^{\mathbb{Q}^{*}}(\mathbb{I}_{\{\tau_{1^{st}} > t\}} Y e^{\Gamma_{1^{st}}^{t}} | \mathcal{F}_t)
\]

Proof:

In view of Lemma 4.5 and its proof and Corollary 4.3, it is enough to mention that

\[
\mathbb{I}_{\{\tau_{1^{st}} > t\}} \mathbb{I}_{\{\tau_{1^{st}} > s\}} = \mathbb{I}_{\{\tau_{1^{st}} > s\}}
\]

to complete the proof.

Changing the direction by inspecting the \( 1^{st} \) time to default as \( \mathcal{F}_s \)-measurable random variable will give another result; as expressed in the next corollary.
Corollary 4.5 (τ^{1st} to Default under $\mathcal{F}_s$)

Let $Y$ be $\mathcal{F}_s$-measurable random variable, then for any $t \leq s$, and under assumption 4.10, the following equality holds:

$$
\mathbb{E}^{Q^*}\left( J_{\{\tau^{\text{1st}} > s\}} Y \big| \mathcal{G}_t \right) = J_{\{\tau^{\text{1st}} > t\}} \mathbb{E}^{Q^*}\left( Y e^{t \tau^{\text{1st}} - r_s^{\text{1st}}} \big| \mathcal{F}_t \right)
$$

Proof:

In consideration of Lemma 4.10 and Lemma 4.9, the chain of equalities holds and proves the corollary:

$$
\mathbb{E}^{Q^*}\left( J_{\{\tau^{\text{1st}} > s\}} Y \big| \mathcal{G}_t \right) = J_{\{\tau^{\text{1st}} > t\}} \mathbb{E}^{Q^*}\left( J_{\{\tau^{\text{1st}} > s\}} Y e^{r_s^{\text{1st}}} \big| \mathcal{F}_t \right) = J_{\{\tau^{\text{1st}} > t\}} \mathbb{E}^{Q^*}\left( Q^*(t > s|\mathcal{F}_s) Y e^{t \tau^{\text{1st}} - r_s^{\text{1st}}} \big| \mathcal{F}_t \right) = J_{\{\tau^{\text{1st}} > t\}} \mathbb{E}^{Q^*}\left( (1 - F_s) Y e^{t \tau^{\text{1st}}} \big| \mathcal{F}_t \right)
$$

It is significant to remark that when $Y$ is $\mathcal{G}_s$-measurable random variable, rather than $\mathcal{F}_s$-measurable random variable, Corollary 4.5 is still valid and could be rephrased as the subsequent corollary.

Corollary 4.6 (τ^{1st} to Default under $\mathcal{G}_s$)

For any $\mathcal{G}_s$-measurable random variable $Y$, any $t \leq s$ and under Assumption 4.10, the subsequent equality is valid:

$$
\mathbb{E}^{Q^*}\left( J_{\{\tau^{\text{1st}} > s\}} Y \big| \mathcal{G}_t \right) = \mathbb{E}^{Q^*}\left( J_{\{\tau^{\text{1st}} > s\}} Y \big| \mathcal{G}_t \right)
$$

Where $\mathcal{G} = \mathcal{F} \vee \mathbb{H}^{\text{1st}}$ and $\mathbb{H}^{\text{1st}}$ is generated by the process $\mathbb{H}_t^{\text{1st}} = J_{\{\tau^{\text{1st}} \leq t\}}$, in other words $\mathcal{G}$ represents the filtration associated with $\tau^{\text{1st}}$. 
4.6 Mathematical Summary

Basic Definitions of Measure, Stochastic Process, and Martingale

- \( \sigma \)-algebra is a collection \( \mathcal{D} \in \mathfrak{D} \), \( \emptyset \in \mathcal{D} \). If \( \{ X_n : X_n \in \mathcal{D}, n \geq 1 \} \), then \( \bigcup_{n \geq 1} X_n \in \mathcal{D} \), and \( \forall X \in \mathcal{D}, X^c \in \mathcal{D} \).

- Its measurable space is \( (\mathfrak{D}, \mathcal{D}) \).

- When \( \alpha : \mathcal{D} \rightarrow [0, \infty] \), \( \alpha(\emptyset) = 0 \), and \( \forall \{ X_n, n \geq 1 \} \) of disjoint elements of \( \mathcal{D} \), \( \alpha(\bigcup_{n \geq 1} X_n) = \sum_{n \geq 1} \alpha(X_n) \) then \( \alpha \) is called the Measure.

- This measure is finite when \( \alpha(\mathfrak{D}) < \infty \), where its mass is equal to the quantity \( \alpha(\mathfrak{D}) \).

- If \( \alpha(\mathfrak{D}) = 1 \), \( \alpha \) is the probability measure and \( (\mathfrak{D}, \mathcal{D}, \alpha) \) is its probability space.

- On \( (\mathfrak{D}, \mathcal{D}) \) and \( (\mathcal{G}, \mathcal{G}) \), \( f : \mathfrak{D} \rightarrow \mathcal{G} \) is a measurable function if \( \forall X \in \mathcal{D}, \exists f^{-1}(X) = \{ x \in \mathfrak{D} : f(x) \in X \} \in \mathcal{G} \)

- On \( (\mathfrak{D}, \mathcal{D}, \alpha) \) and \( (\mathcal{G}, \mathcal{G}) \), \( X : \mathfrak{D} \rightarrow \mathcal{G} \) is a random variable.

- On \( (\mathfrak{D}, \mathcal{D}, \alpha) \), \( \mathcal{X} = \{ X_t, 0 \leq t \leq T \} \) is a stochastic process.

- \( \mathcal{X} = \{ X_t \}_{t \in [0, T]} \) is a \( \mathcal{F}_t \) martingale if \( \mathcal{X} \) is adapted to the filtration \( \{ \mathcal{F}_t \}_{t \in \mathbb{R}^+} \).

- \( \mathbb{E}[|X_t|] \) is \( < \infty \) for each \( t \), \( \forall s, t : s \leq t, \exists \mathbb{E}[X_t | \mathcal{F}_s] = X_s \).

Assumptions and Definitions

- The random default times \( \tau_i \) to model the \( n \) underlying credit entities, and defined on \( (\Omega, \mathcal{G}, \mathbb{Q}^*) \), where \( \mathcal{G} = \mathcal{F} \vee \mathcal{H} \).

- Counter Process: \( \mathcal{N}^i_t = \mathcal{I}_{(\tau_i \leq t)}, \mathcal{N}_t = \sum_{i=1}^n \mathcal{N}^i_t, \) and \( \mathcal{N}_t^{(-i)^{th}} = \sum_{j \neq i} \mathcal{N}_t^j \)

- \( \mathcal{N}_t^i \) is equal under \( \mathcal{F}_T \vee \mathcal{H}_t \) and \( \mathcal{F}_t \vee \mathcal{H}_t^i \)

- \( \mathbb{Q}^*(\tau_i > t|\mathcal{F}_T) = \mathbb{Q}^*(\tau_i > t|\mathcal{F}_U), \forall t_i \in [0, T], and u \in [0, T] \)

- Default process \( F_t = \mathbb{Q}^*\{\tau \leq t|\mathcal{F}_T\} \) and survival process \( G_t := 1 - F_t \)
- **F-Hazard Process and its intensity** \( \Gamma_t = -\ln(1 - F_t) = \int_0^t \gamma_u du \) are equal to the \((F, G)\)-martingale hazard process and its intensity \( \Lambda_t = \int_0^t \lambda_u du \).

- \( \tau_i = \inf\{t \in \mathbb{R}^+: \Gamma_t^i \geq -\ln \xi_i\} \)

**Conditionally Independent Default Times**

\[
\mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_\infty) = \prod_{i=1}^n \mathbb{Q}^*(\tau_i > t_i | \mathcal{F}_T) = \prod_{i=1}^n \mathbb{Q}^*(\tau_i > t_i | \mathcal{F}_\infty)
\]

- \( \mathbb{P} = \prod_{i=1}^n e^{-r_i t_i} \)

**Dynamically Conditionally Independent**

\[
\mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_\infty \vee \mathcal{G}_t) = \mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathcal{F}_T \vee \mathcal{G}_t)
\]

- \( \mathbb{P} = \prod_{i=1}^n \mathbb{Q}^*(\tau_i > t_i | \mathcal{F}_T \vee \mathcal{H}_t) = \prod_{i=1}^n \mathbb{Q}^*(\tau_i > t_i | \mathcal{F}_T \vee \mathcal{H}_t) e^{-\sum_{i=1}^n (r_i - r_i t_i)} \)

*The 4th equality means that \( \tau_i \) are conditionally independent iff they are dynamically conditionally independent*

**Minimum of Default Times**

- \( \Gamma_t^{1st} = \sum_{i=1}^n \Gamma_t^i \)
- \( \forall t \leq s, \text{If } Y \text{ a measurable random variable of:} \)

\[
\begin{align*}
\mathcal{G}_s : & \quad \mathbb{E}^* \left( \mathcal{J}_{[\tau_1^{1st} > s]} \right) \mathbb{E}^* (Y) = \mathcal{J}_{[\tau_1^{1st} > t]} \mathbb{E}^* \left( \mathcal{J}_{[\tau_1^{1st} > s]} \right) Y e^{\Gamma_t^{1st}} | \mathcal{F}_t \\
\mathcal{F}_s : & \quad \mathbb{E}^* \left( \mathcal{J}_{[\tau_2^{1st} > s]} \right) \mathbb{E}^* (Y) = \mathcal{J}_{[\tau_1^{1st} > t]} \mathbb{E}^* (Y) e^{\Gamma_t^{1st} - r_{2^{1st}}} | \mathcal{F}_t \\
\mathcal{G}_s : & \quad \mathbb{E}^* \left( \mathcal{J}_{[\tau_3^{1st} > s]} \right) \mathbb{E}^* (Y) = \mathbb{E}^* \left( \mathcal{J}_{[\tau_1^{1st} > s]} \right) \mathbb{E}^* (Y) | \mathcal{G}_t
\end{align*}
\]
Chapter Five

New Approach of the Linearly Correlated, Stochastically Correlated, and Randomly Loaded Factor Copula Models via Lévy Process

5.1 Outline

- Introduction
- Brownian Motion
- Lévy Process
- Lévy Factor Model
- Lévy Factor Copula Model
- Lévy Stochastic Correlated Factor Copula Model
  - Stochastic Correlation: Binary Structure Case
  - Stochastic Correlation: Symmetric Dependence Structure Case
- Lévy Random Factor Loading Copula Model
- Mathematical Summary
5.2 Introduction

The valuation of first to default swaps has primarily relied on reduced form models, as in (Duffie, 1998b), and produces simple expressions of prices. Nevertheless, no such simple consequences could be originated for more general basket credit derivatives or even CDO’s. According to (Duffie and Gârleanu, 2001), because of the dependence of default times is modelled through correlated stochastic risk intensities, Monte Carlo is a reasonable approach to achieve the levels of dependencies needed and could introduce jumps and pricing of CDO’s. An alternative approach is rooted by a multivariate extension of the Cox process approach, which was pioneered in (Lando, 1998). This has consequences in a series of models for instance the Gaussian copula approach initiated for the pricing of basket credit derivatives in (Li, 1999) and (Li, 2000). The multivariate exponential copula of (Marshall and Olkin, 1967), see as well (Duffie and Singleton, 1998a), (Li, 2000), (Kijima, 2000), presents an alternative framework that takes into consideration simultaneous defaults and is associated with non-smooth joint distribution functions. (Schönbucher and Schubert, 2001) has review the dynamics of default intensities and demonstrate that Clayton copulas, an associate of the Archimedean copula family, are interrelated to the dependent intensities approaches of (Kusuoka, 1999), (Davis and Lo, 2001), (Jarrow and Yu, 2000). Subsequently, (Bouyé et al., 2000), (Schmidt and Ward, 2002), (Gregory and Laurent, 2003), and (Laurent and Gregory, 2005) as well consider a number of copulas pricing of basket credit derivatives models.

Conversely, latent factor models have been broadly intended for the computation of default events as well as to loan loss distributions (see (Koyluoglu and Hickman, 1998), (Belkin et al., 1998), (Finger, 1999), (Crouhy et al., 2000b), (Merino and Nyfeler, 2002), (Gordy, 2002), and (Schönbucher, 2002)). The one factor Gaussian model was
compatible for analytical computation of loss distributions as noticed by (Vasicek, 1997). In the same track, the recent Basel agreement relies on such models. The statistical literature has methodically studied the Latent factor models. In the credit zone, (Frey et al., 2001) has associate factor with copula approaches. The foremost feature of these models is that default events are independent, conditionally on some latent state variables. This simplifies the computation of aggregate loss distributions due to dimensionality reduction. This factor method is harmonious for huge dimensional problems. Since semi-explicit expressions of most relevant quantities can be obtained, it provides an alternative route to Monte Carlo approaches, while the later could be utilised when useful.

As Li (2000) became the market standard Model “Gaussian Copula Model”, which is a linear correlation approach with deterministic parameter, for valuation of nth to default CDS, CDO’s, and some other credit derivatives products, various problems has faced practitioners and especially researchers, for instance it was unqualified to fit the market tranches, where it over-prices the mezzanine tranche and under-prices the equity and senior tranches. As a consequence, practitioners and principally researchers have enriched the literature18 trying to integrate skewness features within the standard model, where this route has made some improvements on the standard models and minimise its deficiencies. But sill further enhancements were needed to overcome these limitations.

Conversely to the linear correlation approach with deterministic parameter, i.e. “Gaussian Copula Model”, correlated skewed models have enhanced some of based model drawbacks, where it was prospected that it is going be the next generation credit derivative valuation models (Burtschell et al., 2007). In this direction, in (Burtschell et al., 2005, Burtschell et al., 2008) and (Schloegl, 2005) the standard model have been

18 See chapter six “Lévy Factor Copula and its Skewed Version from Theory to Application” for more details.
skewed the model’s correlation by a stochastic correlation, where in (Andersen and Sidenius, 2005) by a stochastic risk exposure.

As stated previously, numerous models have incorporate skew futures in contradictory to the normality assumption. But almost none of them, except in (Brunlid, 2006) the copula was managed by the asset default instead of the time to defaults, have generalised this model in a way that take this model out of this assumption, where they had tried to replace the Gaussian distribution with skewed one.

This chapter proposes enlarging and standardising the microscopic way of investigating this problem. Firstly, by introducing the Lévy processes as a base model, which is called “Lévy Factor Copula Model”, in preference to “Gaussian Factor Copula Model”.

Secondly, by expanding the Lévy Factor Copula Model to incorporate the enhanced correlation skewed models mentioned previously. Consequently to these proposed enlargements, the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ of the Gaussian Factor Copula Model could be replaced by any distribution, which admits Lévy process definition and its properties with three restrictions: firstly, they have to be infinitely divisible distributions. Secondly they have zero mean and, finally, they have equal finite variance.

This chapter starts by reviewing and building the geometric Brownian motion model in Subsection 5.3 and then independently presenting the Lévy processes in Subsection 5.4 and concluding it by proving that the Brownian motion is a Lévy processes. Subsequently, the latent factor model is launched and connected to the Copula Model as the first main block, called “Lévy Factor Copula Model”, in Subsections 5.5 and Subsection 5.6. Consequently, the second block is built in Subsection 5.7 and called as “Stochastic Correlated Lévy Factor Copula Model”. It is built in two different manners: Binary Structure Case and Symmetric Dependence Structure Case. The third and final
block is built in Subsection 5.8 and called the “Lévy Random Factor Loading Copula Model”.

**5.3 Brownian Motion**

Bachelier (1900) had utilised stochastic processes to model the financial market; as he modelled the stocks prices of Paris Bourse as a stochastic process that has an independent and stationary increments, where these increments follows a Gaussian distributions, called Brownian motion or Wiener-process in his PhD thesis (Bachelier, 1900) and translated later in English in (Cootner, 1964) and (Bachelier et al., 2006). (Samuelson, 1965) proposed the geometric Brownian motion that models a logarithmic Brownian motion of the stock prices, which was proven as more suitable than the earlier model.

This section starts by reviewing the definitions and properties of the Brownian motion and geometric Brownian motion and concluding by Merton’s (1974) model.

This section will start with the Gaussian distribution as a first block to build the Brownian motion. The Gaussian distribution or called the Normal distribution is one of the most commonly used distributions. It could be applied to many areas and fits many situations. The Gaussian distribution looks like a classical bell-shaped curve.

**Definition 5.1 (Gaussian Density Function)**

A random variable \( X \) is said to be Gaussian, denoted by \( X \sim \mathcal{N}(\mu, \sigma) \), if its density is given by the subsequent equality:

\[
f_{X \sim \mathcal{N}(\mu, \sigma)}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

The Gaussian distribution exist on the real line, its mean belongs to the real line, its variance is always bigger than zero, it is symmetric around the mean, and always has a kurtosis equal to three. For more details and mathematical representation, see
Subsection 6.3.1.

The second block needed to build the Brownian motion is to define the stationary and independent increment. This will be achieved by defining them separately and then combine them as a result.

**Definition 5.2 (Stationary Increment)**

Let \( (W_t)_{t \in \mathbb{R}} \) be a stochastic process that admits Definition 4.10, then \( W \) is said to have a stationary increment when the distribution function of \( W_{s+t} - W_s \) is equal for every \( s \in \mathcal{T} \), i.e. \( s + t \in \mathcal{T} \).

Here the point is that the process generated in each increment has an equal distribution.

**Definition 5.3 (Independent Increments)**

Let \( (W_t)_{t \in \mathbb{R}} \) be a stochastic process defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{Q}) \) and \( s, t \in \mathbb{R}^+ \), where \( t > s \), then it is said to be an independent increment if \( W_t - W_s \) is independent of \( \mathcal{F}_s \).

Just to combine the previous definition with the fundamental probability laws, recall that if there are two events \( X \) and \( Y \). These two events are said to be independent if and only if the occurrence of \( X \) does not affect the occurrence of \( Y \), i.e. \( \mathbb{P}(X \cap Y) = \mathbb{P}(X)\mathbb{P}(Y) \).

The next lemma constructs the stationary independent increments concept from its main components.

**Lemma 5.1 (Stationary Independent Increments)**

Let \( (W_t)_{t \in \mathbb{R}} \) be a stochastic process that possesses, both, stationary increments that admit Definition 5.2, and independent increments that follows Definition 5.3, then \( W_t \) is said to have stationary independent increments.

It is important to investigate the Brownian motion’s path; if its variation is finite or infinite. This concept is defined in the following definition.
Definition 5.4 (Finite and Infinite Variation)

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a stochastic process, \([a, b]\) be a generic time interval, which is partitioned by \(n\) intervals, i.e. \(\mathcal{T} = \{a = t_1 < \cdots < t_{n+1} = b\}\), and finally \(\mathcal{V}\) as the variation of \(\mathcal{W}\) over the partitioned time intervals \(\mathcal{T}\), i.e. \(\mathcal{V}(\mathcal{T}) = \sum_{t=1}^{n} |\mathcal{W}(t_{i+1}) - \mathcal{W}(t_i)|\), then:

i. If \(\sup_{\mathcal{T}} \mathcal{V}(\mathcal{T}) = \infty\), \(\mathcal{W}_t\) is said to be infinite variation

ii. If \(\sup_{\mathcal{T}} \mathcal{V}(\mathcal{T}) < \infty\), \(\mathcal{W}_t\) is said to be finite variation

Definition 5.5 (Brownian Motion)

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a stochastic process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) with \(\mathcal{W}_0 = 0\), then \(\mathcal{W}_t\) is said to be a Brownian motion\(^{19}\) if the subsequent conditions hold:

1. The process has a stationary independent increment that admits Lemma 5.1.

2. For \(s, t \in \mathbb{R}^+\) and \(t > s\), the increments \(\mathcal{W}_t - \mathcal{W}_s\) are distributed in accordance with the Gaussian Distribution \(f_{X_0(t-s)}\).

Recalling Definition 5.4, the paths of Brownian motion are continuous but very unpredictable. Moreover, it can be validated that they are of infinite variation on any closed time interval.

There are other important properties that a Brownian motion encloses. The following are some of these properties.

Property D5.5.1 (Brownian Motion: Scaling)

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a Brownian motion that admits Definition 5.5, then any scaled \(\mathcal{W}\), denoted by \(\widehat{\mathcal{W}}\), is also a Brownian motion, i.e. \((\widehat{\mathcal{W}}_t)_{t \in \mathbb{R}^+} = \left(\frac{\alpha \mathcal{W}_t}{\alpha t}\right)_{t \in \mathbb{R}^+, \alpha \neq 0}\).

Scaling property is a simple consequence of the definition. It is an important property and one of the block those easy the process of sampling and simulating the path in some

\(^{19}\) Or called also Wiener process.
cases. The following property, i.e. Brownian motion is martingale, is also an immediate result from the Brownian motion definition.

**Property D5.5.2 (Brownian Motion: Martingale)**

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a Brownian motion that admits Definition 5.5. Then \(\mathcal{W}_t\) is a martingale, i.e. \(\mathbb{E}[\mathcal{W}_t | \mathcal{F}_s] = \mathcal{W}_s\).

**Property D5.5.3 (Brownian Motion: Discretising)**

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a Brownian motion that admits Definition 5.5, \(\Delta t\) as a small length step, where \(\Delta t \to 0\), and \(v\) be the value of the sampled Brownian motion random Standard Gaussian variables. If \(\mathcal{W}_0 = 0\), then \(\mathcal{W}_{n\Delta t}\) is given by the subsequent equality:

\[
\mathcal{W}_{n\Delta t} = \mathcal{W}_{(n-1)\Delta t} + \sqrt{\Delta t} \, v_n
\]

The discretising property is an important step required to sample and simulates the Brownian motion path.

By applying Definition 5.4, i.e. Finite and Infinite Variation, on Definition 5.5, i.e. Brownian motion, the subsequent property is an immediate result.

**Property D5.5.4 (Brownian Motion: Finite and Infinite Variation)**

Let \((\mathcal{W}_t)_{t \in \mathbb{R}^+}\) be a Brownian motion that admits Definition 5.5, then it follows Definition 5.4 of having a finite or an infinite variation.

As introduced, there is another process, i.e. Geometric Brownian motion, which is closely associated to Brownian motion. It was proposed in (Samuelson, 1965). It is one of the most popular processes in finance. It has many implications such as: it models a logarithmic Brownian motion of the stock prices and it is the foundation of the Black-Scholes model for stock-price dynamics in continuous time, which was proven as more suitable than the earlier model.
Definition 5.6 (Geometric Brownian)

Let \((W_t)_{t \in \mathbb{R}^+}\) be a standard Brownian motion that admits Definition 5.5, \((A_t)_{t \in \mathbb{R}^+}\) be a stochastic process, \(r\) be drift parameter, and \(\sigma\) as a standard deviation, then \(A_t\) is said to be a geometric Brownian Motion if for \(A_0 > 0\), the subsequent stochastic differential equation hold:

\[dA_t = A_t (rdt + \sigma dW_t)\]

Subsequently, to the previous definition, a direct result could be obtained by applying the exponential laws.

Lemma 5.2 (Geometric Brownian)

Let \((W_t)_{t \in \mathbb{R}^+}\) be a standard Brownian motion that admits Definition 5.5, \((A_t)_{t \in \mathbb{R}^+}\) be a geometric Brownian Motion that admits Definition 5.6, \(r\) be drift parameter, and \(\sigma^2\) is the volatility parameter, then \(A_t\) has the unique solution given by the subsequent equality:

\[A_t = A_0 e^{\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}\]

Proof:

Applying the Ito’s lemma with \(f(A) = \log(A)\), the following sequence of equalities hold:

\[d \log(A) = f'(A)dA + \frac{1}{2}f''(A)A^2 \sigma^2 dt\]
\[= \frac{1}{A}(A dW + rA dt) - \frac{1}{2} \sigma^2 dt\]
\[= \sigma dW + \left(r - \frac{\sigma^2}{2}\right) dt\]

And it follows that

\[\log(A_t) - \log(A_0) = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\]

Finally, by taking the exponential of the pervious equation, the lemma is proved.
In order to standardise the previous Lemma in the context of this paper, the subsequent Theorem “Merton’s Model” will be introduced as in (Merton, 1974).

**Theorem 5.1 (Merton’s model)**

Let \((\mathcal{A}_t)_{t \in [0,T], i \in \mathbb{K}}\) be the firm value process that follows a geometric Brownian motion defined under a common space \((\Omega, \mathcal{F}, \mathbb{Q}^*)\), that admits Lemma 5.2, with \(r\) as a constant drift, \(\sigma\) as a constant standard deviation and \((W_t)_{t \in [0,T]}\) as a Brownian motion, that admits Definition 5.5. Then the value of a firm \(i\) at time \(t\) is given by the subsequent equality:

\[
\mathcal{A}_{t_i} = \mathcal{A}_{0_i} e^{\left(\left(r-\frac{\sigma^2}{2}\right)t + \sigma W_t\right)}
\]

### 5.4 Lévy Process

Contradictory to the normality assumption and Gaussian distributed increments that are given by the process that admits Brownian motion definition, Paul Lévy (1886-1971) has pioneered non-Gaussian processes with independent and stationary increments, which was named after him as “Lévy processes”. Lévy process is connected with the infinitely divisible laws, through “Lévy-Khintchine formula” their distributions are characterised, and their structure are described by “Lévy-Itô decomposition”. Paul Lévy scientific work has been gathered in (Loève, 1973).

This subsection is built on the “Brownian motion” subsection and connected to it in some parts. Previous description of the Lévy process outlines the structure of this subsection. In this subsection only the necessary definitions, theorems, and their proofs are stated. There are many good monographs that review carefully the Lévy process, for example: (Sato, 1999), (Schoutens, 2003), and (Cont and Tankov, 2004); as this subsection is mainly referred to, while for comprehensive overviews of its applications...
see (Prabhu, 1998), (Barndorff-Nielsen et al., 2001), (Kyprianou et al., 2005), and (Kyprianou, 2006).

In probability theory, and as could be seen in the subsequent definition, the distribution and the density functions of a random variable are completely defined by their corresponding characteristic function. Therefore it provides the foundation of an alternative method to analytically evaluating the probability of occurrence instead of working directly with the probability distribution and density functions.

There are principally simple consequences for the characteristic functions of distributions described by the weighted sums of random variables. Additionally to univariate distribution functions of a random variable, the characteristic functions can represent vector-valued or matrix-valued random variables. Moreover, it could be extended to incorporate with more complicated and generic situations. These and some other properties will be stated below.

**Definition 5.7 (Characteristic Function \( \varphi \))**

Let \( F_X(x) \) and \( f_X(x) \) be, respectively, a distribution and density function of a random variable \( X \) those, respectively, admit Definition 3.6 and Definition 3.7, and \( j \) be the imaginary number, i.e. \((i^2 = -1)\), then the characteristic function \(^{20}\) of a random variable \( X \), denoted by \( \varphi_X \), is given by the subsequent equalities:

\[
\varphi_X(u) = \mathbb{E}[e^{(jux)}] = \int_{-\infty}^{\infty} e^{(jux)} dF(x) = \int_{-\infty}^{\infty} e^{(jux)} f_X(x) dx
\]

Furthermore, the probability distribution of the random variable \( X \)’s behaviour and properties are completely determined by its corresponding characteristic function. The

---

\(^{20}\) It could be seen as a Fourier-Stieltjes Transform of the distribution function.
two methods are analogous in the sense that knowing one of them leads to find the corresponding one. However, each of them may introduce different insight for understanding the features of the random variable $X$.

The next corollary gives an example of how the characteristic function could be extract from its corresponding density function, i.e. applying the characteristic function, which is defined in Definition 5.7, on the Gaussian density function, which is defined in Definition 5.1, to extract the corresponding Gaussian characteristic function.

**Corollary 5.1 (Gaussian Characteristic Function)**

Let $X_{\mathcal{G}(\mu, \sigma)}$ be a Gaussian random variable that admits Definition 5.1, then its characteristic function, which admits Definition 5.7, is given by the subsequent equality:

$$
\phi_{X_{\mathcal{G}(\mu, \sigma)}}(u) = \int_{-\infty}^{\infty} e^{(jux)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = e^{(j\mu u - \frac{1}{2}\sigma^2 u^2)}
$$

Some of the characteristic function properties are articulated below.

**Property D5.7.1 (Characteristic Function: Existence)**

For any random variable $X$, there exists a continuous corresponding characteristic function $\phi_X$.

When the characteristic function is considered as a function of a real-valued argument, it always exists, unlike the moment-generating function. The existence of the moment-generating and density functions depends on the behaviour and properties of the characteristic function. Therefore, the characteristic function could be seen as a strong alternative as stated above to the distribution function.

**Property D5.7.2 (Characteristic Function: Independent Sequence)**

Let $X_n$ be a sequence of independent random variables, where $n \in \mathbb{N}$, then the
characteristic function $\varphi_X$, which admits Definition 5.7, is given by the subsequent equality:

$$\varphi_X(u) = \varphi_{X_1}(u)\varphi_{X_2}(u) \cdots \varphi_{X_n}(u)$$

The independence sequence property of the characteristic function is one of its most important properties; this method is principally valuable when analysing linear combinations of independent random variables. Additionally, it has an important application when a sequence of random variables is needed to be decomposed. Furthermore, it is the base for another important concept, i.e. the “infinitely divisible distribution”.

The next three properties, i.e. cumulant function, moment generating function, cumulant characteristic function, are some related functions those correspond to the characteristic function and often appear in the literature.

**Property D5.7.3 (Characteristic Function: Cumulant Function)**

Let $\varphi_X$ be the characteristic function of a random variable $X$ that admits Definition 5.7, then the cumulant function, denoted by $\kappa_X$, is related to the characteristic function by the subsequent equality:

$$\varphi_X(ju) = e^{\kappa_X(u)} = e^{\mathbb{E}[e^{-juX}]}$$

**Property D5.7.4 (Characteristic Function: Moment Generating Function)**

Let $\varphi_X$ be the characteristic function of a random variable $X$ that admits Definition 5.7, then the moment generating function, denoted by $\psi_X$, is related to the characteristic function by the subsequent equality:

$$\varphi_X(-j\bar{u}) = e^{\psi_X(u)} = e^{\mathbb{E}[e^{juX}]}$$

As stated previously, the moment-generating function does not always exist and thus the characteristic function is a better representation.
**Property D5.7.5 (Characteristic Function: Cumulant Characteristic Function)**

Let \( \varphi_X \) be the characteristic function of a random variable \( X \) that admits Definition 5.7, then the cumulant characteristic function or characteristic exponent function, denoted by \( \phi_X \), is related to the characteristic function by the subsequent equality:

\[
\varphi_X(u) = e^{\phi_X(u)} = e^{\ln(e^{(juX)})}
\]

The following property manipulates the one-to-one correspondence between the characteristic function and the density function or the distribution function. Therefore, it is possible to it is always possible to obtain one of them when the other one is known. This property is known as the characteristic inversion method or the extraction method.

**Property D5.7.6 (Characteristic Function: Extracting the Density Function)**

Let \( f_X(x) \) and \( \varphi_X(u) \) be, respectively, a density and a characteristic function of a random variable \( X \) those, respectively, admit Definition 3.7 and Definition 5.7, and \( j \) be the imaginary number, i.e. \( j^2 = -1 \). Then the density function of a random variable \( X \) could be extract from its corresponding characteristic function by the subsequent equality:

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jux} \varphi_X(u) \, dx
\]

The pervious definition states that the characteristic function and its corresponding density function are paired, i.e. each of them could be seen as a Fourier Transform of the other. This depends on the existence of the density.

The characteristic function definition and this property will be the access point to the computational chapter, i.e. Chapter 8.

**Definition 5.8 (Infinitely Divisible Distribution)**

Let \( \varphi_X \) be the characteristic function of a random variable \( X \) that admits Definition
5.7, then \( \varphi_X \) is said to be an infinitely divisible if and only if for every \( n \in \mathbb{N} \) there exist an \( n^{th} \) power corresponding characteristic function, which could be given by the subsequent equality:

\[
\varphi_X(u) = \left( \varphi_{X,\frac{1}{n}}(u) \right)^n
\]

The above definition intuitively means that if a random variable \( Z \) could be characterised as a sum of two independent random variables with identically distributions, then \( Z \) is said to be infinitely divisible.

In general to prove that a distribution is infinitely divisible, it has to be examined throughout its sequence and it has to be true on each and every point. The next corollary shows that the Gaussian distribution is an infinitely divisible.

**Corollary 5.2 (Gaussian: Infinitely Divisible Distribution)**

Let \( X_{\bar{G}(\mu,\sigma)} \) be a Gaussian random variable that admits Definition 5.1 then \( \varphi_{X_{\bar{G}(\mu,\sigma)}} \), as the Gaussian characteristic function that follows Corollary 5.1, is infinitely divisible.

i.e.

\[
\varphi_{X_{\bar{G}(\mu,\sigma)}}(u) = \left( \varphi_{X_{\bar{G}(\mu,\sigma)}}(u) \right)^n
\]

**Definition 5.9 (Càdlàg Function)**

Let \( f \) be a function defined on \( \mathbb{R} \), or subset of \( \mathbb{R} \), then \( f \) is called a càdlàg function if and only if it is a right-continuous function with left limit.

The blocks needed to build the Lévy process are completed and thus it is represented in the subsequent definition.

**Definition 5.10 (Lévy Process)**

Let \( (X_t)_{t \in \mathbb{R}^+} \) be a càdlàg stochastic process defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \)
that admits Definition 5.9, where $\mathcal{X}_t$ is a continuous process $\mathbb{P}$-almost surely\(^{21}\) and has a stationary independent increments that admits Lemma 5.1, with $\mathcal{X}_0 = 0$. Then $\mathcal{X}_t$ is said to be Lévy process.

In view of Definition 5.8 and Definition 5.10 the next Lemma describes the one to one association of Lévy processes and infinitely divisible distributions.

**Lemma 5.3 (Infinite Divisibility of Lévy processes)**

Let $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$ be a Lévy process that admits Definition 5.10, then for every $t$, $\mathcal{X}_t$ has a consequent infinitely divisible distribution $F_{\mathcal{X}}$ that admits Definition 5.6.

Lemma 5.3 could be seen as contrary result of the next corollary where having an infinitely divisible distribution consequence a Lévy process.

**Corollary 5.3 (Infinite Divisibility of Lévy processes)**

If $F_{\mathcal{X}}$ is an infinitely divisible distribution, then there exists a Lévy process $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$.

The next corollary is a direct implication of the Lévy process when defined through the characteristic function.

**Corollary 5.4 (Characteristic function of Lévy processes)**

Let $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$ be a Lévy process that admits Definition 5.10 with corresponding infinitely divisible distribution $F_{\mathcal{X}}$ that admits Definition 3.6, where both follow Lemma 5.3, and $\varphi_{\mathcal{X}}$ be a characteristic function that admits Definition 5.7, then for $s, t \in \mathbb{R}^+$, the distribution of the increment $\mathcal{X}_{t+s} - \mathcal{X}_t$ is given by the characteristic function $(\varphi_{\mathcal{X}}(u))^t$.

Taking into consideration Property D5.7.5, and Definition 5.10, Corollary 5.4 could be rephrased to be represented through its corresponding cumulant characteristic function.

**Corollary 5.5 (Characteristic function of Lévy processes)**

Let $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$ be a Lévy process that admits Definition 5.10, $\varphi_{\mathcal{X}}$ be a characteristic function that admits Definition 5.10, where $\varphi_{\mathcal{X}}$ is a characteristic function $\varphi_{\mathcal{X}}(u) = \mathbb{E}[e^{iu\mathcal{X}_t}]$ for $u \in \mathbb{R}$.

---

\(^{21}\) At a given time $t$, the probability of having a jump is 0. It does not mean the paths of Lévy processes are continuous.
function that admits Definition 5.7, and $\phi_X$ be the cumulant characteristic function that admits Property D5.7.5. Then for any $t$, the corresponding characteristic function is given by the subsequent equality:

$$\varphi_X(u) = \mathbb{E}[\exp(juX_t)] = \exp(t\phi(u))$$

The above corollary intuitively means that the Lévy process law at time $t$ is completely characterised by the law of $X_1$.

The subsequent theorem presents a complete explanation of random variables with infinitely divisible distributions through their characteristic functions, which is the notable Lévy-Khintchine formula.

**Theorem 5.2 (Lévy-Khintchine formula)**

Let $F_X$ be distribution function of a random variable $X$, and $\phi_X$ be the cumulant characteristic function that admits Property D5.7.5, then $F_X$ is infinitely divisible if and only if there exists a triplet $[\gamma, \zeta^2, \nu(dx)]$ with $\gamma \in \mathbb{R}$, $\zeta^2 \in \mathbb{R}^+$, $I_A$ as the indicator function of $A$, $\nu$ as a measure that satisfies $\nu(\{0\}) = 0$ and $\int \mathbb{R}^+ (1 \wedge x^2) \nu(dx) < \infty$, and its cumulant characteristic function is given by the subsequent equality:

$$\phi_X = j\gamma u - \frac{\zeta^2}{2} u^2 + \int_{-\infty}^{\infty} (e^{jux} - 1 - jux I_{\{|x| < 1\}}) \nu(dx)$$

**Proof.**

In (Sato, 1999) Theorem 8.1.

**Definition 5.11 (Lévy-Khintchine formula Components)**

Let $F_X$ be infinitely divisible distribution function of a random variable $X$, that satisfies Theorem 5.2, then its component are defined as follow:

i. $[\gamma, \zeta^2, \nu(dx)]$: The Lévy or characteristic triplet.

ii. $\phi_X$: The Lévy or characteristic exponent.
iii. \( \gamma \): The drift term.

iv. \( \xi^2 \): The Gaussian or diffusion coefficient.

v. \( \nu \): The Lévy measure.

Notwithstanding Corollary 5.5 and Theorem 5.2, the characteristic function is fully defined as shown in the subsequent corollary.

**Corollary 5.6 (Characteristic function of Lévy processes)**

Let \((X_t)_{t \in \mathbb{R}^+}\) be a Lévy process that admits Definition 5.10 with a corresponding infinitely divisible distribution function \(F_X\) that admits Theorem 5.2, \(\varphi_X\) be its characteristic function that admits Definition 5.7, and \(\phi_X\) its Lévy exponent that admits Definition 5.11. Then for any \(t\), the corresponding characteristic function is given by the subsequent equality:

\[
\varphi_X(u) = \mathbb{E}[\exp(juX_t)] \\
= e^{t\phi(u)} \\
= e^{t\left(jyu - \frac{\xi^2}{2}u^2 + \int_{-\infty}^{\infty} (e^{(ju|x|)-1} - 1)\nu(dx)\right)}
\]

where \(\phi(u) = \phi_1(u)\) and \(X_t := X\).

This theorem means that the law of \(X_1\) of a Lévy process determines the law at time \(t\) of its Lévy process. As a consequence, the only Lévy process part that could be specified is its distribution at a single time.

When decomposing the Lévy-Khintchine formula theorem, the Lévy process could be seen as four independent elements: a linear deterministic element, a Brownian element, and two pure jump elements. These elements are summarised in the subsequent corollary.

**Theorem 5.3 (The Lévy-Itô Decomposition)**

Let \((X_t)_{t \in \mathbb{R}^+}\) be a Lévy process that admits Definition 5.10, and \([\gamma, \xi^2, \nu(dx)]\) \(\gamma, \xi^2, \nu\) and \(\nu\) be respectively, as the Lévy triplet, the drift term, the Gaussian coefficient, and the Lévy measure, those admits Definition 5.11 and follows Theorem 5.2. Then there
exists four independent Lévy processes, denoted by $\mathcal{X} = \sum_{i=1}^{4} \mathcal{X}^i$, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

i. $\mathcal{X}^1 = \gamma u$: is a constant drift

ii. $\mathcal{X}^2 = \frac{\zeta^2}{2} u^2$: is a Brownian motion

iii. $\mathcal{X}^3 = \int_{|x| \geq 1} (e^{jux} - 1)\nu(dx)$: is a compound Poisson process

iv. $\mathcal{X}^4 = \int_{|x| < 1} (e^{jux} - 1 - jux)\nu(dx)$: is a square integrable martingale, conditionally on existence of an almost surely countable number of jumps on each finite time interval is of magnitude less than 1.

**Proof.**

In (Sato, 1999), chapter 4.

In the next corollary, the Brownian motion defined in Definition 5.5 could be seen as a special case of the Lévy process when Lévy measure is equal to zero.

**Corollary 5.7 (Brownian Motion Lévy Triplet)**

Let $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$ be a Lévy process that admits Definition 5.10, and $[\gamma, \zeta^2, \nu(dx)]$ the Lévy triplet, the drift term, the Gaussian coefficient, and the Lévy measure, those admit Definition 5.11 and follow Theorem 5.2. Then the Lévy triplet of the Brownian motion, that admits Definition 5.5, is $[\gamma, \zeta^2, 0]$.

Another important feature needed to be examined is when the Lévy has a finite or infinite variation.

**Lemma 5.4 (Lévy process: Finite and Infinite Variation)**

Let $(\mathcal{X}_t)_{t \in \mathbb{R}^+}$ be a Lévy process that admits Definition 5.10 with a corresponding infinitely divisible distribution function $F_X$ that admits Theorem 5.2 and Theorem 5.3, then it has a finite variation if:
i. \( \gamma^2 = 0 \) and \( \int_{-1}^{+1} |x| \nu(dx) < \infty \).

And has an infinite variation if:

i. \( \gamma^2 \neq 0 \).

ii. \( \gamma^2 = 0 \) and \( \int_{-1}^{+1} |x| \nu(dx) = \infty \)

Proof:

In view of Property D5.5.4, Definition 5.4, Theorem 5.2, and Theorem 5.3, Lemma 5.4 is proved; since the Brownian motion is of infinite variation, a Lévy process with a Brownian component, i.e. \( \gamma^2 \neq 0 \) is of infinite variation. Conversely, when \( \gamma^2 = 0 \), then the component \( \int_{-1}^{+1} |x| \nu(dx) \) distinguish, upon Definition 5.4, if it is finite or infinite variation.

Definition 5.12 (Subordinator)

Let \( (X_t)_{t \in \mathbb{R}^+} \) be a Lévy process that admits Definition 5.10, and \( [\gamma, \gamma^2, \nu(dx)] \) \( \gamma \), \( \gamma^2 \), and \( \nu \) be respectively, as the Lévy triplet, the drift term, the Gaussian coefficient, and the Lévy measure, those admits Definition 5.11 and follows Theorem 5.2. Then it is said to a subordinate if the subsequent conditions hold:

i. \( \gamma^2 = 0 \)

ii. \( \int_{(-\infty,0)} \gamma x \nu(dx) = 0 \)

iii. \( \int_{(0,1)} \gamma x \nu(dx) < \infty \)

iv. \( \gamma + \int_{(0,1)} \gamma x \nu(dx) > 0 \)

In other words the previous definition means, that if a Lévy process has no Brownian component and the drift and the increments always lies on \( \mathbb{R}^+ \), then it is called a subordinate.
Chapter Five: From Lévy Process to Lévy Factor Copula Model

5.5 Lévy Factor Model

In this subsection, the extension that links the Merton’s structural firm value model, which has been launched in Theorem 5.1 and depends on (Merton, 1974), to the one factor model will be introduced. This extension was first introduced in (Vasicek, 1987).

As mentioned in the introduction, its limitation was that it still depends on the Gaussian assumption. To end with endless extensions, the Lévy Factor Model will be introduced.

Assumption 5.1 (Lévy: Merton Model)

Let \((A_{t_i})_{t \in [0,T]}\) be the i’s firm value process that follows a geometric Brownian motion defined under a common space \((\Omega, \mathcal{F}, \mathbb{Q}^*)\), which admits Theorem 5.1, and \((X_{t_i})_{t \in \mathbb{R}^+}\) be a Lévy process with finite variance that admits Definition 5.10 and follows Lemma 5.4. Then it is supposed that \(X_t = \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t\) and thus satisfies the subsequent equation:

\[
A_{t_i} = A_{0_i} e^{(X_{t_i})}
\]

After introducing the Lévy process as a base process for the Merton’s structural firm value model, this model could be manipulated in order to incorporate another important component when valuing an asset, i.e. the probability of default. Taking into consideration Theorem 5.1 and Assumption 5.1, the probability of default could be introduced by the next Lemma.

Lemma 5.5 (Lévy: Probability of Default)

Let \((A_{t_i})_{t \in [0,T]}\) be the firm value process that follows a geometric Brownian motion defined under a common space \((\Omega, \mathcal{F}, \mathbb{Q}^*)\), which admits Theorem 5.1, \((X_{t_i})_{t \in \mathbb{R}^+}\) be a Lévy process that admits Assumption 5.1, and \(D\) as a given default barrier. Then the probability of default for the firm \(i\) under \(D\), denoted by \(p_{i}^{d}\), is given by the cumulative distribution function of a normalised stationary increment of \(X_{t_i}\), denoted by \(F_{X_{t_i}}\), such
that:

\[ p^d_{t_i} = F_{\Delta t_i} \left( \frac{\ln \left( \frac{D_i}{A_{0i}} \right)}{\sqrt{t}} \right) \]

**Proof:**

With Assumption 5.1 valid, and \( F_{\Delta t_i} \) as normalized stationary increment that follow the subsequent equation

\[ x_i = \frac{x_{t+s} - x_t}{\sqrt{s}} \]

the probability of default for firm \( i \) is given by:

\[ p^d_{t_i} = \mathbb{Q}^* \left[ x_{t_i} < \ln \left( \frac{D_i}{A_{0i}} \right) \right] \]

\[ = F_{\Delta t_i} \left( \frac{\ln \left( \frac{D_i}{A_{0i}} \right)}{\sqrt{t}} \right) \]

With Lemma 5.5 being held and \( x_{t_i} \) follows a Lévy process, the proposed model, Lévy Factor Model, will be established through the correlation structure that is produced by a universal risk factors and an idiosyncratic risk. The universal risk factors are the factors that may produce a credit default event across all reference credit names. On the contrary, the idiosyncratic risk is specific factor for a specific reference credit name which may produces a credit default event. Correspondingly, this structure, through the universal risk factors and the idiosyncratic risk, have some other constrains those insure that the proposed model is still a Lévy Factor Model. This could be summarised as the following assumptions and then completed by Lemma 5.6.

**Assumption 5.2 (Structure of Lévy’s Process)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process that follows Lemma 5.5, then \( X_{t_i} \) is assumed to be structured by two kind of risk factors:
i. $M_t$ “Systematic Market Risk Factor”: this factor represents the universal risk between all firms. Furthermore, $M_t$ is assumed to admit a certain distribution function, denoted by $F_{M_t}$.

ii. $J_t$ “Idiosyncratic Risk Factors”: these factors represent each firm’s risks independently from each other. Moreover, $M_t$ is assumed to admit a certain distribution function, denoted by $F_{J_t}$.

where $M_t$ and $J_t$ have to:

i. Admits the Lévy process definition and its properties and be infinitely divisible distributions.

ii. Independent.

iii. Have zero mean.

iv. Have equal finite variance.

In view of $(X_{t|})_{t\in\mathbb{R}^+}$ that follows Assumption 5.2 of each firm $i$ at time $t$, the only random variable that affect its status is the correlation structure between the firms’ default probabilities, which is introduced in the subsequent Assumptions.

Assumption 5.3 (Correlation Coefficient Equal at all Time)

Let $(X_{t|i})_{t\in\mathbb{R}^+, i\in K}$ and $(X_{t|j})_{t\in\mathbb{R}^+, j\in K}$ be any two Lévy process those follow Assumption 5.2, where $i \neq j$, with correlation Coefficient between them, denoted by $\rho_{t(i,j)}$. The correlation coefficient is assumed to be the same at all time, and it could be represented as $\rho_{(i,j)} = \rho_{t(i,j)}$.

For simplicity the correlation coefficient will be held as the same between each and every two pairs of firms, as in Assumption 5.4 and then will be relaxed as needed.

Assumption 5.4 (Correlation Coefficient between all Companies)

Let $(X_{t|i})_{t\in\mathbb{R}^+, i\in K}$ and $(X_{t|j})_{t\in\mathbb{R}^+, j\in K}$ be any two Lévy process those follow Assumption 5.2, with correlation Coefficient between them, denoted by $\rho_{t(i,j)}$. The correlation coefficient is assumed to be the same at all time, and it could be represented as $\rho_{(i,j)} = \rho_{t(i,j)}$. 

For simplicity the correlation coefficient will be held as the same between each and every two pairs of firms, as in Assumption 5.4 and then will be relaxed as needed.
5.2, where \( i \neq j \), with correlation Coefficient between them that follows Assumption 5.3, denoted by \( \rho_{(i,j)} \). The correlation coefficient is assumed to be the same at all time and between all companies, and it could be represented as \( \rho = \rho_{(i,j)} \).

**Lemma 5.6 (Lévy Factor Model with Equal Correlation)**

Let \( \{X_{ti}\}_{t \in \mathbb{R}^+} \) be any Lévy process, \( M_t \) be the systematic market risk factor, and \( J_{ti} \) be the idiosyncratic risk factors those admits Assumption 5.2, with a correlation factor \( \rho \) that admits Assumption 5.4, then the Lévy Factor Model of \( X_{ti} \) is represented by the subsequent equality:

\[
X_{ti} = \rho M_t + \sqrt{1 - \rho^2} J_{ti}
\]

**Proof:**

Firstly, Let \( \alpha^2 + \beta^2 = 1 \) and the Lévy process is represented by the subsequent factor model:

\[
X_{ti} = \alpha M_t + \beta J_{ti}
\]

Secondly, by Itô calculus it could be obtain that,

1. with \( i \neq j \)

\[
\mathbb{E}^{Q^*} \left[ dX_{ti}, dX_{tj} \right] = \alpha^2 \sigma^2 dt
\]

\[
= \rho^2 \sigma^2 dt
\]

\[
\Rightarrow \alpha = \rho
\]

2. and

\[
\mathbb{E}^{Q^*} \left[ (dX_{ti})^2 \right] = (\alpha^2 + \beta^2) \sigma^2 dt
\]

\[
= \sigma^2 dt
\]

\[
\Rightarrow \beta = \sqrt{1 - \rho^2}
\]

which proves the equality in the Lemma as

\[
X_{ti} = \rho M_t + \sqrt{1 - \rho^2} J_{ti}
\]
With Assumption 5.4 being relaxed, the subsequent corollary is an immediate result of Lemma 5.6, which is going to be valid, unless explicitly stated otherwise.

**Corollary 5.8 (Lévy Factor Model)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( M_t \) be the systematic market risk factor, and \( J_{t_i} \) be the idiosyncratic risk factors those admits Assumption 5.2, with a correlation factor \( \rho_i \) that follows Assumption 5.3, then the Lévy Factor Model of \( X_{t_i} \) is represented by the subsequent equality:

\[
X_{t_i} = \rho_i M_t + \sqrt{1 - \rho_i^2} J_{t_i}
\]

**Lemma 5.7 (Lévy Factor Model: Probability of Default)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( M_t \) be the systematic market risk factor, and \( J_{t_i} \) be the idiosyncratic risk factors those admits Corollary 5.8, \( p^d_{t_i} \) be the probability of default for the firm \( i \) under a given default barrier \( D_i \) that follows Lemma 5.5, \( F_{X_{t_i}}^{-1} \) be the inverse cumulative distribution function, which could be derived by Lemma 3.6, of the cumulative distribution function \( F_{X_{t_i}} \), that acts upon Lemma 5.5. Then the probability of default conditioned upon the Systematic Market Risk Factor \( M_t \), denoted by \( p^d_{t_i | M_t} \), is given by the subsequent equality:

\[
p^d_{t_i | M_t} = F_{J_{t_i}} \left( \frac{F_{X_{t_i}}^{-1}(p^d_{t_i}) - \rho_i M_t}{\sqrt{1 - \rho_i^2}} \right)
\]

**Proof:**

Let \( B_{t_i} = \frac{\ln \left( \frac{D_i}{X_{t_i}} \right)}{\sqrt{\kappa}} \) and as \( p^d_{t_i} = F_{X_{t_i}}(B_{t_i}) \) it is possible to write \( B_{t_i} = F_{X_{t_i}}^{-1}(p^d_{t_i}) \) that follows Lemma 5.5, then the probability of default conditioned upon \( M_t \) is proved to hold the proposed equation through the subsequent chain of equalities:
\[ p_{t_i}^{d|\mathcal{M}_t} = \mathbb{Q}^*\left( X_{t_i} < B_{t_i} \mid \mathcal{M}_t \right) \]
\[ = \mathbb{Q}^*\left( \rho_i M_t + \sqrt{1 - \rho_i^2} J_{t_i} < B_{t_i} \mid \mathcal{M}_t \right) \]
\[ = \mathbb{Q}^*\left( J_{t_i} < \frac{B_{t_i} - \rho_i M_t}{\sqrt{1 - \rho_i^2}} \mid \mathcal{M}_t \right) \]
\[ = F_{J_{t_i}}\left( \frac{B_{t_i} - \rho_i M_t}{\sqrt{1 - \rho_i^2}} \right) \]
\[ = F_{J_{t_i}}\left( \frac{F_{X_{t_i}}^{-1}(p_{t_i}^{d}) - \rho_i M_t}{\sqrt{1 - \rho_i^2}} \right) \]

With this brief introduction about the Lévy factor model, the preceding model will be introduced through the Copula function.

### 5.6 Lévy Factor Copula Model

In view of Lévy factor model introduced in the preceding subsection, Chapter 3, which cover the copula, and through Chapter 4, which was focussed on the individual time to default and individual survival probabilities, this subsection will be developed to get tractable of \( n^{th} \) to default CDS premiums and CDO tranches and to take into account the default dependency among entities in the economy. In this subsection the main block of this thesis will be introduced, by generalising the standard “Gaussian Factor Copula Model” through the Lévy processes and called the “Lévy Factor Copula Model”.

In this subsection, the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. Furthermore, the default times will be assumed to be conditionally independent upon some enlargement space, which is going to be valid, unless explicitly stated otherwise.
Assumption 5.5 (Conditional Independency between Default Time & Market Risk)

Let $\tau_i$ be the random times those admit Definition 4.20 and $\mathcal{M}_t$ be the systematic market risk factor that admits Assumption 5.2. It is assumed that $\tau_i$ is conditionally independent with respect to an enlarged filtration $\mathbb{F} \vee \sigma(\mathcal{M}_t)$ under $\mathbb{Q}^*$, where $\mathcal{M}_t$ is $\mathcal{G}$-measurable random variable that follows a given distribution function $F_{\mathcal{M}_t}$. And for every $T \in \mathbb{R}^+$ and arbitrary $t_i \in [0, T], i \in \mathbb{K}$ the following statement satisfy:

$$
\mathbb{Q}^* (\tau_1 > t_1, \cdots, \tau_n > t_n | \mathcal{F}_T \vee \sigma(\mathcal{M}_t)) = \prod_{i=1}^{n} \mathbb{Q}^* (\tau_i > t_i | \mathcal{F}_T \vee \sigma(\mathcal{M}_t))
$$

Assumption 5.6 ($\mathcal{M}_t$ as Latent Mixing Variable)

Conditional on $\mathcal{M}_t$ as not $\mathcal{F}_\infty$ measurable, it can be seen as a latent mixing variable that corresponds to an unobserved random effects.

To concentrate on the dependence through the latent factor in this stage, $\gamma^i_t$ in Definition 4.16, Corollary 4.4 will be assumed to hold the following.

Assumption 5.7 (Deterministic and Continuous $\mathcal{F}$-Intensity)

The $\mathcal{F}$-intensity $\gamma^i_t$ of $\tau$, with hazard process $\Gamma^i_t$ that follows Assumption 4.13, is assumed to be deterministic and continuous.

At this point, the barrier assumptions that have been driven in the previous section and its consequences of having a probability of default upon given default barrier $D_i$, will be generalised in the next corollary to be held under Assumption 4.12, and Assumption 4.13 and Definition 4.20.

Corollary 5.9 (Default Barrier: Lévy Factor Copula Model)

Let $(X_t)_{t \in \mathbb{R}^+}$ be a Lévy process, $\mathcal{M}_t$ be the systematic market risk factor, and $\mathcal{J}_t$ be the idiosyncratic risk factors those admits Lemma 5.7, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, and Assumption 5.5 and Assumption 5.6 being hold.
Then the probability of default conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$, denoted by $p_{t_i}^{\xi_i|\mathcal{M}_t}$, is given by the subsequent equality:

$$p_{t_i}^{\xi_i|\mathcal{M}_t} = \mathbb{Q}^* \left( X_{t_i} < F_{\mathcal{M}_t}^{-1}(\xi_i) \big| \mathcal{M}_t \right) = F_{\mathcal{M}_t} \left( \frac{F_{X_{t_i}}^{-1}(\xi_i) - \rho_t \mathcal{M}_t}{\sqrt{1 - \rho_t^2}} \right)$$

The previous corollary, associates the factor model with the factor copula model, which is going to be clear in the next Lemma. In addition, an immediate consequence could be found, under the copula function, to map $X_{t_i}$ to the $t_i$ using a percentile-to-percentile transformation. Corollary 5.9 could be rephrased as the subsequent corollary.

**Corollary 5.10 (Default Time: Lévy Factor Copula Model)**

Let $\left( X_{t_i} \right)_{t \in \mathbb{R}^+}$ be a Lévy process, $\mathcal{M}_t$ be the systematic market risk factor, and $\mathcal{J}_{t_i}$ be the idiosyncratic risk factors those follow Corollary 5.9, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, $\tau_i$ be the default time that admits Definition 4.20, and $p_{t_i}^{\xi_i|\mathcal{M}_t}$ and $q_{t_i}^{\xi_i|\mathcal{M}_t}$ be, respectively, the probability of default time of $\tau_i$ and the survival time, conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$. Then $p_{t_i}^{\xi_i|\mathcal{M}_t}$ of the Lévy Factor Copula Model is given by the subsequent equality:

$$p_{t_i}^{\xi_i|\mathcal{M}_t} = \mathbb{Q}^* (\tau_i < t_i | \mathcal{M}_t) = F_{\mathcal{M}_t} \left( \frac{F_{X_{t_i}}^{-1}(\xi_i) - \rho_t \mathcal{M}_t}{\sqrt{1 - \rho_t^2}} \right)$$

and $q_{t_i}^{\xi_i|\mathcal{M}_t}$ of the Lévy Factor Copula Model the is given by the subsequent equality:

$$q_{t_i}^{\xi_i|\mathcal{M}_t} = 1 - p_{t_i}^{\xi_i|\mathcal{M}_t}$$

Assumption 5.6 and the subsequent corollaries intuitively mean that the Idiosyncratic Risk Factors $\mathcal{J}_{t_i}$ become independent of each other, when the Systematic Market Risk...
Factor $\mathcal{M}_t$ is predetermined. These results introduce the core theorem in this section by associating the Lévy Factor Model with the Lévy Factor Copula Model.

**Theorem 5.4 (Lévy Factor Copula Model)**

Let $(X_{t_i})_{t \in \mathbb{R}^+, i \in \mathbb{K}}$ be a Lévy process, $\mathcal{M}_t$ be the systematic market risk factor, and $\mathcal{J}_{t_i}$ be the idiosyncratic risk factors those follows Corollary 5.9. $p_{t_i}^{\xi|m|\mathcal{M}_t}$ be the probability of default conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$, and $C$ as an $n$-dimensional copula function that follows Theorem 3.2 and Theorem 3.3. Then the Lévy Factor Copula Model could be represented by the subsequent equality:

$$
C(\xi_1, ..., \xi_n) = \int \left( \prod_{i=1}^{n} p_{t_i}^{\xi|m|\mathcal{M}_t} \right) f_{\mathcal{M}_t}(m) dm
$$

where $f_{\mathcal{M}_t}$ is the associated density of the conditional factor $\mathcal{M}_t$.

**Proof:**

Initiating the chain of equalities by Theorem 3.2, followed by applying the iterated expectation theorem and upon the $X_{t_i}$ that follows Corollary 5.8, by the random time $\tau_i$ being conditionally independent on $\mathcal{M}_t$, from Assumption 5.5 and Assumption 5.6, and finally by substituting the $p_{t_i}^{\xi|m|\mathcal{M}_t}$ that admits Corollary 5.9, the next sequence of equalities hold:

$$
C(\xi_1, ..., \xi_n) = \mathbb{Q}^* \left( X_1 < F_{X_1}^{-1}(\xi_1), ..., X_n < F_{X_n}^{-1}(\xi_n) \right)
$$

$$
= \mathbb{E}^{\mathbb{Q}^*} \left( \mathbb{Q}^* \left( X_1 < F_{X_1}^{-1}(\xi_1), ..., X_n < F_{X_n}^{-1}(\xi_n) \right) \right| \mathcal{M}_t)
$$

$$
= \int \left( \prod_{i=1}^{n} F_{\mathcal{J}_{t_i}} \left( \frac{F_{X_i}^{-1}(\xi_i) - \rho_i m}{\sqrt{1 - \rho_i^2}} \right) f_{\mathcal{M}_t}(m) dm
$$

$$
= \int \left( \prod_{i=1}^{n} p_{t_i}^{\xi|m|\mathcal{M}_t} \right) f_{\mathcal{M}_t}(m) dm
$$

where $f_{\mathcal{M}_t}$ is the associated density of the conditional factor $\mathcal{M}_t$.  

Chapter Five: From Lévy Process to Lévy Factor Copula Model
In view of the mutuality independency property of the default probability conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$, Lemma 5.5, as well as its applications in Corollary 5.9 and Corollary 5.10, starts overcoming the first proposed problem by reducing the dimensionality by giving a univariate marginal distribution functions. Then it can be utilised in the copula function, as in Theorem 5.4, to calculate joint distributions of a credit portfolio in a reduced dimension. Final step could be achieved by employing Theorem 3.2, Sklar’s Theorem, as it is summarised in the following corollary.

**Corollary 5.11 (Lévy Factor Copula Model, Default & Survival Distributions)**

Let $\left( X_t \right)_{t \in \mathbb{R}^+}$ be a Lévy process, $\mathcal{M}_t$ be the systematic market risk factor, and $\mathcal{J}_{t_i}$ be the idiosyncratic risk factors those follows Corollary 5.9, $p_{t_i}^{\xi \mid \mathcal{M}_t}$ be the probability of default and $q_{t_i}^{\xi \mid \mathcal{M}_t}$ be the probability of survival, where both are conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$ that follows Corollary 5.10, and $f_{\mathcal{M}_t}$ as the associated density of the conditional factor $\mathcal{M}_t$. Then the Joint Lévy Factor Copula Model Distribution Function could be represented by the subsequent equality:

$$F(t_1, \ldots, t_n) = \mathbb{Q}^*(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n) = \int \left( \prod_{i=1}^{n} \frac{F_{X_{t_i}}^{-1} \left( F_{t_i} (t_i) - \rho_i m \right)}{1 - \rho_i^2} \right) f_{\mathcal{M}_t}(m) \, dm$$

$$= \int \left( \prod_{i=1}^{n} p_{t_i}^{\xi \mid \mathcal{M}_t} \right) f_{\mathcal{M}_t}(m) \, dm$$

and the Survival Lévy Factor Copula Model Distribution Function could be represented by the subsequent equality:

For shortness, it will be from this point as the Lévy Factor Model.
\[ G(t_1, \ldots, t_n) = \mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n) = \int \left( \prod_{i=1}^{n} q_{t_i}^{x_i \mid m} \right) f_M(t)(m) dm \]

### 5.7 Lévy Stochastic Correlated Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. This subsection introduces two Stochastic Correlation implementations: Binary Structure Case and Symmetric Dependence Structure Case.

**Assumption 5.8 (Stochastic Correlation \( \tilde{\rho} \))**

Let \((X_{t_i})_{t \in \mathbb{R}^+, i \in \mathbb{K}}\) and \((X_{t_j})_{t \in \mathbb{R}^+, j \in \mathbb{K}}\) be any two Lévy process those follow Assumption 5.2, and \(i \neq j\), with stochastic correlation coefficient between them, denoted by \(\tilde{\rho}_{t_{(i,j)}}\), where \(\tilde{\rho}_{t_{(i,j)}} \in \mathbb{I}\).

At this point the, the Lévy factor model correlation structure is refined to incorporate a stochastic correlation instead of having a linear and deterministic correlation. This model is so called the Lévy stochastic correlated factor model.

**Corollary 5.12 (Lévy Stochastic Correlated Factor Model)**

Let \((X_{t_i})_{t \in \mathbb{R}^+}\) be a Lévy process, \(M_t\) be the systematic market risk factor, and \(\mathcal{J}_{t_i}\) be the idiosyncratic risk factors those follows Assumption 5.2, with a Stochastic correlation factor \(\tilde{\rho}_t\) that follows Assumption 5.8, then the Lévy Factor Model of \(X_{t_i}\) is represented by the subsequent equality:

\[ X_{t_i} = \tilde{\rho}_t M_t + \sqrt{1 - \tilde{\rho}_t^2} \mathcal{J}_{t_i} \]
5.7.1 Stochastic Correlation: Binary Structure Case

The first Stochastic Correlation implementation is structured by the binary distribution. This assumption will be considered by rephrasing the Lévy Factor Copula Model framework.

**Assumption 5.9 (Binary Structure Case)**

Let \( \tilde{\rho}_i \) be a stochastic correlation coefficient that follows Assumption 5.8, structured by two constants \( \rho_1, \rho_2 \in \mathbb{I} \) and Bernoulli random variables \( B_i \) such that:

\[
B_i = \begin{cases} 
0, & \text{with probability } 1 - q \\
1, & \text{with probability } q 
\end{cases}
\]

and \( \tilde{\rho}_i \) is structured as given in the subsequent equality:

\[
\tilde{\rho}_i = (1 - B_i) \rho_1 + B_i \rho_2.
\]

In view of Assumption 5.8 and Assumption 5.9, Corollary 5.12 could be rephrased as in the subsequent corollary.

**Corollary 5.13 (Lévy Binary Stochastic Correlated Factor Copula Model)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( \mathcal{M}_t \) be the systematic market risk factor, and \( \mathcal{J}_{t_i} \) be the idiosyncratic risk factors those follows Corollary 5.12, with a stochastic correlation factor \( \tilde{\rho}_i \) that follows Assumption 5.9, then the Lévy Factor Model of \( X_{t_i} \) is represented by the subsequent equality:

\[
X_{t_i} = \left( (1 - B_i) \rho_1 + B_i \rho_2 \right) \mathcal{M}_t + \sqrt{1 - \left( (1 - B_i) \rho_1 + B_i \rho_2 \right)^2} \mathcal{J}_{t_i}
\]

A direct result could be obtained when injecting the structure of the Lévy binary stochastic correlated factor copula model, which is expressed in Corollary 5.13, into the conditional independent representation of Lévy factor copula this model.

**Lemma 5.8 (Lévy Binary Stochastic Correlated Factor Copula Model)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( \mathcal{M}_t \) be the systematic market risk factor, and \( \mathcal{J}_{t_i} \) be the idiosyncratic risk factors those follows Corollary 5.13, \( \xi_i \) be as supposed in
Assumption 4.14 and admits Definition 4.20, \( \tau_i \) be the default time that admits Definition 4.20, and \( p_{t_i}^{\xi|M_t} \) be the probability of default time \( \tau_i \), conditioned upon the Systematic Market Risk Factor \( M_t \). Then \( p_{t_i}^{\xi|M_t} \) of the Single Binary Stochastic Correlated Lévy Factor Copula Model is given by the subsequent equality:

\[
p_{t_i}^{\xi|M_t} = (1 - q)F_{\xi_{t_i}} \left( \frac{F_{X_{t_i}}^{-1} \left( F_{t_i}(t) - \rho_1 M_t \right)}{\sqrt{1 - \rho_1^2}} \right) + qF_{\xi_{t_i}} \left( \frac{F_{X_{t_i}}^{-1} \left( F_{t_i}(t) - \rho_2 M_t \right)}{\sqrt{1 - \rho_2^2}} \right)
\]

**Proof**

With proof of Lemma 5.7 and the use of Corollary 5.11, it is enough to show the effect of replacing \( \rho_i \) by the Binary structure of the \( \tilde{\rho}_i \).

Firstly: when \( B_i = 0 \), \( p_{t_i}^{\xi|M_t,B_i=0} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi|M_t,B_i=0} = \mathbb{Q}^*(\tau_i < t_i | M_t, B_i = 0)\mathbb{Q}^*(B_i = 0)
= (1 - q)\mathbb{Q}^* \left( \rho_1 M_t + \sqrt{1 - \rho_1^2} \xi_{t_i} < F_{X_{t_i}}^{-1} \left( F_{t_i}(t_i) \right) \right) M_t
= (1 - q)F_{\xi_{t_i}} \left( \frac{F_{X_{t_i}}^{-1} \left( F_{t_i}(t_i) - \rho_1 M_t \right)}{\sqrt{1 - \rho_1^2}} \right)
\]

Secondly: when \( B_i = 1 \), \( p_{t_i}^{\xi|M_t,B_i=1} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi|M_t,B_i=1} = \mathbb{Q}^*(\tau_i < t_i | M_t, B_i = 1)\mathbb{Q}^*(B_i = 1)
= q\mathbb{Q}^* \left( \rho_2 M_t + \sqrt{1 - \rho_2^2} \xi_{t_i} < F_{X_{t_i}}^{-1} \left( F_{t_i}(t_i) \right) \right) M_t
= qF_{\xi_{t_i}} \left( \frac{F_{X_{t_i}}^{-1} \left( F_{t_i}(t_i) - \rho_2 M_t \right)}{\sqrt{1 - \rho_2^2}} \right)
\]

Finally: by adding the first and second result the Lemma is proved.

### 5.7.2 Stochastic Correlation: Symmetric Dependence Structure Case

Implementing the stochastic correlation by the binary distribution case overcomes partially the under-pricing problem of the equity and senior tranches and over-pricing problem of the mezzanine tranche, but still further consideration is required, i.e. it is
noticeable that the senior tranche premiums are significant, despite the consequences of its capability standing tens of credit events. In (Tavares et al., 2004) and Trinh et al. [2005] the correlated model that incorporates the events of the systemic risk have been described. As a consequence of these approach limitations, i.e. the systemic risk is associated to the default of the whole credit portfolio regardless of its size, a stochastic correlation will be implemented through the symmetric dependence structure proposed by (Burtschell et al., 2005) and (Burtschell et al., 2008).

This approach permits defaults ordering, where the entities could be grouped depending on their symmetric risk factor, i.e. risky credit entities could be anticipated to default prior to less riskier ones and therefore they could be, without any technical problems, separated and grouped within the same credit portfolio in spite of their spreads. As a consequence, this method will overcome the problem of under-pricing the equity and senior tranches.

This method will be generalised within the Lévy Factor Copula Model framework and will be called the Lévy Symmetric Stochastic Correlated Factor Copula Model.

**Assumption 5.10 (Symmetric Dependence Structure Case)**

Let \( \tilde{\rho}_i \) be a stochastic correlation coefficient that follows Assumption 5.8 and structured by \( \rho \in \mathbb{I} \), \( B_i \) is a Bernoulli random variables that follows Assumption 5.9, and \( B_s \) is a Bernoulli random variables, where \( B_s \) is given by:

\[
B_s = \begin{cases} 
0, & \text{with probability } 1 - \tilde{q} \\
1, & \text{with probability } \tilde{q}
\end{cases}
\]

and \( \tilde{\rho}_i \) is structured as given in the subsequent equality:

\[
\tilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s
\]

Taking into consideration Assumption 5.8 and Assumption 5.10, Corollary 5.12 could be rearticulated as in the consequent corollary.
Corollary 5.14 (Lévy Symmetric Stochastic Correlated Factor Copula Model)

Let \((X_t)_{t \in \mathbb{R}^+}\) be any Lévy process that \(\xi_t\) be a Lévy process, \(M_t\) be the systematic market risk factor, and \(J_{t_i}\) be the idiosyncratic risk factors that follows Corollary 5.12, with a Stochastic correlation factor \(\tilde{\rho}_t\) that follows Assumption 5.10, then the Lévy Factor Model of \(X_{t_i}\) is represented by the subsequent equality:

\[
X_{t_i} = \left((1 - B_s)(1 - B_t)\rho + B_s\right)M_t + \left((1 - B_s)(1 - \rho^2 + B_t)\right)J_{t_i}
\]

A straightforward consequence could be achieved once inserting the structure of the Lévy systematic stochastic correlated factor copula model, which is expressed in Corollary 5.14, into the conditional independent representation of Lévy factor copula this model.

Lemma 5.9 (Lévy Symmetric Stochastic Correlated Factor Copula Model)

Let \((X_t)_{t \in \mathbb{R}^+}\) be a Lévy process, \(M_t\) be the systematic market risk factor, and \(J_{t_i}\) be the idiosyncratic risk factors that follows Corollary 5.14, \(\xi_t\) be as supposed in Assumption 4.14 and admits Definition 4.20, \(\tau_i\) be the default time that admits Definition 4.20, then the probability of default time \(\tau_i\) conditioned upon the Systematic Market Risk Factor \(M_t\), denoted by \(p_{\xi_{t_i}}^{\tau_i|M_t}\), is given by the subsequent equality:

\[
p_{\xi_{t_i}}^{\tau_i|M_t} = \tilde{q}F_{M_t}\left(F_{X_{t_i}}^{-1}\left(F_{t_i}(t)\right)\right) \\
+ (1 - \tilde{q}) \left[(1 - q)F_{J_{t_i}}\left(F_{X_{t_i}}^{-1}\left(F_{t_i}(t)\right) - \rho M_t\right)\sqrt{1 - \rho^2}\right] \\
+ qF_{J_{t_i}}\left(F_{X_{t_i}}^{-1}\left(F_{t_i}(t)\right)\right)
\]

**Proof:**
With proof of Lemma 5.7 and Lemma 5.8 and the use of Corollary 5.11, it is enough to show the effect of replacing \( \rho_i \) by the Symmetric Dependence Structure of the \( \tilde{\rho}_i \).

Firstly: when \( B_s = 1 \), \( p_{t_i}^{\xi|M_t,B_s=1} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi|M_t,B_s=1} = Q^*(\tau_i < t_i | M_t, B_s = 1)Q^*(B_s = 1) = \tilde{q}Q^*(M_t < F_{X_t}^{-1}(F_{t_i}(t_i)) | M_t) = \tilde{q}F_M(t_i)
\]

Secondly: when \( B_s = B_i = 0 \), \( p_{t_i}^{\xi|M_t,B_s,B_i=0} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi|M_t,B_s,B_i=0} = Q^*(\tau_i < t_i | M_t, B_s = 0, B_i = 0)Q^*(B_s = 0)Q^*(B_i = 0) = (1 - \tilde{q})(1 - q)Q^*(\rho M_t + \sqrt{1 - \rho^2} | J_{t_i} < F_{X_t}^{-1}(F_{t_i}(t_i)) | M_t) = (1 - \tilde{q})(1 - q)F_{\tilde{\rho}_{t_i}} \left( \frac{F_{X_t}^{-1}(F_{t_i}(t)) - \rho M_t}{\sqrt{1 - \rho^2}} \right)
\]

Thirdly: when \( B_s = 0 \) and \( B_i = 1 \), \( p_{t_i}^{\xi|M_t,B_s=0,B_i=1} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi|M_t,B_s=0,B_i=1} = Q^*(\tau_i < t_i | M_t, B_s = 0, B_i = 1)Q^*(B_s = 0)Q^*(B_i = 1) = (1 - \tilde{q})qQ^*(J_{t_i} < F_{X_t}^{-1}(F_{t_i}(t_i)) | M_t) = (1 - \tilde{q})qF_{\rho_{t_i}} \left( F_{X_t}^{-1}(F_{t_i}(t_i)) \right)
\]

Finally: by adding the first, second, and third results the Lemma is proved.

When the distribution function of the systematic market risk factor, idiosyncratic risk factors, and the \( X_t \), where all these distribution admits the Lévy process, are equal, the previous lemma could be remodelled as follow.

**Corollary 5.15 (Lévy Symmetric Stochastic Correlated Factor Copula Model)**

Let \( (X_t)_{t \in \mathbb{R}^+} \) be a Lévy process, \( M_t \) be the systematic market risk factor, and \( J_{t_i} \) be the idiosyncratic risk factors those follows Corollary 5.14, the probability of default time \( \tau_i \) conditioned upon the Systematic Market Risk Factor \( M_t \), denoted by \( p_{t_i}^{\xi|M_t} \)
follows Lemma 5.9 and \( F_{\mathcal{M}_t} = F_{\mathcal{J}_{t_i}} = F_{\mathcal{X}_{t_i}} \), then \( p_{t_i}^{\xi M_t} \) is given by the subsequent equality:

\[
p_{t_i}^{\xi M_t} = \tilde{q} F_{t_i}(t) + (1 - \tilde{q}) \left[ (1 - q) F_{\mathcal{J}_{t_i}} \left( \frac{F_{\mathcal{X}_{t_i}}^{-1}(F_{t_i}(t)) - \rho \mathcal{M}_t}{\sqrt{1 - \rho^2}} \right) + q F_{t_i}(t_i) \right]
\]

### 5.8 Lévy Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] have, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a generalised approach within the Lévy Factor Copula Model framework and will be called the Lévy Random Factor Loading Copula Model.

**Assumption 5.11 (Random Factor Loading Structure)**

Let \( \rho_t(\mathcal{M}_t) \in \mathbb{R} \) be a random factor loading that is structured by a constant \( \kappa \in \mathbb{R}^+ \), and some input parameters \( \ell_1, \ell_2 \in \mathbb{R}^+ \). Then it is assumed that \( \rho_t(\mathcal{M}_t) \) is structured by the subsequent equality:

\[
\rho_t(\mathcal{M}_t) = \ell_1 \mathcal{I}_{(\mathcal{M}_t < \kappa)} + \ell_2 \mathcal{I}_{(\mathcal{M}_t \geq \kappa)}
\]

**Corollary 5.16 (Lévy Random Factor Loading Copula Model)**

Let \( (\mathcal{X}_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( \mathcal{M}_t \) be the systematic market risk factor, and \( \mathcal{J}_{t_i} \) be the idiosyncratic risk factors those follows Corollary 5.11, with a random factor loading \( \rho_t(\mathcal{M}_t) \) that follows Assumption 5.11, then the Lévy Factor Model of \( \mathcal{X}_{t_i} \) is represented by the subsequent equality:

\[
\mathcal{X}_{t_i} = \rho_t(\mathcal{M}_t) \mathcal{M}_t + \nu_t \mathcal{J}_{t_i} + \kappa_i
\]
where \( v_i = \sqrt{1 - \mathbb{V}[(\rho_t(M_t)M_t)]} \) and \( \kappa_i = -\mathbb{E}[(\rho_t(M_t)M_t)] \).

**Lemma 5.10 (Lévy Random Factor Loading Copula Model)**

Let \( (X_{t_i})_{t \in \mathbb{R}^+} \) be a Lévy process, \( M_t \) be the systematic market risk factor, and \( J_{t_i} \) be the idiosyncratic risk factors those follows Corollary 5.16, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, \( \tau_i \) be the default time that admits Definition 4.20, \( p_{t_i}^{\xi_i|M_t} \) be the probability of default time \( \tau_i \), conditioned upon the Systematic Market Risk Factor \( M_t \) and let the Systematic Market Risk Factor be denoted by \( M_t^1 \) when \( M_t < \kappa \) and \( M_t^2 \) when \( M_t \geq \kappa \). Then \( p_{t_i}^{\xi_i|M_t^1} \) of the Lévy Random Factor Loading Copula Model is given by the subsequent equality:

\[
p_{t_i}^{\xi_i|M_t^1} = F_{\gamma_{t_i}} \left( \frac{F_{X_{t_i}^{-1}}(F_{t_i}(t)) - \kappa_i - \ell_1 M_t^1}{\sqrt{1 - \ell_1^2}} \right)
\]

and \( p_{t_i}^{\xi_i|M_t^2} \) of the Lévy Random Factor Loading Copula Model is given by the subsequent equality:

\[
p_{t_i}^{\xi_i|M_t^2} = F_{\gamma_{t_i}} \left( \frac{F_{X_{t_i}^{-1}}(F_{t_i}(t)) - \kappa_i - \ell_2 M_t^2}{\sqrt{1 - \ell_2^2}} \right)
\]

**Proof**

Firstly: when \( p_{t_i}^{\xi_i|M_t^1} \) is given by the subsequent dual equalities:

\[
p_{t_i}^{\xi_i|M_t^1} = \mathbb{Q}^*/(\tau_i < t_i|M_t^1)
= \mathbb{Q}^*/(\rho_t(M_t^1)M_t^1 + \nu J_{t_i} + \kappa_i < F_{X_{t_i}^{-1}}(F_{t_i}(t_i))|M_t^1)
= F_{\gamma_{t_i}} \left( \frac{F_{X_{t_i}^{-1}}(F_{t_i}(t)) - \kappa_i - \ell_1 M_t^1}{\sqrt{1 - \ell_1^2}} \right)
\]

Secondly: when \( p_{t_i}^{\xi_i|M_t^2} \) is given by the subsequent dual equalities:
In Lemma 5.10 the conditional probability of default is partitioned into two segments. This property of the Lévy Factor Loading Copula Model leads to more complications; especially when coming to the distribution properties and its parameters and the model’s mean and variance.

**Theorem 5.5 (Lévy Random Factor Loading Copula Model: The Unconditional Accumulated Loss \( \varphi_{\xi_t} \))**

Let \( \mathcal{N}_t \) be a default counter process that admits Definition 4.13, \( p_{\xi,t|M_t} \) be the conditional default upon the systematic market risk factor \( M_t \) in the “Lévy Random Factor Loading Copula” model, those follows Lemma 5.10. Then the unconditional number of default’s characteristic function, denoted by \( \varphi_{\mathcal{N}_t} \), is given by:

\[
\varphi_{\mathcal{N}_t}(u) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi,t|M_t} \right) f_{M_t}(m) dm + \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi,t|M_t} \right) f_{M_t}(m) dm
\]

Where \( \kappa_i = -\ell_1 \int_{-\infty}^{\infty} mf_{M_t}(m) dm - \ell_2 \int_{-\infty}^{\infty} mf_{M_t}(m) dm \)

\[p_{\xi,t|M_t} = \mathbb{Q}^*(\tau_i < t_i | M_t^2) = \mathbb{Q}^* \left( \rho_i (M_t^2) M_t^2 + \nu J_t_i + \kappa_i < F_{X_i}^{-1} \left( F_{\xi_i}(t_i) \right) | M_t^2 \right) = F_{\beta_t}^2 \left( \frac{F_{X_i}^{-1} \left( F_{\xi_i}(t) \right) - \kappa_i - \ell_2 M_t^2}{\sqrt{1 - \ell_2^2}} \right)
\]
5.9 Mathematical Summary

Part I: Brownian motion

- \((W_t)_{t \in \mathbb{R}^+}\), i.e. \(\mathcal{V}_T(W) = \sum_{i=1}^n |W(t_{i+1}) - W(t_i)|\), then \(W_t\) is **infinite variation** if 
  \(\sup_T \mathcal{V}_T(W) = \infty\) and **finite variation** if \(\sup_T \mathcal{V}_T(W) < \infty\).

- \((W_t)_{t \in \mathbb{R}^+}\) is **Brownian motion** if \(W_0 = 0\) has stationary independent increment, where these increments has Gaussian Distribution \(f_{\mathcal{N}(0,t-s)}\)
  
  - Its scaled is still Brownian motion, i.e. \((\overline{W}_t)_{t \in \mathbb{R}^+} = (\frac{\alpha}{\sqrt{\sigma^2}} W_t)_{t \in \mathbb{R}^+; \alpha \neq 0}\), it is a **martingale**, i.e. \(\mathbb{E}[W_t|\mathcal{F}_s] = W_s\), it could be **discretised**, i.e. \(W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t} \nu_n\), and it could has **finite** or an **infinite variation**

- \((A_t)_{t \in \mathbb{R}^+}\) is a **geometric Brownian Motion** if for \(A_0 > 0\), and standard Brownian motion \((W_t)_{t \in \mathbb{R}^+}\) with \(r\) be drift parameter, and \(\sigma\) as a standard deviation, 
  \(dA_t = A_t(r dt + \sigma dW_t)\) and it has a unique solution i.e. \(A_t = A_0 \exp((r - \frac{\sigma^2}{2})t + \sigma W_t)\)

Part II: Lévy process

- The characteristic function is given by 
  \[ \varphi_X(u) = \mathbb{E}[e^{iux}] = \int_{-\infty}^{\infty} e^{iux} f(x) dx = \int_{-\infty}^{\infty} e^{iux} f_X(x) dx. \]

  - \(\forall X, \exists \varphi_X\), if \(X_n\) are independent random sequence, then \(\varphi_{X_n}(u) = \varphi_{X_1}(u) \cdots \varphi_{X_n}(u)\), and it is **infinitely divisible** iff \(\varphi_X(u) = \left(\varphi_{X_{1/n}}(u)\right)^n\)

- \(f\) is called a **càdlàg function** iff it is a right-continuous function with left limit.

- \((X_t)_{t \in \mathbb{R}^+}\) is **Lévy process** if \(X_0 = 0\), it is a càdlàg and continuous process \(\mathbb{P}\)-almost surely and has a stationary independent increments.

  - Has an **infinitely divisible** distribution \(F_X\), \((\varphi_X(u))^t\) is the distribution of \(X_{t+s} - X_t\), and \(\varphi_X(u) = \mathbb{E}[\exp(juX_t)] = \exp(t\varphi(u))\).

- **Lévy-Khintchine formula**: \(F_X\) is infinitely divisible iff \(\exists [\gamma, \zeta^2, \nu(dx)]\) with
\[ \gamma \in \mathbb{R}, \, \zeta^2 \in \mathbb{R}^+, \text{ indicator function: } I_A, \text{ a measure: } \nu([0]) = 0 \text{ and } \int_{\mathbb{R}} (1 \land x^2)\nu(dx) < \infty, \text{ then } \phi_X = yu - \frac{\zeta^2}{2}u^2 + \int_{-\infty}^{\infty} \left(e^{(iux)} - 1 - iuxI_{|x|<1}\right)\nu(dx) \]

- \((X_t)_{t \in \mathbb{R}^+}\) is a Lévy process with an infinitely divisible \(F_X\). Then \(\forall t, \varphi_X(u) = \exp \left[t \left(yu - \frac{\zeta^2}{2}u^2 + \int_{-\infty}^{\infty} \left(e^{(iux)} - 1 - iuxI_{|x|<1}\right)\nu(dx)\right)\right] \)

- The Lévy-Itô Decomposition: if \((X_t)_{t \in \mathbb{R}^+}\) is a Lévy process. Then \(\exists X = \sum_{i=1}^{4} X^i\), where \(X^i\) are independent i.e. constant drift: \(X^1 = yu\), Brownian motion: \(X^2 = \frac{\zeta^2}{2}u^2\), compound Poisson process: \(X^3 = \int_{|x|\geq1} \left(e^{(iux)} - 1 - iux\right)\nu(dx)\), square integrable martingale: \(X^4 = \int_{|x|<1} \left(e^{(iux)} - 1\right)\nu(dx)\)

- The Lévy process is a Brownian motion if its triplet \([\gamma, \zeta^2, 0]\).

- a Lévy process \((X_t)_{t \in \mathbb{R}^+}\) has a finite variation if \(\zeta^2 = 0\) and \(\int_{-1}^{+1} |x|\nu(dx) < \infty\) and an infinite variation if \(\zeta^2 \neq 0\) or \(\zeta^2 = 0\) and \(\int_{-1}^{+1} |x|\nu(dx) = \infty\).

**Part III: Lévy Factor Model**

- If \((A_{t_i})_{t \in [0,T]}\) is geometric Brownian motion and \((X_{t_i})_{t \in \mathbb{R}^+}\) is Lévy process with finite variance and supposing \(X_t = (r - \sigma^2/2)t + \sigma \mathcal{W}_t\) then \(A_{t_i} = A_{0_i}e^{(X_t)}\). And its probability of default under \(D_t\) is given by \(F_{X_{t_i}}\) of a normalised stationary increment of \(X_{t_i}\) denoted by, i.e.\( p_{t_i}^D = F_{X_{t_i}} \left( \ln(D_t/A_{0_i})/\sqrt{t} \right) \)

- \((X_{t_i})_{t \in \mathbb{R}^+}\) is a Lévy process structured by two independent risk factors: \(\mathcal{M}_t\) “Systematic Market Risk Factor”, and \(\mathcal{J}_{t_i}\) “Idiosyncratic Risk Factors”.

- **Lévy Factor Model**: \(X_{t_i} = \rho_i \mathcal{M}_t + \sqrt{1 - \rho_i^2} \mathcal{J}_{t_i}\), where \(\rho_i\) is the correlation coefficient. Its probability of default conditioned upon the Systematic Market Risk Factor \(\mathcal{M}_t\) is given by the subsequent equality:
\[ P_{d|M_t}^{d|M_t} = F_{\tilde{\gamma}_t}(\left(F_{X_{\gamma}^{-1}(\gamma_{\tilde{\gamma}}^{d|M_t})} - \rho_{t|M_t}\right)/\left(1 - \rho_{t}^2\right)) \]

**Part IV: Lévy Factor Copula Model**

- **It is assumed** that \( \tau_i \) is **conditionally independent** with respect to an enlarged filtration \( \mathbb{F} \vee \sigma(M_t) \) under \( \mathbb{Q}^* \), where \( M_t \) is \( \mathcal{G} \) -measurable, i.e.

  \[ \mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n | \mathbb{F}_T \vee \sigma(M_t)) = \prod_{i=1}^{n} \mathbb{Q}^*(\tau_i > t_i | \mathbb{F}_T \vee \sigma(M_t)) \]

- **It is assumed** that the \( \mathbb{F} \)-intensity \( \gamma^i \) of the hazard process \( \Gamma^i \) are **deterministic** and **continuous**.

- **Lévy Factor Copula Model**:

  \[ P_{\tilde{M}_t}^{\xi|M_t} = \mathbb{Q}^* \left( X_{\gamma} < F_{X_{\gamma}}^{-1}(\xi) \right) | M_t \) \]

  \[ = \mathbb{Q}^*(\tau_i < t_i | M_t) = F_{\tilde{\gamma}_t} \left( \frac{F_{X_{\gamma}}^{-1}(\xi) - \rho_{t|M_t}}{\sqrt{1 - \rho_{t}^2}} \right) \]

- **Copula**: \( C(\xi_1, \ldots, \xi_n) = \int \left( \prod_{i=1}^{n} P_{\tilde{M}_t}^{\xi|M_t} \right) f_{M_t}(m) dm \)

- **Default**: \( F(t_1, \ldots, t_n) = \mathbb{Q}^*(\tau_1 \leq t_1, \ldots, \tau_n \leq t_n) = \int \left( \prod_{i=1}^{n} P_{\tilde{M}_t}^{\xi|M_t} \right) f_{M_t}(m) dm \)

- **Survival**: \( G(t_1, \ldots, t_n) = \mathbb{Q}^*(\tau_1 > t_1, \ldots, \tau_n > t_n) = \int \left( \prod_{i=1}^{n} P_{\tilde{M}_t}^{\xi|M_t} \right) f_{M_t}(m) dm \)

**Part V: Stochastic Correlated Lévy Factor Copula Model**

- **Stochastic Correlated Lévy Factor Model** with a **Stochastic correlation factor** \( \bar{\rho}_t \) is given by:

  \[ X_{\gamma} = \bar{\rho}_t M_t + \sqrt{1 - \bar{\rho}_t^2} \tilde{\sigma}_t \]

- **Binary Stochastic Correlated Lévy Factor Model**

  - \( \bar{\rho}_t \) is **Structured** by two constants \( \rho_1, \rho_2 \in \mathbb{I} \) and Bernoulli random variables \( B_t \), i.e. \( \mathbb{P}(B_t = 0) = 1 - q \) and \( \mathbb{P}(B_t = 1) = q \) such that: \( \bar{\rho}_t = (1 - B_t)\rho_1 + B_t\rho_2 \)

  - \( X_{\gamma} = ((1 - B_t)\rho_1 + B_t\rho_2) M_t + \sqrt{1 - ((1 - B_t)\rho_1 + B_t\rho_2)^2} \tilde{\sigma}_t \)
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

Chapter Five: From Lévy Process to Lévy Factor Copula Model

• \[ p_{t_i}^{\xi|M_t} = (1 - q) F_{J_{t_i}} \left( \frac{F_{\xi M_t}^{-1}(F_{\xi M_t}(t)) - \rho_1 M_t}{\sqrt{1 - \rho_1^2}} \right) + q F_{J_{t_i}} \left( \frac{F_{\xi M_t}^{-1}(F_{\xi M_t}(t)) - \rho_2 M_t}{\sqrt{1 - \rho_2^2}} \right) \]

- Symmetric Dependence Stochastic Correlated Lévy Factor Model

• \( \tilde{p}_i \) is Structured by two constants \( \rho_1, \rho_2 \in \mathbb{I} \), and two Bernoulli random variables \( B_i \) and \( B_s \), i.e. \( \mathbb{P}(B_i = 0) = 1 - q \) \( \mathbb{P}(B_i = 1) = q \) \( \mathbb{P}(B_i = 0) = 1 - \tilde{q} \), and \( \mathbb{P}(B_i = 1) = \tilde{q} \), such that: \( \tilde{p}_i = (1 - B_s)(1 - B_i)\rho + B_s \)

• \[ x_{t_i} = ((1 - B_s)(1 - B_i)\rho + B_s) M_t + (1 - B_s) \left( (1 - B_i) \sqrt{1 - \rho^2} + B_i \right) J_{t_i} \]

• \[ p_{t_i}^{\xi|M_t} = \tilde{q} F_{M_t} \left( F_{X_{t_i}^{-1}}(F_{\xi M_t}(t)) \right) + (1 - \tilde{q}) \left[ (1 - q) F_{J_{t_i}} \left( \frac{F_{\xi M_t}^{-1}(F_{\xi M_t}(t)) - \rho M_t}{\sqrt{1 - \rho^2}} \right) + q F_{J_{t_i}} \left( F_{\xi M_t}^{-1}(F_{\xi M_t}(t)) \right) \right] \]

Part VI: Random Factor Loading Lévy Factor Copula Model

- \( \rho_i(M_t) \in \mathbb{R} \) is a random factor loading that is structured by a constant \( \kappa \in \mathbb{R}^+ \), and some input parameters \( \ell_1, \ell_2 \in \mathbb{R}^+ \), such that: \( \rho_i(M_t) = \ell_1 J_{(M_t<\kappa)} + \ell_2 J_{(M_t>\kappa)} \)

- Random Factor Loading Copula Model

• \[ x_{t_i} = \rho_i(M_t) M_t + \nu_i J_{t_i} + \kappa_i \], where \( \nu_i = \sqrt{1 - \mathbb{V}[\rho_i(M_t) M_t]} \) and \( \kappa_i = -\mathbb{E} [\rho_i(M_t) M_t] \).

• \[ p_{t_i}^{\xi|M_t} = F_{J_{t_i}} \left( \frac{F_{X_{t_i}^{-1}}(F_{\xi M_t}(t)) - \kappa_i - \ell_1 M_t}{\sqrt{1 - \ell_1^2}} \right) \]

• \[ p_{t_i}^{\xi|M_t} = F_{J_{t_i}} \left( \frac{F_{X_{t_i}^{-1}}(F_{\xi M_t}(t)) - \kappa_i - \ell_2 M_t}{\sqrt{1 - \ell_2^2}} \right) \]

• Where \( \kappa_i = -\ell_1 \int_{-\infty}^{\kappa} m f_{M_t}(m) dm - \ell_2 \int_{-\infty}^{\kappa} m f_{M_t}(m) dm \).
Chapter Six

Lévy Factor Copula and its Skewed Version from Theory to Application

6.1 Outline

- Introduction

- Lévy Skew Alpha-Stable Distribution
  - A Gaussian Factor Copula
  - A Normalized Mixture Gaussian Factor Copula
  - A Standard Lévy Alpha Skewed Factor Copula

- Generalized Hyperbolic Distribution
  - A Skewed t Factor Copula
  - A Normalised Fractional t Factor Copula
  - A Mixture Standard Gaussian and Normalised Fractional-t Factor Copula
  - Variance Gamma Factor Copula
  - A Mixture Gaussian and Variance Gamma Factor Copula
  - A Mixture Normalised Fractional-t and Variance Gamma Factor Copula
  - A Normal Inverse Gaussian Factor Copula
  - A Mixture Gaussian and Normal Inverse Gaussian Factor Copula
  - A Mixture Fractional-t and Normal Inverse Gaussian Factor Copula

- Conclusion
6.2 Introduction

In Chapter 5, the Gaussian Factor Copula Model has been extended to a Lévy one, where the Lévy Factor Copula Model has been proposed. Subsequently, two cases of the Stochastic Correlated Factor Copula Models have been proposed, i.e. the Lévy Binary Stochastic Correlated Factor Copula Model and the Lévy Symmetric Stochastic Correlated Factor Copula Model. Finally, in the context of the risk exposure by loading its factor, the Lévy Random Factor Loading Copula Model is proposed.

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by a Student-t as in (Andersen et al., 2003), (Embrechts et al., 2003), (Frey and McNeil, 2003), (Mashal et al., 2003), (Mashal R. and Zeevi A., 2003), (Greenberg et al., 2004), (Demarta and McNeil, 2005), (Schloegl and O’Kane, 2005). However, this extended model’s features have the same drawbacks as the original. Other distributions where considered also in the credit domain, i.e. Clayton copula and Marshall-Olkin. Clayton copula was introduced in (Schönbucher and Schubert, 2001), (Schönbucher, 2002), (Rogge and Schönbucher, 2003), (Madan et al., 2004), (Friend and Rogge, 2005), (Laurent and Gregory, 2005), and (Schloegl and O’Kane, 2005), where the Marshall-Olkin copula was introduced in (Duffie and Singleton, 1998b), (Li, 2000),
In the same direction, several additive factor copula models came to extend the Gaussian Factor Copula, such as the Double Student-t Copula model as in (Hull and White, 2004) and (Burtschell et al., 2009) and discussed in (Cousin and Laurent, 2008a) and (Cousin and Laurent, 2008b), the Normal Inverse Gaussian Factor copula model in (Guegan and Houdain, 2005), the Double Normal Inverse Gaussian Factor copula model in (Brunlid, 2006), (Ferrarese, 2006), and (Kalemanova et al., 2007), the double variance gamma Factor copula model in (Brunlid, 2006) and (Moosbrucker, 2006), the Generalised Hyperbolic skewed Student-t Factor Copula Model in (Brunlid, 2006), the $\alpha$-stable copula in (Ferrarese, 2006) and (Prange and Scherer, 2006), the double smoothly truncated stable copula in (Wang et al., 2006), and the double fraction Student-t distribution copula model (Wang et al., 2006) and (Wang et al., 2007).

Recently, various researchers observed how mixture factor copula models could overcome the pricing problem, for example the mixture of multi-Gaussian copula model introduced in (Xu, 2006), the double mixture of $t$ and Gaussian Factor copula model in (Wang et al., 2006) and (Wang et al., 2007), the double mixture Gaussian copula model in (Wang et al., 2006), and the double mixture of Gaussian and Normal Inverse Gaussian factor copula model in (Yang et al., 2009). Some of these factor copula models has been compared in the context of CDO in (Burtschell et al., 2009).

Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

Gaussian Factor model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In
(Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions. On the other hand, extending the Gaussian Factor model by a stochastic risk exposure by loading its factor is extended in (Andersen and Sidenius, 2005) and discussed in (Bürtschell et al., 2009) and (Lüscher, 2005). Some authors have extend this model further, for example, the Normal Inverse Gaussian Random Factor Loading Copula Model in (Lüscher, 2005) and the Normal Inverse Gaussian Random Factor Loading Copula Model by (Yang et al., 2009).

However, all extended models, which are cited above and others who have incorporate skew futures in contradictory to the normality assumption, have not generalised this model in a way that take this model out of this atomic representation as they have tried replacing the Gaussian distribution with skewed one in both the linear representation, except in (Brunlid, 2006), where it extend the Gaussian Assumption to the Generalised Hyperbolic Lévy Assumption but depending on the asset price as a threshold, or the stochastic one.

This chapter inherits Chapter 5’s proposed models and apply the distributions those admits the Lévy process, where there is an infinite alternatives of copulas. It starts by representing the Lévy Skew Alpha-Stable Distribution and Generalized Hyperbolic Distribution. Then, they are specialised as limiting and mixture cases.

Most of these limiting and mixture cases are newly proposed in this thesis, see Table 6.1. Since, each subsection is built to be as self-contained as possible, when the reader is interested in a specific model, he could jump directly to the required subsection. Oppositely, when the reader is going through the whole chapter, he could skip the subsections’ introductions and jump directly to their statements once he pass through at least one of them.
### Table 6.1: Proposed Models

<table>
<thead>
<tr>
<th>Lévy Skew Alpha-Stable</th>
<th>Limiting or Mixture Case</th>
<th>Linear</th>
<th>Stochastic Correlated</th>
<th>Factor Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Binary</td>
<td>Symmetric</td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
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<tr>
<td>Norm. $\mathcal{M}$ Gaussian</td>
<td>NP</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Stand. Lévy Alpha Skewed</td>
<td>NP</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Skewed $t$</td>
<td>NP</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Fractional $t$</td>
<td>NP</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>$\mathcal{M}$ (Stand. Gaussian, Fractional-$t$)</td>
<td>NP</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Variance Gamma</td>
<td>NP</td>
<td>P</td>
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<td>P</td>
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<tr>
<td>$\mathcal{M}$ (Gaussian, Variance Gamma)</td>
<td>P</td>
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</tr>
<tr>
<td>$\mathcal{M}$ (Fractional-$t$, Variance Gamma)</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Normal Inverse Gaussian</td>
<td>NP</td>
<td>P</td>
<td>P</td>
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</tr>
<tr>
<td>$\mathcal{M}$ (Gaussian, Normal Inverse Gaussian)</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>$\mathcal{M}$ (Fractional-$t$, Normal Inverse Gaussian)</td>
<td>P</td>
<td>P</td>
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<td>P</td>
</tr>
</tbody>
</table>

6.3 Lévy Skew Alpha-Stable Distribution

Statistically supported by the Central Limit Theorem (CLT), i.e. the sum of a large number of independent and identical distributed variables those admit a finite-variance distribution will tend to be normally distributed, the financial assets are modelled by the Gaussian distribution; as stated by the pioneer work in (Bachelier, 1900). Nonetheless, based on empirical verifications in (Mandelbrot, 1963) and (Fama, 1965), financial asset returns usually have heavier tails than what Gaussian distribution can provide.

Accordingly, the Lévy Skewed Alpha Stable distributions, which were introduced in (Lévy, 1925), were proposed as an alternative framework in (Mandelbrot, 1963) and (Fama, 1965). Those distributions are supported by at least two strong factors. The first is that these distributions are supported by the Generalised Central Limit Theorem.
(GCLT), i.e. the only possible limit distributions for accurately normalised and centred
sums of independent and identically distributed random variables are the Lévy stable
laws (Laha and Rohatgi, 1979). The second is that they are leptokurtic.

Since Mandelbrot works regarding modelling asset returns in (Mandelbrot, 1960),
(Mandelbrot, 1961), and (Mandelbrot, 1963) rich literature has follow supporting this
alternative framework, for instance, in (Fama, 1963), (Fama, 1965), (Fama and Roll,
1968), (Leitch and Paulson, 1975), (Stuck, 1976), (Peters, 1994), (McCulloch, 1996),
(McCulloch, 1997), (Bidarkota and McCulloch, 1998), (Walter, 1999), (Belkacem et al.,
2000), (Rachev and Mittnik, 2000), (Nolan, 2003), (Rachev, 2003), (Kozubowski et al.,
2003), (Borak et al., 2005), (Haas et al., 2005), (Lombardi and Calzolari, 2005),
(Ortobelli and Rachev, 2005), (Martin et al., 2006), and (Frain, 2007), where in
connection to copulas theory in (Kallsen and Tankov, 2006) and (Prange and Scherer,
2006).

In this subsection the Lévy Skewed Alpha Stable distribution will be represented as in
(Nolan, 2009) and then specialised as limiting and mixture cases.

**Definition 6.1 (Lévy Skew Alpha-Stable Distribution $\mathcal{L}$)**

A random variable $X$ is said to be Lévy Skewed Alpha-Stable$^{24}$, denoted by $X_{\mathcal{L}(\alpha, \beta, \gamma, \delta; 1)}$, if its characteristic function is given by the subsequent equality:

$$
X_{\mathcal{L}(\alpha, \beta, \gamma, \delta; 1)}(x) = \begin{cases} 
  e^{-\gamma|\alpha|(1-i\beta(\text{sign}(x))(\tan(\frac{\pi\alpha}{\alpha})))+i\delta x), & \alpha \neq 1 \\
  e^{-\gamma|x|\left(1+i\beta_2^2(\text{sign}(x))(\frac{\pi\alpha}{\alpha})\right)+i\delta x), & \alpha = 1 
\end{cases}
$$

where its probability density function is recovered through the Inverse Fourier
Transform$^{25}$. With the parameters:

i. **Characteristic exponent**: $\alpha \in ]0,2]$.$^{26}$

---

$^{24}$ It is also called $\alpha$-stable, stable Paretian or Lévy stable

$^{25}$ See Section 8.3 and Section 8.5, for the FFT and VFFT could be used as explained.

$^{26}$ Decreasing $\alpha$ fats the tails.
ii. **Skewness parameter:** \( \beta \in [-1,1] \).

iii. **Scale parameter:** \( \gamma \in \mathbb{R}^+ \).

iv. **Location parameter:** \( \delta \in \mathbb{R} \)

And the \( \text{sign}(x) \) function is defined as following

\[
\text{sign}(x) = \begin{cases} 
-1 & , x < 0 \\
0 & , x = 0 \\
1 & , x > 0 
\end{cases}
\]

It could not be expressed in close form except for three special cases: Gaussian, Cauchy, and Lévy distributions.

**Property D6.1.1 (L: Convolution)**

If \( X \) and \( Y \) are Lévy Skewed Alpha-Stable random variables those follow Definition 6.1, then they are stable under convolution, as shown in the subsequent equality:

\[
X_L(\alpha_1, \beta_1, \gamma_1, \delta_1; 1) + Y_L(\alpha_2, \beta_2, \gamma_2, \delta_2; 1) \sim Z_{L(\alpha_1, \beta_1, \gamma_1, \delta_1; 1) + \gamma_2, \delta_2; 1)}
\]

**Property D6.1.2 (L: Scaling)**

If \( X \) is Lévy Skewed Alpha-Stable random variables those follow Definition 6.1, then for any \( a \neq 0 \) and \( b \in \mathbb{R} \) the subsequent equality hold:

\[
aX_L(\alpha, \beta, \gamma, \delta; 1) + b \sim \begin{cases} 
Y_L(\alpha, \beta, \gamma, \delta; 1), & a \neq 1 \\
Y_L(1, \beta, \gamma, \delta; 1), & a = 1 
\end{cases}
\]

when \( a = \alpha_1 = \alpha_2 \)

**Property D6.1.3 (L: Continuity and Infinity Devisable)**

All Lévy Skewed Alpha-Stable are continuous distributions with an infinitely differentiable density.

**6.3.1 A Gaussian Factor Copula**

The Gaussian Distribution has been brought forward; since, in the literature, it is the baseline that all other Factor Copula cases are compared to. The Gaussian Factor Copula model, which was introduced in (Li, 2000), is the fastest and easiest to model.
and manipulate. It also overcomes the dimensionality and complexity of the credit risk derivatives products. Gaussian Factor Copula model became the market’s standard model even though it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

The first direction is to replace the Gaussian distribution by another distribution that contains more skewness. Conversely, the second direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor. This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Gaussian distribution that admits the Lévy process. This Distribution is introduced as a limiting case of the Lévy Skew Alpha-Stable Distribution articulated in Definition 6.1.

This subsection starts by stating the Gaussian distribution and its properties and subsequently applying it to the proposed models.

### 6.3.1.1 $\mathcal{G}$ Distribution and its Properties

This subsection states the Gaussian distribution and introduces it as a limiting case of the Lévy Skew Alpha-Stable Distribution articulated in Definition 6.1. Subsequently, this distribution is formalised by stating its definition and followed by it properties. The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors
distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Gaussian distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Limiting Case 6.1 (Lévy Skew Alpha-Stable with \( \alpha = 2, \beta = 0 \))**

Let \( X \) be Lévy Skewed Alpha-Stable random variable that admits Definition 6.1, \( \alpha = 2 \) and \( \beta = 0 \), then \( X_{2,0,2,1} \) is said to be a “Normal” or Gaussian, denoted by \( X_{2,2,1} \). For general representation it will be denoted by \( X_{2,2,1} \).

It is proved as a limiting case of the Lévy skewed alpha stable by Property D6.1.2, by substituting \( a = \frac{\sigma}{\sqrt{2}}, b = \mu, \alpha = 2 \) and \( \beta = 0 \) (Nolan, 2009). In most of the literature, the Gaussian distribution is called the “Normal distribution” and also abbreviated as \( N(\mu, \sigma^2) \). In order to standardise the representation, the previous notation and name will be used.

**Definition 6.2 (Gaussian Distribution \( \mathcal{G} \))**

A random variable \( X \) is said to be Gaussian, denoted by \( X_{\mathcal{G}(\mu, \sigma)} \), if its density is given by the subsequent equality:

\[
f_{X_{\mathcal{G}(\mu, \sigma)}}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}
\]

Its moment generating function, denoted by \( \psi(u)_{\mathcal{G}(\mu, \sigma)} \) is given by the following equality:

\[
\psi(u)_{\mathcal{G}(\mu, \sigma)} = e^{\left(\mu u + \frac{\sigma^2 u^2}{2}\right)}
\]
Property D6.2.1 (G: Mean, Variance)

If $X$ is a Gaussian random variable that follows Definition 6.2, then $X_{G(\mu, \sigma)}$ Mean and Variance, respectively, are given by the subsequent equality:

i. $\mathbb{E}[X_{G(\mu, \sigma)}] = \mu$

ii. $\mathbb{V}[X_{G(\mu, \sigma)}] = \sigma^2$

iii. $\mathbb{S}[X_{G(\mu, \sigma)}] = 0$

iv. $\mathbb{K}[X_{G(\mu, \sigma)}] = 3$

A strong argument for using the $G$ distribution with the one factor Lévy copula is its closeness under convolution property, given by the next property.

Property D6.2.2 (G: Convolution)

If $X$ and $Y$ are Gaussian random variables those follow Definition 6.2, then they are stable under convolution, as shown in the subsequent equality:

$$X_{G(\mu_1, \sigma_1)} + Y_{G(\mu_2, \sigma_2)} \sim Z_{G(\mu_{1+2}, \sigma_{1+2})}$$

Property D6.2.3 (G: Scaling)

If $X$ is a Gaussian random variable that follows Definition 6.2, then $X_{G(\mu, \sigma)}$ can be scaled by a constant $c$, as shown in the subsequent equality:

$$X_{G(\mu c, \sigma c^2)}$$

6.3.1.2 G Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits
it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Gaussian Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6.

By admitting Assumption 5.2 and the Gaussian Distribution, \( \mathcal{M}_t, \mathcal{J}_t, \) and \( \mathcal{X}_t \) are given by the following Lemma.

**Lemma 6.1** \((\mathcal{M}_t^\mathcal{G}, \mathcal{J}_t^\mathcal{G}, \) and \( \mathcal{X}_t^\mathcal{G} \) with \( SG \) Distribution Functions)\)

\[
\text{Let } \left( \mathcal{X}_t^\mathcal{G} \right)_{t \in \mathbb{R}^+} \text{ be a Lévy process that follows Corollary 5.11 and specialised upon Limiting Case 6.1 as a } \mathcal{G} \text{ that admits Definition 6.2, with } \mathcal{M}_t^\mathcal{G}, \mathcal{J}_t^\mathcal{G}, \text{ and } \mathcal{X}_t^\mathcal{G} \text{ as, respectively, the } \mathcal{G} \text{ systematic market risk factor, and the } \mathcal{G} \text{ idiosyncratic risk factors. Then for } \mathcal{M}_t^\mathcal{G}, \mathcal{J}_t^\mathcal{G}, \text{ and } \mathcal{X}_t^\mathcal{G} \text{ to admits Assumption 5.2 conditions, their parameters has to be set as following:}
\]

\[
i. \quad \mathcal{M}_t^\mathcal{G}(\mu=0, \sigma=1) = \mathcal{M}_t^\mathcal{SG} \\
ii. \quad \mathcal{J}_t^\mathcal{G}(\mu=0, \sigma=1) = \mathcal{J}_t^\mathcal{SG} \\
iii. \quad \mathcal{X}_t^\mathcal{G}(\mu=0, \sigma=1) = \mathcal{X}_t^\mathcal{SG}
\]

where \( \mathcal{SG} \) is the Standard Gaussian Distribution.

To standardise the notation of the previous Lemma, the Standard Gaussian Distribution is given as a limiting case of the Gaussian Distribution and then defined upon it.

**Limiting Case 6.2** \((\text{Gaussian Distribution } \mathcal{G} \text{ with } \mu = 0, \sigma = 1)\)

Let \( \mathcal{X} \) be Gaussian random variable that admits Definition 6.2, \( \mu = 0 \) and \( \sigma = 1 \), then \( \mathcal{X}_{\mathcal{G}(0,1)} \) is said to be a “Standard Normal” or Standard Gaussian, denoted by \( \mathcal{X}_{\mathcal{SG}(0,1)} \).

or for shortness purposes \( \mathcal{X}_{\mathcal{SG}} \).
**Definition 6.3 (Standard Gaussian Distribution \( \mathcal{G} \))**

A random variable \( X \) is said to be Standard Gaussian, denoted by \( X_{\mathcal{G}} \), if its density is given by the subsequent equality:

\[
f_{X_{\mathcal{G}}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}
\]

Its moment generating function, denoted by \( \psi(u)_{X_{\mathcal{G}}} \) is given by the following equality:

\[
\psi(u)_{X_{\mathcal{G}}} = e^{\left(\frac{u^2}{2}\right)}
\]

**Corollary 6.1 (\( \mathcal{G} \) Factor Copula Model)**

Let \( (X_{\mathcal{G}}^{t_1})_{t \in \mathbb{R}^+}, M_{\mathcal{G}}^{t_1}, \) and \( \xi_{\mathcal{G}}^{t_1} \) be, respectively, a Lévy process specialised as a \( \mathcal{G} \), the \( \mathcal{G} \) systematic market risk factor, and the \( \mathcal{G} \) idiosyncratic risk those follow Lemma 6.1, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit Assumption 4.12 and Definition 4.15, and \( p_{\tau_i|M_{\mathcal{G}}^{t_1}}^{\xi_i} \) be the probability of \( \tau_i \) conditioned upon \( M_{\mathcal{G}}^{t_1} \). Then \( p_{\tau_i|M_{\mathcal{G}}^{t_1}}^{\xi_i} \) of the \( \mathcal{G} \) Factor Copula Model is given by the subsequent equality:

\[
p_{\tau_i|M_{\mathcal{G}}^{t_1}}^{\xi_i} = F_{\mathcal{G}}^{t_1}\left(\frac{F_{X_{\mathcal{G}}^{-1}}^{t_1}\left(F_{\xi_i}^{t_1}(t)\right) - \rho_i M_{\mathcal{G}}^{t_1}}{\sqrt{1 - \rho_i^2}}\right)
\]

### 6.3.1.3 Binary Stochastic Correlated \( \mathcal{G} \) Factor Copula Model

As articulated earlier, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is inapplicable to fit the market tranches; since it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005).
In this subsection the stochastic Correlation implementation is structured by the binary
distribution, where a direct result could be obtained when injecting the Gaussian
Distribution in the Lévy Binary Stochastic Correlated Factor Copula Model that was
articulated in Subsection 5.7.1. This model overcomes the limitation of the standard
model; it contains more tail dependence and thus it is proposed as better alternative.

Remark 6.1 ($\mathcal{M}_{SG}^t$, $\mathcal{J}_{SG}^t$, and $\mathcal{X}_{SG}^t$ with $\mathcal{S}\mathcal{G}$ Distribution Functions)

In this subsection, the parameters of $\left(\mathcal{X}_{SG}^t\right)_{t \in \mathbb{R}_+}$, $\mathcal{M}_{SG}^t$, and $\mathcal{J}_{SG}^t$ are admitting Lemma 6.1 but structured as in Lemma 5.8 instead of Corollary 5.11.

Corollary 6.2 (Binary Stochastic Correlated $\mathcal{S}\mathcal{G}$ Factor Copula Model)

Let $\left(\mathcal{X}_{SG}^t\right)_{t \in \mathbb{R}_+}$, $\mathcal{M}_{SG}^t$, and $\mathcal{J}_{SG}^t$ be, respectively, a Lévy process specialised as a $\mathcal{S}\mathcal{G}$, the $\mathcal{S}\mathcal{G}$ systematic market risk factor, and the $\mathcal{S}\mathcal{G}$ idiosyncratic risk factor those follow Lemma 6.1 and structured by the Binary stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{t_i}^{\xi_i|\mathcal{M}_{SG}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{SG}^t$. Then $p_{t_i}^{\xi_i|\mathcal{M}_{SG}^t}$ of the Binary Stochastic Correlated $\mathcal{S}\mathcal{G}$ Factor Copula Model is given by the subsequent equality:

$$
p_{t_i}^{\xi_i|\mathcal{M}_{SG}^t} = (1 - q) F_{\mathcal{J}_{SG}^t}^{-1} \left( F_{\mathcal{X}_{SG}^t}^{-1} \left( F_{\mathcal{X}_{SG}^t}^{-1} (t) - \rho_1 \mathcal{M}_{SG}^t \right) \right) \frac{1}{\sqrt{1 - \rho_1^2}}
+ q F_{\mathcal{J}_{SG}^t}^{-1} \left( F_{\mathcal{X}_{SG}^t}^{-1} \left( F_{\mathcal{X}_{SG}^t}^{-1} (t) - \rho_2 \mathcal{M}_{SG}^t \right) \right) \frac{1}{\sqrt{1 - \rho_2^2}}
$$

6.3.1.4 Symmetric Stochastic Correlated $\mathcal{G}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”,
which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it
over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005).

In this subsection a straightforward consequence could be achieved once inserting the Gaussian Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative.

Remark 6.2 ($\mathcal{M}_S^t$, $\mathcal{N}_S^t$, and $\mathcal{X}_S^t$ with $SG$ Distribution Functions)

In this subsection, the parameters of $\left(\mathcal{X}_S^t\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_S^t$, and $\mathcal{N}_S^t$ are admitting Lemma 6.1 but structured as in Lemma 5.9 instead of Corollary 5.11.

Corollary 6.3 (Symmetric Stochastic Correlated $SG$ Factor Copula Model)

Let $\left(\mathcal{X}_S^t\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_S^t$, and $\mathcal{N}_S^t$ be, respectively, a Lévy process specialised as a $SG$, the $SG$ systematic market risk factor, and the $SG$ idiosyncratic risk factor those follow Lemma 6.1 and structured by the symmetric stochastic correlation, $\xi$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi \mid \mathcal{M}_S^t}$ be the probability of $\tau$ conditioned upon $\mathcal{M}_S^t$. Then $p_{\xi \mid \mathcal{M}_S^t}$ of the Symmetric Stochastic Correlated $SG$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi \mid \mathcal{M}_S^t} = \hat{q} F_{\xi \mid \mathcal{M}_S^t}(t) + (1 - \hat{q}) \left[ (1 - q) F_{\xi \mid \mathcal{M}_S^t} \left( \frac{F_{\xi \mid \mathcal{M}_S^t}^{-1}(F_{\xi \mid \mathcal{M}_S^t}(t)) - \rho \mathcal{M}_S^t}{\sqrt{1 - \rho^2}} \right) + q F_{\xi \mid \mathcal{M}_S^t}(t) \right]$$
6.3.1.6 \( G \) Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome difference between the markets’ and the based model’s loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Gaussian Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8.

**Remark 6.3 \( (M_{SG}^{t}, \vartheta_{SG}^{t}, \text{and } X_{SG}^{t} \text{ with } SG \text{ Distribution Functions}) \)**

In this subsection, the parameters of \( \left( X_{SG}^{t} \right)_{t \in \mathbb{R}^{+}}, M_{SG}^{t}, \text{and } \vartheta_{SG}^{t} \) are admitting Lemma 6.1 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

**Corollary 6.4 \( (SG \text{ Random Factor Loading Copula Model}) \)**

Let \( \varphi_{N_{t}} \) be the unconditional number of default’s characteristic function that follows Theorem 5.5. \( \left( X_{SG}^{t} \right)_{t \in \mathbb{R}^{+}}, M_{SG}^{t}, \text{and } \vartheta_{SG}^{t} \) be, respectively, a Lévy process specialised as a \( SG \), the \( SG \) systematic market risk factor, and the \( SG \) idiosyncratic risk factor those follow Lemma 6.1 and structured by the random factor loading. Then \( \varphi_{N_{t}}^{SG} \) of the \( SG \) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi_{N_{t}}^{SG}(u) = \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu}) F_{\vartheta_{SG}^{t}}^{\text{v}^{-1} \left( F_{t_{i}}(t) - \vartheta_{t_{i}} - z_{1} \right)} \left( \frac{\vartheta_{SG}^{t} \left( F_{t_{i}}(t) - \vartheta_{t_{i}} - z_{1} \right)}{1 - z_{1}^{2}} \right) \right) f_{M_{SG}^{t}}(m) dm + \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu}) F_{\vartheta_{SG}^{t}}^{\text{v}^{-1} \left( F_{t_{i}}(t) - \vartheta_{t_{i}} - z_{2} \right)} \left( \frac{\vartheta_{SG}^{t} \left( F_{t_{i}}(t) - \vartheta_{t_{i}} - z_{2} \right)}{1 - z_{2}^{2}} \right) \right) f_{M_{SG}^{t}}(m) dm
\]
Where \( k_i = -\ell_1 \int_{-\infty}^{\infty} mf_{\mathcal{M}_{\mathcal{S}_{\mathcal{G}}}^i}(m) dm - \ell_2 \int_{\mathcal{K}} mf_{\mathcal{M}_{\mathcal{S}_{\mathcal{G}}}^i}(m) dm. \)

6.3.2 A Normalized Mixture Gaussian Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply mixture Gaussian distribution that admits the Lévy process.

This Distribution is introduced as a mixture distribution of two or more Gaussian distributions. For simplicity, we consider the case of a mixture distribution of two Gaussian distributions which have a zero mean.

This subsection starts by stating the mixture Gaussian distribution and its properties and subsequently applying it to the proposed models.
6.3.2.1 $\mathbb{M}^N_\mathcal{G}$ Distribution and its Properties

This subsection states the mixture Gaussian distribution and introduces it as a mixture of two Gaussian distributions. Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admit Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the mixture Gaussian distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Mixture Case 6.1 (Normalized Double Mixture Gaussian $\mathbb{M}^N_\mathcal{G}$)**

Let $f_{\mathcal{G}(\mu_1, \sigma_1)}$ and $f_{\mathcal{G}(\mu_2, \sigma_2)}$ be two independent Gaussian Distributions, where $\mathcal{G}(\mu_1, \sigma_1)$ and $\mathcal{G}(\mu_2, \sigma_2)$ admits Definition 6.2, and

i. $\sigma_1 > \sigma_2$

ii. $\mathbb{E} \left[ y_{\mathcal{G}(\mu_1, \sigma_1)} \right] = \mu_1 = 0$

iii. $\mathbb{E} \left[ w_{\mathcal{G}(\mu_2, \sigma_2)} \right] = \mu_2 = 0$

iv. $p \in (0,1) \text{ as the probability of occurrence}$

v. $\sigma^2 = \mathbb{V} \left[ p \left( y_{\mathcal{G}(0, \sigma_1)} \right) + (1 - p) \left( w_{\mathcal{G}(0, \sigma_2)} \right) \right]$

\[ = p \sigma_1^2 + (1 - p) \sigma_2^2 \]

Then the normalization factor is given by $\kappa = \frac{1}{\sigma}$ and the normalized mixture is said to be Normalized Double Mixture Gaussian, denoted by $\mathcal{X}_{\mathbb{M}^N_\mathcal{G}(\sigma_1, \sigma_2; p)}$. 
Definition 6.4 (Normalized Double Mixture Gaussian $\mathbb{M}_g^{\mathcal{N}}$)

A random variable $X$ is said to be Normalized Double Mixture Gaussian, denoted by $X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}$, with a normalization factor $\kappa = \frac{1}{\sigma}$ if its density is given by the subsequent equality:

$$f_{X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}}(x) = \frac{p\sigma}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}(\frac{x}{\sigma_1})^2} + \frac{(1-p)\sigma}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}(\frac{x}{\sigma_2})^2}$$

Property D6.4.1 ($\mathbb{M}_g^{\mathcal{N}}$: Inheritance)

Since $X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}$ is a mixture of two independent Gaussian random variables that follows Definition 6.2, then each $X_{\mathbb{M}_g^{\mathcal{N}}(\mu,\sigma)}$ inherits the properties of the Gaussian Distribution.

Property D6.4.2 ($\mathbb{M}_g^{\mathcal{N}}$: Mean, Variance)

If $X$ is a Normalized Double Mixture Gaussian random variable that follows Definition 6.4, then by definition of $X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}$, the Mean and Variance, respectively, are given by the subsequent equality:

i. $\mathbb{E}[X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}] = 0$

ii. $\mathbb{V}[X_{\mathbb{M}_g^{\mathcal{N}}(\sigma_1,\sigma_2;p)}] = 1$

6.3.2.2 $\mathbb{M}_g^{\mathcal{N}}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must
admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

Recently, various researchers observed how mixture factor copula models could overcome the pricing problem. The mixture of multi-Gaussian copula model is introduce in (Xu, 2006). This model could be seen as an immediate result when applying the mixture Gaussian distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6.

By admitting Assumption 5.2 and the $\mathcal{M}_G^N$ Distribution $\mathcal{M}_t$, $\mathcal{J}_t$, and $\mathcal{X}_t$ are given by the following Lemma.

**Lemma 6.2 ($\mathcal{M}_G^N$, $\mathcal{J}_G^N$, and $\mathcal{X}_G^N$ with $\mathcal{M}_G^N$ Distribution Functions)**

Let $\left(\mathcal{X}_{t,i}^{G,N}\right)_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.1 as a $\mathcal{M}_G^N$ that admits Definition 6.4, with $\mathcal{M}_G^t$, $\mathcal{J}_G^t$, and $\mathcal{X}_G^t$ as, respectively, the $\mathcal{M}_G^N$ systematic market risk factor, and the $\mathcal{M}_G^N$ idiosyncratic risk factor. Then by definition of $\mathcal{M}_G^N$, the parameters $\sigma_1$, $\sigma_2$, and $\rho$ are not restricted$^{27}$ in order for $\mathcal{M}_G^t$, $\mathcal{J}_G^t$, and $\mathcal{X}_G^t$ to admit Assumption 5.2.

**Corollary 6.5 ($\mathcal{M}_G^N$ Factor Copula Model)**

Let $\left(\mathcal{X}_{t,i}^{G,N}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_G^t$, and $\mathcal{J}_G^t$ be, respectively, a Lévy process specialised as a $\mathcal{M}_G^N$, the $\mathcal{M}_G^N$ systematic market risk factor, and the $\mathcal{M}_G^N$ idiosyncratic risk factor those follow Lemma 6.2, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{t_i}$ be the

$^{27}$These parameters have only the definition’s restrictions.
probability of $\tau_i$ conditioned upon $M^{t \mid M_g^N}_{g^N}$. Then $p_{t_i} = F^{t_i}_{M_g^N}$ of the $\mathcal{NDG}$ Factor Copula Model is given by the subsequent equality:

$$p_{t_i} = F^{t_i}_{M_g^N} \left( \frac{F^{-1}\chi^{t_i}_{M_g^N}(F_{t_i}(t)) - \rho_t M^{t \mid M_g^N}}{\sqrt{1 - \rho_t^2}} \right)$$

### 6.3.2.3 Binary Stochastic Correlated $\mathcal{M}_g^N$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005).

In this subsection a direct result could be obtained when injecting the normalised mixture Gaussian distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative.

**Remark 6.4 ($\mathcal{M}_g^N$, $\mathcal{J}_g^N$, and $\mathcal{X}_g^N$ with $\mathcal{M}_g^N$ Distribution Functions)**

In this subsection, the parameters of $\left(\mathcal{X}_g^N, \mathcal{M}_g^N, \mathcal{J}_g^N\right)$ are admitting Lemma 6.2 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.6 (Binary Stochastic Correlated $\mathcal{M}_g^N$ Factor Copula Model)**

Let $\left(\mathcal{X}_g^N, \mathcal{M}_g^N, \mathcal{J}_g^N\right)$ be, respectively, a Lévy process specialised as a $\mathcal{M}_g^N$, the $\mathcal{M}_g^N$ systematic market risk factor, and the $\mathcal{M}_g^N$ idiosyncratic risk factor those
follow Lemma 6.2 and structured by the Binary stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \)

admit Assumption 4.12 and Definition 4.15, and \( p_{\tau_i} \) be the probability of \( \tau_i \)

conditioned upon \( \mathcal{M}^t \). Then \( p_{\tau_i} \) of the Binary Stochastic Correlated \( \mathbb{M}^N \) Factor Copula Model is given by the subsequent equality:

\[
\xi_i \big| \mathcal{M}^t \mathbb{M}^N = (1 - q) F_{\mathcal{J}^t \mathcal{M}^t \mathbb{M}^N} \left( \frac{F_{-1}^{-1} \left( F_{\tau_i} (t) - \rho_1 \mathcal{M}^t \mathbb{M}^N \right)}{\sqrt{1 - \rho_1^2}} \right) \\
+ q F_{\mathcal{J}^t \mathcal{M}^t \mathbb{M}^N} \left( \frac{F_{-1}^{-1} \left( F_{\tau_i} (t) - \rho_2 \mathcal{M}^t \mathbb{M}^N \right)}{\sqrt{1 - \rho_2^2}} \right)
\]

6.3.2.4 Symmetric Stochastic Correlated \( \mathbb{M}^N \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranche, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005).

In this subsection a straightforward consequence could be achieved once inserting the normalised mixture Gaussian distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative.
Remark 6.5 \((\mathcal{M}_{\frac{N}{G}}^{t}, \mathcal{J}_{\frac{N}{G}}^{t}, \text{ and } \mathcal{X}_{\frac{N}{G}}^{t}) \text{ with } \mathcal{G}^{N}_{\frac{G}} \text{ Distribution Functions})\)

In this subsection, the parameters of \(\left(\mathcal{X}_{\frac{N}{G}}^{t}\right)_{t \in \mathbb{R}^{+}}, \mathcal{M}_{\frac{N}{G}}^{t}, \text{ and } \mathcal{J}_{\frac{N}{G}}^{t}\) are admitting Lemma 6.2 but structured as in Lemma 5.9 instead of Corollary 5.11.

Corollary 6.7 (Symmetric Stochastic Correlated \(\mathcal{M}_{\frac{N}{G}}^{t}\) Factor Copula Model)

Let \(\left(\mathcal{X}_{\frac{N}{G}}^{t}\right)_{t \in \mathbb{R}^{+}}, \mathcal{M}_{\frac{N}{G}}^{t}, \text{ and } \mathcal{J}_{\frac{N}{G}}^{t}\) be, respectively, a Lévy process specialised as a \(\mathcal{M}_{\frac{N}{G}}^{t}\), the \(\mathcal{M}_{\frac{N}{G}}^{t}\) systematic market risk factor, and the \(\mathcal{M}_{\frac{N}{G}}^{t}\) idiosyncratic risk factor those follow Lemma 6.2 and structured by the symmetric stochastic correlation, \(\xi_{t}\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_{t}\) admit Assumption 4.12 and Definition 4.15, and \(p_{t_{i}}\) be the probability of \(\tau_{t}\) conditioned upon \(\mathcal{M}_{\frac{N}{G}}^{t}\). Then \(p_{t_{i}}\) of the Symmetric Stochastic Correlated \(\mathcal{M}_{\frac{N}{G}}^{t}\) Factor Copula Model is given by the subsequent equality:

\[
p_{t_{i}} = q F_{\mathcal{M}_{\frac{N}{G}}^{t}}^{-1} \left( F_{\mathcal{X}_{\frac{N}{G}}^{t}}^{-1} \left( F_{t_{i}}(t) \right) \right) \\
+ (1 - q) \left[ \left(1 - q\right) F_{\mathcal{J}_{\frac{N}{G}}^{t}}^{\mathcal{M}_{\frac{N}{G}}^{t}} \left( \frac{F_{\mathcal{M}_{\frac{N}{G}}^{t}}^{-1} \left( F_{t_{i}}(t) \right) - \rho \mathcal{M}_{\frac{N}{G}}^{t}}{\sqrt{1 - \rho^{2}}} \right) \right] \\
+ q F_{\mathcal{J}_{\frac{N}{G}}^{t}}^{\mathcal{M}_{\frac{N}{G}}^{t}} \left( F_{\mathcal{X}_{\frac{N}{G}}^{t}}^{-1} \left( F_{t_{i}}(t_{i}) \right) \right)
\]

6.3.2.5 \(\mathcal{M}_{\frac{N}{G}}^{t}\) Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005]

has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor
Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the normalised mixture Gaussian distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8.

**Remark 6.6**

In this subsection, the parameters of \( \mathcal{M}_{M_g^N}^t, \mathcal{J}_{M_g^N}^{t_i}, \) and \( X_{M_g^N}^{t_i} \) with \( \mathcal{M}_G^N \) Distribution Functions

In this subsection, the parameters of \( (X_{M_g^N}^{t_i})_{t \in \mathbb{R}^+}, \mathcal{M}_M^N, \) and \( \mathcal{J}_{M_g^N}^{t_i} \) are admitting Lemma 6.2 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

**Corollary 6.8** (\( \mathcal{M}_{M_g^N}^N \) Random Factor Loading Copula Model)

Let \( \varphi_{N_1} \) be the unconditional number of default’s characteristic function that follows

Theorem 5.5, \( (X_{M_g^N}^{t_i})_{t \in \mathbb{R}^+}, \mathcal{M}_M^N, \) and \( \mathcal{J}_{M_g^N}^{t_i} \) be, respectively, a Lévy process specialised as a \( \mathcal{M}_M^N \), the \( \mathcal{M}_M^N \) systematic market risk factor, and the \( \mathcal{M}_M^N \) idiosyncratic risk factor those follow Lemma 6.2 and structured by the random factor loading. Then \( \varphi_{N_1} \) of the \( \mathcal{M}_M^N \) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\begin{align*}
\varphi_{N_1}(u) &= \int_{-\infty}^\infty \left( 1 - (1 - e^{iu}) \right) f_{\mathcal{J}_{M_g^N}^{t_i}} \left( \frac{F^{-1}_{X_{M_g^N}^{t_i}}(F_{t_i}(t)) - \ell_i - \ell_1 m}{\sqrt{1 - \ell_1^2}} \right) \int_{\mathcal{M}_M^N} dm \\
&\quad + \int_{\infty}^\infty \left( 1 - (1 - e^{iu}) \right) f_{\mathcal{J}_{M_g^N}^{t_i}} \left( \frac{F^{-1}_{X_{M_g^N}^{t_i}}(F_{t_i}(t)) - \ell_i - \ell_2 m}{\sqrt{1 - \ell_2^2}} \right) \int_{\mathcal{M}_M^N} dm
\end{align*}
\]
6.3.3 A Standard Lévy Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Standard Lévy distribution that admits the Lévy process. This Distribution is introduced as a limiting case of the Lévy Skew Alpha-Stable Distribution articulated in Definition 6.1.

This subsection starts by stating the Standard Lévy distribution and its properties and subsequently applying it to the proposed models.

6.3.2.1 SL Distribution and its Properties

This subsection states the Standard Lévy distribution and introduces it as a limiting case of the Lévy Skew Alpha-Stable Distribution articulated in Definition 6.1.

\[ \mathcal{K}_i = -\ell_1 \int_{-\infty}^{\infty} m f_{\mathcal{M}_{\mathcal{K}_i}}(m) dm - \ell_2 \int_{\mathcal{K}}^{\infty} m f_{\mathcal{M}_{\mathcal{K}_i}}(m) dm. \]
Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Standard Lévy distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Limiting Case 6.3 (Lévy Skew Alpha-Stable with γ = 1, δ = 0)**

Let \( X \) be Lévy Skewed Alpha-Stable random variable that admits Definition 6.1, \( γ = 1 \) and \( δ = 0 \), then \( X_{\nu(\alpha, \beta, 1, 0; 1)} \) is said to be a Standard Lévy Skew Alpha-Stable or Standard Lévy, denoted by \( X_{\nu(\alpha, \beta)} \). For general representation it will be denoted by \( X_{\nu(\alpha, \beta)} \).

**Definition 6.5 (Standard Lévy \( \nu \))**

A random variable \( X \) is said to be Standard Lévy, denoted by \( X_{\nu(\alpha, \beta)} \), if its characteristic function is given by the subsequent equality:

\[
X_{\nu(\alpha, \beta)}(x) = \begin{cases} 
-1^{\alpha}|x|^\alpha \left(1-i\beta \text{sign}(x) \left(\tan\left(\frac{\pi\alpha}{2}\right)\right)\right), & \alpha \neq 1 \\
1-|x| \left(1+i\beta \frac{\pi}{2} \text{sign}(x) \left(\tan\left(\frac{\pi\alpha}{2}\right)\right)\right), & \alpha = 1
\end{cases}
\]

where its probability density function is recovered throw the Inverse Fourier Transform. With a parameters:

v. **Characteristic exponent:** \( \alpha \in ]0,2[ \).

vi. **Skewness parameter:** \( \beta \in [-1,1] \).
And the sign(x) function is defined as following

\[
\text{sign}(x) = \begin{cases} 
-1, & x < 0 \\
0, & x = 0 \\
1, & x > 0 
\end{cases}
\]

Remark 6.7 (SL: Mean & Skewness)

i. Since \( \alpha \in [0,2] \), the classical variance, skewness, kurtosis and further moments are not defined for non-Gaussian Lévy.

ii. The mean if \( \alpha \leq 1 \) is undefined

iii. \( \beta \) is not a classical skewness parameter.

Property D6.5.1 (SL: Inheritance)

Since \( X_{\text{SL}(\alpha,\beta)} \) is a special case of the Lévy Skewed Alpha-Stable that admits Definition 6.1, then each \( X_{\text{SL}(\alpha,\beta)} \) inherits its properties, i.e. convolution, scaling, continuity.

Property D6.5.2 (SL: Reflection)

If \( X_{\text{SL}(\alpha,\beta)} \) be is Standard Lévy that admits Definition 6.5, then for any \( \alpha \) and \( \beta \) the subsequent equalities hold:

i. \( X_{\text{SL}(\alpha,-\beta)} = -X_{\text{SL}(\alpha,\beta)} \)

ii. \( f_{X_{\text{SL}(\alpha,\beta)}}(x) = f_{X_{\text{SL}(\alpha,-\beta)}}(-x) \)

iii. \( F_{X_{\text{SL}(\alpha,\beta)}}(x) = 1 - F_{X_{\text{SL}(\alpha,-\beta)}}(-x) \)

Since the moments of the Standard Lévy Distribution do not always exists, as explained briefly in Remark 6.7, the use of fractional absolute moments will replace the classical moments.

Property D6.5.3 (SL: Fractional Absolute Moments)

If \( X \) is a Standard Lévy random variable that follows Definition 6.5, then its fractional absolute moments are given by the subsequent equality:

\[
E \left[ \left| X_{\text{SL}(\alpha,\beta)} \right|^\phi \right] = \int_{-\infty}^{\infty} |x|^\phi f_{X_{\text{SL}(\alpha,\beta)}}(x) dx
\]
where $0 < \theta < \alpha$

6.3.3.2 $SL$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Standard Lévy Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. This Standard Lévy Factor Copula Model was proposed in (Ferrarese, 2006) and (Prange and Scherer, 2006).

As a consequence of the nonexistence of the Standard Lévy Distributions moments, structuring the dependence, i.e. “correlation”, of the Lévy process that follows Corollary 5.11 and its extensions with the Standard Lévy Distributions is not applicable. Therefore, the dependence will be structured by the use of the Fractional Absolute Moments as in Property D6.5.3 (Prange and Scherer, 2006) and by taking into account Assumption 5.2 and the $SL$ Distribution definition.

Assumption 6.1 (Dependence: Coefficients)

Let $(X_t)_{t \in \mathbb{R}^+}$ be a Lévy process, $\mathcal{M}_t$ be the systematic market risk factor, and $\mathcal{J}_t$ be the idiosyncratic risk factor factors those admits Assumption 5.2, with a coefficients
factor $\rho_i$ that follows Assumption 5.3, then the Lévy Factor Model of $X_{t_i}$ is represented by the subsequent equality:

$$X_{t_i} = \rho_i \mathcal{M}_t + \sqrt{1 - \rho_i^2} \mathcal{J}_{t_i}$$

**Lemma 6.3 (\(\mathcal{M}_{SL}^t, \mathcal{J}_{SL}^{t_i}, \text{and} \ X_{SL}^{t_i}\) with \(\mathcal{S}\mathcal{L}\) Distribution Functions)**

Let \(\left( X_{SL}^{t_i} \right)_{t \in \mathbb{R}^+} \) be a Lévy process that follows Corollary 5.11 with restructuring the dependence as in Assumption 6.1 and specialised upon Limiting Case 6.3 as a \(\mathcal{S}\mathcal{L}\) that admits Definition 6.5, with \(\mathcal{M}_{SL}^t\), \(\mathcal{J}_{SL}^{t_i}\) as, respectively, the \(\mathcal{S}\mathcal{L}\) systematic market risk factor, and the \(\mathcal{S}\mathcal{L}\) idiosyncratic risk factor. Then by definition of \(\mathcal{S}\mathcal{L}\) and its properties, the parameters $\alpha$ and $\beta$ do not have any restrictions, other than those given by definition of \(\mathcal{M}_{SL}^t\), \(\mathcal{J}_{SL}^{t_i}\), and \(X_{SL}^{t_i}\) to admits Assumption 5.2 conditions. However, $\alpha$ and $\beta$ of \(\mathcal{M}_{SL}^t\) and \(\mathcal{J}_{SL}^{t_i}\) are assumed to be equal in sequence to have \(X_{SL}^{t_i}\) with the same parameters, i.e. \(\mathcal{S}\mathcal{L}(\alpha, \beta)\).

**Corollary 6.9 (\(\mathcal{S}\mathcal{L}\) Factor Copula Model)**

Let \(\left( X_{SL}^{t_i} \right)_{t \in \mathbb{R}^+}, \mathcal{M}_{SL}^t, \text{and} \mathcal{J}_{SL}^{t_i}\) be, respectively, a Lévy process specialised as a \(\mathcal{S}\mathcal{L}\), the \(\mathcal{S}\mathcal{L}\) systematic market risk factor, and the \(\mathcal{S}\mathcal{L}\) idiosyncratic risk factor those follow Lemma 6.3, $\xi_t$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i|\mathcal{M}_{SL}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{SL}^t$. Then $p_{\xi_i|\mathcal{M}_{SL}^t}$ of the \(\mathcal{S}\mathcal{L}\) Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i|\mathcal{M}_{SL}^t} = F_{\mathcal{J}_{SL}^{t_i}} \left( \frac{F_{X_{SL}^{t_i}}^{-1} \left( F_{\xi_i}(t) \right) - \rho_i \mathcal{M}_{SL}^t}{\sqrt{1 - \rho_i^2}} \right)$$
6.3.3.3 Binary Stochastic Correlated \( SL \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Standard Lévy Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. This Standard Lévy Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

Remark 6.8 (\( M_{SL}^t, J_{SL}^t, \) and \( X_{SL}^{ti} \) with \( SL \) Distribution Functions)

In this subsection, the parameters of \( (X_{SL}^{ti})_{t \in \mathbb{R}^+}, M_{SL}^t, \) and \( J_{SL}^t \) are admitting Lemma 6.3 but structured as in Lemma 5.8 instead of Corollary 5.11.

Corollary 6.10 (Binary Stochastic Correlated \( SL \) Factor Copula Model)

Let \( (X_{SL}^{ti})_{t \in \mathbb{R}^+}, M_{SL}^t, \) and \( J_{SL}^t \) be, respectively, a Lévy process specialised as a \( SL \), the \( SL \) systematic market risk factor, and the \( SL \) idiosyncratic risk factor those follow Lemma 6.3 and structured by the Binary stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit
Assumption 4.12 and Definition 4.15, and $p_{\xi_i|\mathcal{M}_{SL}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{SL}^t$. Then $p_{\xi_i|\mathcal{M}_{SL}^t}$ of the Binary Stochastic Correlated SL Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i|\mathcal{M}_{SL}^t} = (1 - q)F_{\alpha|\mathcal{M}_{SL}^t} \left( \frac{\mathcal{X}_{SL}^{-1}(F_{\xi_i}(t)) - \rho_1 \mathcal{M}_{SL}^t}{\sqrt{1 - \rho_i^2}} \right)$$

$$+ qF_{\alpha|\mathcal{M}_{SL}^t} \left( \frac{\mathcal{X}_{SL}^{-1}(F_{\xi_i}(t)) - \rho_2 \mathcal{M}_{SL}^t}{\sqrt{1 - \rho_i^2}} \right)$$

### 6.3.3.4 Symmetric Stochastic Correlated SL Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In this subsection a straightforward consequence could be achieved once inserting the Standard Lévy Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Standard Lévy Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.9 ($\mathcal{M}_{SL}^t$, $\mathcal{X}_{SL}^{t_i}$, and $\mathcal{J}_{SL}^{t_i}$ with SL Distribution Functions)**

*In this subsection, the parameters of $(\mathcal{X}_{SL}^{t_i})_{t \in \mathbb{R}^+}$, $\mathcal{M}_{SL}^t$, and $\mathcal{J}_{SL}^{t_i}$ are admitting Lemma 6.3 but structured as in Corollary 5.15 instead of Corollary 5.11.*
Corollary 6.11 (Symmetric Stochastic Correlated $SL$ Factor Copula Model)

Let $\left( X_{t}^{i} \right)_{t \in \mathbb{R}^+}$, $M_{SL}^{i}$, and $I_{SL}^{t}$ be, respectively, a Lévy process specialised as a $SL$, the $SL$ systematic market risk factor, and the $SL$ idiosyncratic risk factor those follow Lemma 6.3 and structured by the symmetric stochastic correlation, $\xi_1$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{t_i}^{\xi_i|M_{SL}^{i}}$ be the probability of $\tau_i$ conditioned upon $M_{SL}^{i}$. Then $p_{t_i}^{\xi_i|M_{SL}^{i}}$ of the Symmetric Stochastic Correlated $SL$ Factor Copula Model is given by the subsequent equality:

$$
p_{t_i}^{\xi_i|M_{SL}^{i}} = \hat{q}F_{t_i}(t) + (1 - \hat{q}) \left( 1 - q \right)F_{\varphi_{i}}(\frac{\sqrt{1 - \rho_{t_i}^{M_{SL}^{i}}}}{1 - \rho_{t_i}^{M_{SL}^{i}}}) + qF_{\varphi_{i}}(t_i)
$$

6.3.3.5 $SL$ Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Standard Lévy Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Standard Lévy Random Factor Loading Copula Model is introduced as a proposed model.
Remark 6.10 ($\mathcal{M}_{\mathcal{SL}}^t, \mathcal{J}_{\mathcal{SL}}^{t_i}$, and $\mathcal{X}_{\mathcal{SL}}^{t_i}$ with $\mathcal{SL}$ Distribution Functions)

In this subsection, the parameters of $(\mathcal{X}_{\mathcal{SL}}^{t_i})_{t \in \mathbb{R}^+}, \mathcal{M}_{\mathcal{SL}}^t,$ and $\mathcal{J}_{\mathcal{SL}}^{t_i}$ are admitting Lemma 6.3 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

Corollary 6.12 ($\mathcal{SL}$ Random Factor Loading Copula Model)

Let $\varphi_{N_t}$ be the unconditional number of default’s characteristic function that follows Theorem 5.5, $(\mathcal{X}_{\mathcal{SL}}^{t_i})_{t \in \mathbb{R}^+}, \mathcal{M}_{\mathcal{SL}}^t,$ and $\mathcal{J}_{\mathcal{SL}}^{t_i}$ be, respectively, a Lévy process specialised as a $\mathcal{SL}$, the $\mathcal{SL}$ systematic market risk factor, and the $\mathcal{SL}$ idiosyncratic risk factor those follow Lemma 6.3 and structured by the random factor loading. Then $\varphi_{N_t}^{\mathcal{SL}}$ of the $\mathcal{SL}$ Random Factor Loading Copula Model is given by the subsequent equality:

$$\varphi_{N_t}^{\mathcal{SL}}(u) = \int_{-\infty}^\infty \prod_{i=1}^n \left(1 - e^{-iu} \right) F_{\mathcal{J}_{\mathcal{SL}}^{t_i}} \left( \frac{F_{\mathcal{X}_{\mathcal{SL}}^{t_i}}^{-1}(F_{\mathcal{J}_{\mathcal{SL}}^{t_i}}(t)) - \kappa_i - \ell_1 m}{\sqrt{1 - \ell_1^a}} \right) f_{\mathcal{M}_{\mathcal{SL}}^t}(m) dm$$

$$+ \int_{\kappa}^\infty \prod_{i=1}^n \left(1 - e^{-iu} \right) F_{\mathcal{J}_{\mathcal{SL}}^{t_i}} \left( \frac{F_{\mathcal{X}_{\mathcal{SL}}^{t_i}}^{-1}(F_{\mathcal{J}_{\mathcal{SL}}^{t_i}}(t)) - \kappa_i - \ell_2 m}{\sqrt{1 - \ell_2^a}} \right) f_{\mathcal{M}_{\mathcal{SL}}^t}(m) dm$$

Where $\kappa_i = -\ell_1 \int_{-\infty}^\kappa f_{\mathcal{M}_{\mathcal{SL}}^t}(m) dm - \ell_2 \int_{\kappa}^\infty f_{\mathcal{M}_{\mathcal{SL}}^t}(m) dm$.

6.4 Generalized Hyperbolic Distribution

As introduced, the financial assets are modelled as stochastic processes; influenced by distributional assumptions on the dependence structure and its increments. Empirical studies showed that these assets have semi-heavy tails, i.e. their kurtoses are greater than the normal one (Mandelbrot, 1963). This has led to the Lévy Skewed Alpha Stable distributions to be introduced. However, the limitation of the Lévy Skewed Alpha Stable distribution, i.e. it does not always have mean, does not have variance or higher moments except for the Gaussian case Definition 6.2, lead to an alternative distributions known as the Generalised Hyperbolic distributions.
The Generalised Hyperbolic distribution was introduced in (Barndorff-Nielsen, 1977a). This distribution is proved to be infinitely divisible and thus admits the Lévy process (Barndorff-Nielsen, 1977b), have semi-heavy tails that could almost fit the assets returns (Prause, 1999), and its density is identified explicitly. It also has many special cases and limiting cases, for example, Scaled Student-t Distribution introduced in (Barndorff-Nielsen, 1978), Variance Gamma Distribution introduced in (Madan and Seneta, 1990), Normal Inverse Gaussian Distribution introduced in (Barndorff-Nielsen, 1997) and described in details in (Rydberg, 1997) and (Barndorff-Nielsen, 1998), Generalised Hyperbolic Skewed Student-t Distribution introduced in (Aas and Haff, 2006), and their univariate distribution mathematical properties are well-known (Barndorff-Nielsen and Blaesild, 1981, Blæsild, 1999).

In the context of financial assets returns, the Generalised Hyperbolic and its special and limiting cases distributions were introduced in (Eberlein and Keller, 1995), (Bibby and Sørensen, 1997), (Rydberg, 1997), (Barndorff-Nielsen, 1998), and (Eberlein et al., 1998), where in the context of the Factor Copula models it was referenced in the introduction of this chapter.

These facts encouraged applying the Generalised Hyperbolic its special and limiting cases distributions to the Factor Copula models. For more details on the limiting behaviour of the Generalised Hyperbolic Distribution, see (Eberlein and Hammerstein, 2002).

**Definition 6.6 (Generalized Hyperbolic Distribution \( \mathcal{GH} \))**

A random variable \( X \) is said to be Generalised Hyperbolic \(^{28}\), denoted by \( X_{\mathcal{GH}(\lambda, \alpha, \beta, \delta, \mu)} \), represented by the following parameters:

i. The order: \( \lambda \)

\(^{28}\) It is also called Lévy stable Generalized Hyperbolic
ii. Skewness parameter: $|\beta| \in [0, \alpha[.$

iii. Scale parameter: $\delta \in \mathbb{R}^+.$

iv. Location parameter: $\mu \in \mathbb{R}.$

v. Kurtosis: $\delta \sqrt{\alpha}.^{29}$

and restricted as by:

$\begin{cases} 
\delta > 0, |\beta| \leq \alpha, & \text{if } \lambda < 0 \\
\delta > 0, |\beta| < \alpha, & \text{if } \lambda = 0 \\
\delta \geq 0, |\beta| < \alpha, & \text{if } \lambda < 0
\end{cases}$

If its density is given by the subsequent equality:

$$f_{x_{G^H(\lambda, \alpha, \beta, \delta, \mu)}}(x) = a(\lambda, \alpha, \beta, \delta, \mu) (\delta^2 + (x - \mu)^2)^{(\lambda-1)/2} \times K_{\frac{\lambda}{2}} \left( \alpha \sqrt{\delta^2 + (x - \mu)^2} e^{\beta(x - \mu)} \right)$$

and its moment generating function, denoted by $\psi(u)_{G^H(\lambda, \alpha, \beta, \delta, \mu)}^{30},$ when $|\beta + u| < \alpha,$ is given by the following equality:

$$\psi(u)_{G^H(\lambda, \alpha, \beta, \delta, \mu)} = e^{\omega u} \left( \frac{\omega}{\varepsilon_{(\omega, u)}} \right)^{\lambda/2} \frac{K_{\frac{\lambda}{2}} \left( \delta \sqrt{\varepsilon_{(\omega, u)}} \right)}{K_{\frac{\lambda}{2}} \left( \delta \sqrt{\omega} \right)}$$

Where$^{31}$

i. The normalising constant $a(\lambda, \alpha, \beta, \delta, \mu) = \frac{\omega^2}{\sqrt{2\pi a} \frac{\lambda}{2} \delta^\lambda K_{\frac{\lambda}{2}} (\delta \sqrt{\omega})}$

ii. $\omega = (\alpha^2 - \beta^2).$

iii. $\sigma = (\alpha^2 + \beta^2).$

iv. $\varepsilon_{(\omega, u)} = \alpha^2 - (\beta + u)^2$

---

$^{29}$ Decreasing $(\delta \sqrt{\omega})$ increases the kurtosis.

$^{30}$ $\phi(u)_{\mathcal{G}^H} = \phi(iu)_{\mathcal{G}^H}.$ It worth noting that $\phi(u)_{\mathcal{G}^H}^t$ in general have the form of this equation only if $t = 1.$ Thus, in general, the $\mathcal{G}^H$ distribution is not stable under convolution.

$^{31}$ The representation in ii-v is used for shortness purposes.
v. The modified Bessel function of the third kind of order $\lambda$, denoted by $K_\lambda(\cdot)$ is given by:

$$K_\lambda(x) = \frac{1}{2} \int_0^\infty y^{\lambda-1} e^{-\frac{x}{2}(y+y^{-1})} dy, \quad x \in \mathbb{R}^+$$

6.4.1 A Skewed $t$ Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Generalised Hyperbolic Skewed $t$ Distribution that admits the Lévy process. This Distribution is introduced as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6.

The Generalised Hyperbolic Skewed $t$ Distribution, which was introduced in context of Factor Copula model in (Aas and Haff, 2006) is the only Generalised Hyperbolic
subclass that could represent heavy tailed returns, i.e. skewness, and characterised by one exponential tail and one polynomial tail. The later property is proved empirically to be preferable when heavy data is modelled (Aas and Haff, 2006).

This subsection starts by stating the Generalised Hyperbolic Skewed t Distribution and its properties and subsequently applying it to the proposed models.

6.4.1.1 St Distribution and its Properties

This subsection states the Generalised Hyperbolic Skewed t Distribution and introduces it as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6. Subsequently, this distribution is formalised by stating its definition and followed by it properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Generalised Hyperbolic Skewed t Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Limiting Case 6.4 (Generalized Hyperbolic with \( \lambda = -\frac{\nu}{2}, \alpha = |\beta| \))**

Let \( \mathcal{X} \) be Generalised Hyperbolic random variable that admits Definition 6.6, \( \lambda = -\frac{\nu}{2} \) and \( \alpha = |\beta| \). Then \( \mathcal{X}_{\gamma\mathcal{H}(\frac{-\nu}{2} |\beta|, \beta, \delta, \mu)} \) is said to be Skewed Generalized Hyperbolic Student-t or Skewed t, denoted by \( \mathcal{X}_{\text{St}(\nu, \beta, \delta, \mu)} \).

It worth articulating that there are other types of Skewed-t distribution found in the literatures.
Definition 6.7 (Skewed t Distribution \(S_t\))

A random variable \(X\) is said to be Skewed t, denoted by \(X_{St} (\nu, \beta, \delta, \mu)\), with \(\beta \neq 0\), \(\Gamma(x)\) as the Gamma Function, and \(K_\lambda(\cdot)\) as the modified Bessel function of the third kind of order \(\lambda\) that is given in Definition 6.5, if its density is given by the subsequent equality:

\[
f_{X_{St} (\nu, \beta, \delta, \mu)}(x) = \frac{2(\frac{1}{\nu-1})^\delta \nu |\beta|^\frac{1}{\nu+1} e^{\frac{1}{\beta}(x-\mu)}}{\sqrt{\pi} \Gamma\left(\frac{1}{\nu}\right) \left(\sqrt{\frac{\delta^2}{\nu}} + (x - \mu)^2\right)^{\nu+1}} K_{\frac{1}{\nu+1}}\left(\sqrt{\frac{\delta^2}{\nu}} + (x - \mu)^2\right)
\]

where \(n\nu + m = n(\nu + m)\).

Property D6.7.1 (\(S_t\): Mean, Variance, Skewness and Kurtosis)

If \(X\) is a Skewed t random variable that follows Definition 6.7, then \(X_{St} (\nu, \beta, \delta, \mu)\) Mean, Variance, Skewness and Kurtosis, respectively, are given by the subsequent equality:

\[
\begin{align*}
\text{i. } & \mathbb{E}[X_{St} (\nu, \beta, \delta, \mu)] = \mu + \frac{\beta \delta^2}{\nu - 2} \\
\text{ii. } & \mathbb{V}[X_{St} (\nu > 4, \beta, \delta, \mu)] = \frac{2 \beta^2 \delta^4}{(\nu-2)(\nu-4)} + \frac{\delta^2}{\nu - 2} \\
\text{iii. } & \mathbb{S}[X_{St} (\nu > 6, \beta, \delta, \mu)] = \frac{2(\nu-4)^2 \beta \delta}{(2 \beta^2 \delta^2 + (\nu-2)(\nu-4))^2} \left(3(\nu-2) + \frac{9 \beta^2 \delta^2}{\nu-6}\right) \\
\text{iv. } & \mathbb{K}[X_{St} (\nu > 8, \beta, \delta, \mu)] = \frac{6}{(2 \beta^2 \delta^2 + (\nu-2)(\nu-4))^2}. \\
\end{align*}
\]

Property D6.7.2 (\(S_t\): Convolution)

Any Skewed t random variables convolution must be calculated numerically; as a consequence of its instability under convolution.

6.4.1.2 \(S_t\) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the
model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Generalised Hyperbolic Skewed t Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Generalised Hyperbolic skewed Student-t Factor Copula Model in (Brunlid, 2006).

By admitting Assumption 5.2 and the skewed t Distribution $\mathcal{M}_t$ and $\mathcal{J}_t$, and $\mathcal{X}_t$ are given

by the following Lemma.

**Lemma 6.4** ($\mathcal{M}^t_{St}$, $\mathcal{J}^t_{St}$, and $\mathcal{X}^t_{St}$ with $St$ Distribution Functions)

Let $(\mathcal{X}^t_{St})_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Limiting Case 6.4 as a $St$ that admits Definition 6.7, with $\mathcal{M}^t_{St}$, and $\mathcal{J}^t_{St}$ as, respectively, the $St$ systematic market risk factor, and the $St$ idiosyncratic risk factor. Then for $\mathcal{M}^t_{St}$, $\mathcal{J}^t_{St}$, and $\mathcal{X}^t_{St}$ to admits Assumption 5.2, their parameters has to be set as follow:

i. $\mathcal{M}^t_{St} = \frac{1}{2\beta_1} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left( \frac{4v + \sqrt{v^2 - 8v + 16 + 8\beta^2 v - 32\beta^2}}{(v-2)} \right) du$

ii. $\mathcal{J}^t_{St} = \frac{1}{2\beta_1} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left( \frac{4v + \sqrt{v^2 - 8v + 16 + 8\beta^2 v - 32\beta^2}}{(v-2)} \right) du$

iii. $\mathcal{X}^t_{St} = \frac{1}{2\pi} \int e^{(-iut)} \left( \varphi_{\mathcal{M}^t_{St}} (\rho t u) \right) \left( \varphi_{\mathcal{J}^t_{St}} (\sqrt{1 - \rho^2 t u}) \right) du$
where $\nu$ and $\beta$ of $\mathcal{M}_{St}$ and $\mathcal{J}^t_{St}$ may differ.

**Proof:**

Firstly, in case of $\mathcal{M}_{St}$ and $\mathcal{J}^t_{St}$, their $\delta$ and $\mu$ are set in order to satisfy Assumptions 5.2 Conditions, i.e. zero mean and unit variance. This could be achieved by setting:

\[ \delta = \frac{1}{2\beta} \sqrt{4 - \nu + \sqrt{\nu^2 - 8 \nu + 16 + 8 \beta^2 \nu - 32 \beta^2}} (\nu - 2) \]

\[ \mu = \frac{4 - \nu + \sqrt{\nu^2 - 8 \nu + 16 + 8 \beta^2 \nu - 32 \beta^2}}{2(\nu - 2)\beta} \]

Secondly, since $St$ is not close under convolution, it has to be computed numerically. The method used here is inversing Fourier transform of the products of the of $\mathcal{M}_{St}$ and $\mathcal{J}^t_{St}$, where the FFT and the VFFT could be utilised.

For more details on the characteristic function and its inversion see Definition 5.7 and its property, on computational settings see Theorem 8.9, on implement it by the FFT Corollary 8.5, and on implement it by the VFFT Corollary 8.6. Using the FFT and VFFT are much faster and accurate than the Monte Carlo Simulation techniques; especially when optimisation techniques are utilised and the parameters are changed continuously.

**Corollary 6.13 (St Factor Copula Model)**

Let $(\chi^t_{St})_{t \in \mathbb{R}^+}$, $\mathcal{M}_{St}$, and $\mathcal{J}^t_{St}$ be, respectively, a Lévy process specialised as a $St$, the $St$ systematic market risk factor, and the $St$ idiosyncratic risk factor those follow Lemma 6.4, $\xi_t$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_t$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_t | \mathcal{M}^t_{St}}$ be the probability of $\tau_t$ conditioned upon $\mathcal{M}^t_{St}$. Then $p_{\xi_t | \mathcal{M}^t_{St}}$ of the $St$ Factor Copula Model is given by the subsequent equality:
6.4.1.3 Binary Stochastic Correlated $\mathcal{S}_t$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Generalised Hyperbolic Skewed $t$ Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Skewed $t$ Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

Remark 6.11 ($\mathcal{M}^t_{\mathcal{S}_t}$, $\mathcal{J}^t_{\mathcal{S}_t}$, and $\mathcal{X}^t_{\mathcal{S}_t}$ with $\mathcal{S}_t$ Distribution Functions)

In this subsection, the parameters of $(\mathcal{X}^t_{\mathcal{S}_t})_{t\in\mathbb{R}^+, \mathcal{I}\in\mathbb{R}}$, $\mathcal{M}^t_{\mathcal{S}_t}$, and $\mathcal{J}^t_{\mathcal{S}_t}$ are admitting Lemma 6.4 but structured as in Lemma 5.8 instead of Corollary 5.11.

Corollary 6.14 (Binary Stochastic Correlated $\mathcal{S}_t$ Factor Copula Model)

Let $(\mathcal{X}^t_{\mathcal{S}_t})_{t\in\mathbb{R}^+, \mathcal{M}^t_{\mathcal{S}_t}}$, and $\mathcal{J}^t_{\mathcal{S}_t}$ be, respectively, a Lévy process specialised as a $\mathcal{S}_t$, the $\mathcal{S}_t$ systematic market risk factor, and the $\mathcal{S}_t$ idiosyncratic risk factor those follow
Lemma 6.4 and structured by the Binary stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p^{\xi_i|M^{M}_{st}}_{t_i}$ be the probability of $\tau_i$ conditioned upon $M^M_{st}$. Then $p^{\xi_i|M^{M}_{st}}_{t_i}$ of the Binary Stochastic Correlated $\mathcal{S}$t Factor Copula Model is given by the subsequent equality:

$$
p^{\xi_i|M^{M}_{st}}_{t_i} = (1-q)F_{\mathcal{J}^{St}_{M}}^{-1}\left(\frac{F_{\mathcal{J}^{St}_{M}}(t) - \rho_1 M^{M}_{st}}{\sqrt{1 - \rho_1^2}}\right) + qF_{\mathcal{J}^{St}_{M}}^{-1}\left(\frac{F_{\mathcal{J}^{St}_{M}}(t) - \rho_2 M^{M}_{st}}{\sqrt{1 - \rho_2^2}}\right)
$$

6.4.1.4 Symmetric Stochastic Correlated $\mathcal{S}$t Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Generalised Hyperbolic Skewed $t$ Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Skewed $t$ Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.
Remark 6.12 ($\mathcal{M}_{\mathcal{S}_t}^t$, $\mathcal{J}_{\mathcal{S}_t}^t$, and $\mathcal{X}_{\mathcal{S}_t}^t$ with $\mathcal{S}_t$ Distribution Functions)

In this subsection, the parameters of $(\mathcal{X}_{\mathcal{S}_t}^t)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{S}_t}^t$, and $\mathcal{J}_{\mathcal{S}_t}^t$ are admitting Lemma 6.4 but structured as in Lemma 5.9 instead of Corollary 5.11.

Corollary 6.15 (Symmetric Stochastic Correlated $\mathcal{S}_t$ Factor Copula Model)

Let $(\mathcal{X}_{\mathcal{S}_t}^t)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{S}_t}^t$, and $\mathcal{J}_{\mathcal{S}_t}^t$ be, respectively, a Lévy process specialised as a $\mathcal{S}_t$, the $\mathcal{S}_t$ systematic market risk factor, and the $\mathcal{S}_t$ idiosyncratic risk factor those follow Lemma 6.4 and structured by the symmetric stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i|M_{\mathcal{S}_t}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{S}_t}^t$. Then $p_{\xi_i|M_{\mathcal{S}_t}^t}$ of the Symmetric Stochastic Correlated $\mathcal{S}_t$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i|M_{\mathcal{S}_t}^t} = \tilde{q} F_{\mathcal{M}_{\mathcal{S}_t}^t} \left( F_{\mathcal{X}_{\mathcal{S}_t}^t}^{-1} \left( F_{\mathcal{I}_i}(t) \right) \right) + (1 - \tilde{q}) \left[ (1 - q) F_{\mathcal{J}_{\mathcal{S}_t}^t} \left( \frac{F_{\mathcal{X}_{\mathcal{S}_t}^t}^{-1} \left( F_{\mathcal{I}_i}(t) \right) - \rho \mathcal{M}_{\mathcal{S}_t}^t}{\sqrt{1 - \rho^2}} \right) + q F_{\mathcal{J}_{\mathcal{S}_t}^t} \left( F_{\mathcal{X}_{\mathcal{S}_t}^t}^{-1} \left( F_{\mathcal{I}_i}(t_i) \right) \right) \right]$$

6.4.1.5 $\mathcal{S}_t$ Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically,
the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Generalised Hyperbolic Skewed t Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Skewed t Random Factor Loading Copula Model is introduced as a proposed model.

Remark 6.13 (\(\mathcal{M}_t^{t_1}, \mathcal{J}_t^{t_i}, \text{ and } \mathcal{X}_t^{t_i} \text{ with } \mathcal{S}_t \text{ Distribution Functions})

In this subsection, the parameters of \(\mathcal{X}_t^{t_i} \in \mathbb{R} \), \(\mathcal{M}_t\), and \(\mathcal{J}_t^{t_i}\) are admitting Lemma 6.4 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

Corollary 6.16 (\(\mathcal{S}_t \text{ Random Factor Loading Copula Model})

Let \(\varphi_{X_t}\) be the unconditional number of default’s characteristic function that follows Theorem 5.5. \(\mathcal{X}_t^{t_i} \in \mathbb{R} \), \(\mathcal{M}_t\), and \(\mathcal{J}_t^{t_i}\) be, respectively, a Lévy process specialised as a \(\mathcal{S}_t\), the \(\mathcal{S}_t\) systematic market risk factor, and the \(\mathcal{S}_t\) idiosyncratic risk factor those follow Lemma 6.4 and structured by the random factor loading. Then \(\varphi_{X_t}^{S_t}\) of the \(\mathcal{S}_t \text{ Random Factor Loading Copula Model} \) is given by the subsequent equality:

\[
\varphi_{X_t}^{S_t}(u) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{\mathcal{J}_t} \left( \frac{F_{X_t}^{-1} \left( F_{t_i}(t) \right) - \ell_i - \ell_1 m}{\sqrt{1 - \ell_1^2}} \right) \right) f_{\mathcal{M}_t}(m) dm
\]

\[
\varphi_{X_t}^{S_t}(u) = \sum_{\ell_2, \ell} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{\mathcal{J}_t} \left( \frac{F_{X_t}^{-1} \left( F_{t_i}(t) \right) - \ell_i - \ell_2 m}{\sqrt{1 - \ell_2^2}} \right) \right) f_{\mathcal{M}_t}(m) dm
\]

Where \(\ell_i = -\ell_1 \int_{-\infty}^{\infty} m f_{\mathcal{M}_t}(m) dm - \ell_2 \int_{-\infty}^{\infty} m f_{\mathcal{M}_t}(m) dm\).
6.4.2 A Normalised Fractional \( t \) Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Normalised Fractional-\( t \) Distribution that admits the Lévy process. This Distribution is introduced as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6.

This subsection starts by stating the Normalised Fractional-\( t \) Distribution and its properties and subsequently applying it to the proposed models.

6.4.2.1 \( \mathcal{F}t \) Distribution and its Properties

This subsection states the Normalised Fractional-\( t \) Distribution and introduces it as limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6. Subsequently, this distribution is formalised by stating its definition and properties.
The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admit Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Normalised Fractional-t Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Limiting Case 6.5 (Generalized Hyperbolic with \( \lambda = -\frac{\nu}{2}, \alpha = 0, \beta = 0 \))**

Let \( X \) be Generalised Hyperbolic random variable that admits Definition 6.6, \( \lambda = -\frac{\nu}{2}, \alpha = 0, \) and \( \beta = 0. \) Then \( X_{gH(-\frac{\nu}{2}, 0, 0, \delta, \mu)} \) is said to be Scaled and Shifted Student-t or Scaled-t\(^{32} \), denoted by \( X_{\text{Scaled-t}(\nu, \delta, \mu)} \).

**Definition 6.8 (Scaled-t Distribution t)**

A random variable \( X \) is said to be Scaled-t, denoted by \( X_{\text{Scaled-t}(\nu, \delta, \mu)} \), with \( \nu \) as the degree of freedom and \( \Gamma(x) \) as the Gamma Function, if its density is given by the subsequent equality:

\[
f_{X_{\text{Scaled-t}(\nu, \delta, \mu)}}(x) = \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\nu\right)} \left[1 + \frac{(x - \mu)^2}{\delta^2}\right]^{-\nu/2-1}
\]

**Limiting Case 6.6 (Scaled-t with \( \delta^2 = \nu, \mu = 0 \))**

Let \( X \) be Scaled-t random variable that admits Definition 6.8, \( \delta^2 = \nu, \mu = 0, \) and \( \nu \in \mathbb{R}^+ \) then \( X_{t(\nu, \sqrt{\nu}, 0)} \) is said to be Fractional Student-t or Fractional-t, denoted by \( X_{\text{Ft}(\nu)} \).

\(^{32}\) It could be seen as a limiting case of the skewed-t; by setting \( \beta = 0. \)
The Student-t distribution could be generalised to incorporate a fractional degree of freedom as the stated in the next definition; for more details on the Fractional Student-t representation and algorithm, the reader is referred to (Mardia and Zemroch, 1978).

**Definition 6.9 (Fractional-t Distribution $\mathcal{F}_t$)**

A random variable $X$ is said to be Fractional-$t$, denoted by $X_{\mathcal{F}_t(v)}$, with $v \in \mathbb{R}^+$ as the fractional degree of freedom, $\Gamma(x)$ as the Gamma Function, and $K_\lambda(\cdot)$ as the modified Bessel function of the third kind of order $\lambda$ that is given in Definition 6.5, if its density is given by the subsequent equality:

$$
f_{X_{\mathcal{F}_t(v)}}(x) = \frac{\Gamma\left(\frac{1}{2}v+1\right)}{\sqrt{\pi v}\Gamma\left(\frac{1}{2}v\right)} \left[1 + \frac{x^2}{v}\right]^{\frac{1}{2}v-1}
$$

And its moment generating function, denoted by $\psi(u)_{\mathcal{F}_t(v)}$, when $u \in \mathbb{R}$, is given by the following equality:

$$
\psi(u)_{\mathcal{F}_t(v)} = \frac{2\left(\frac{1-v}{2}\right)\left(\sqrt{\pi}|u|\right)\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} K_{\frac{v}{2}}\left(\sqrt{\pi}|u|\right)
$$

**Property D6.9.1 ($\mathcal{F}_t$: Mean, Variance, Skewness and Kurtosis)**

If $X$ is a Fractional-$t$ random variable that follows Definition 6.9, then $X_{\mathcal{F}_t(v)}$ Mean, Variance, Skewness and Kurtosis, respectively, are given by the subsequent equality:

i. $\mathbb{E}[X_{\mathcal{F}_t(v>1)}] = 0$

ii. $\mathbb{V}[X_{\mathcal{F}_t(v>2)}] = \frac{v}{v-2}$

iii. $\mathbb{S}[X_{\mathcal{F}_t(v>3)}] = 0$

iv. $\mathbb{K}[X_{\mathcal{F}_t(v>4)}] = \frac{6}{(v-4)}$

**Property D6.9.2 ($\mathcal{F}_t$: Convolution)**

Any $\mathcal{F}_t$ random variables convolution must be calculated numerically; as a consequence of its instability under convolution.
6.4.2.2 \( N^{\text{Fr}}t \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Normalised Fractional-t Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative.

Student-t Distribution was introduced by many tried to overcome this drawback by extending the Gaussian Copula model as in (Andersen et al., 2003), (Embrechts et al., 2003), (Frey and McNeil, 2003), (Mashal et al., 2003), (Mashal R. and Zeevi A., 2003), (Greenberg et al., 2004), (Demarta and McNeil, 2005), (Schloegl and O’Kane, 2005). On the same direction some other authors introduced a Double Student-t Copula model as in (Hull and White, 2004) and (Burtschell et al., 2009) and discussed in (Cousin and Laurent, 2008a) and (Cousin and Laurent, 2008b), where the one discussed in this subsection, i.e. the Double Fraction Student-t Copula Model (Wang et al., 2006) and (Wang et al., 2007).

By admitting Assumption 5.2 and Fractional-\( t \) Distribution and its properties, it could be normalised by its variance as shown in the next definition.
Definition 6.10 (Normalised Fractional-\(t\) Distribution \(\mathcal{N}^t\))

A random variable \(X\) is said to be Normalised Fractional-\(t\), denoted by \(X_{\mathcal{N}^t(v)}\), with \(v \in \mathbb{R}^+\) as the fractional degree of freedom, and \(\Gamma(x)\) as the Gamma Function, if its density is given by the subsequent equality:

\[
f_{X_{\mathcal{N}^t(v)}}(x) = \left(\frac{v}{v-2}\right)^{\frac{1}{2}} \sqrt{\frac{\Gamma\left(\frac{1}{2}v+1\right)}{\Gamma\left(\frac{1}{2}v\right)}} \left[1 + \frac{x^2}{v}\right]^{-\frac{1}{2}v-1}.
\]

Lemma 6.5 (\(\mathcal{M}^t_{\mathcal{N}^t}, \mathcal{J}^t_{\mathcal{N}^t}\), and \(X^t_{\mathcal{N}^t}\) with \(\mathcal{N}^t\) Distribution Functions)

Let \((X^t_{\mathcal{N}^t})_{t \in \mathbb{R}^+}\) be a Lévy process that follows Corollary 5.11 and specialised upon Limiting Case 6.6 as a \(\mathcal{N}^t\) that admits Definition 6.10, with \(\mathcal{M}^t_{\mathcal{N}^t}\), and \(\mathcal{J}^t_{\mathcal{N}^t}\) as, respectively, the \(\mathcal{N}^t\) systematic market risk factor, and the \(\mathcal{N}^t\) idiosyncratic risk factor, and \(v \in \mathbb{R}^+\) as the fractional degree of freedom of \(\mathcal{M}^t_{\mathcal{N}^t}\) and \(\mathcal{J}^t_{\mathcal{N}^t}\). Then for \(\mathcal{M}^t_{\mathcal{N}^t}, \mathcal{J}^t_{\mathcal{N}^t}\), and \(X^t_{\mathcal{N}^t}\) to admit Assumption 5.2 conditions, their parameters has to be set as follow:

i. \(\mathcal{M}^t_{\mathcal{N}^t(v)}\), where \(v \in \mathbb{R}^+\).

ii. \(\mathcal{J}^t_{\mathcal{N}^t(v)}\), where \(v \in \mathbb{R}^+\).

iii. \(X^t_{\mathcal{N}^t} = \frac{1}{2\pi} \int e^{-iu}\left(\varphi_{\mathcal{M}^t_{\mathcal{N}^t}(v)}(\rho_i u) \cdot \varphi_{\mathcal{J}^t_{\mathcal{N}^t}(v)}(\sqrt{1 - \rho_i^2 u})\right) du\)

Proof: (See Lemma 6.4).

Corollary 6.17 (\(\mathcal{N}^t\) Factor Copula Model)

Let \((X^t_{\mathcal{N}^t})_{t \in \mathbb{R}^+}, \mathcal{M}^t_{\mathcal{N}^t}, \mathcal{J}^t_{\mathcal{N}^t}\) be, respectively, a Lévy process specialised as a \(\mathcal{N}^t\), the \(\mathcal{N}^t\) systematic market risk factor, and the \(\mathcal{N}^t\) idiosyncratic risk factor those follow Lemma 6.5, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and
\( p_{t_i}^{\xi_i|\mathcal{M}_{N\mathcal{F}t}^t} \) be the probability of \( \tau_i \) conditioned upon \( \mathcal{M}_{N\mathcal{F}t}^t \). Then \( p_{t_i}^{\xi_i|\mathcal{M}_{N\mathcal{F}t}^t} \) of the \( \mathcal{N}\mathcal{F}t \) Factor Copula Model is given by the subsequent equality:

\[
p_{t_i}^{\xi_i|\mathcal{M}_{N\mathcal{F}t}^t} = \frac{F_{\mathcal{M}_{N\mathcal{F}t}^t}^{-1}\left(F_{t_i}(t) - \rho_t\mathcal{M}_{N\mathcal{F}t}^t\right)}{\sqrt{1 - \rho_t^2}}
\]

### 6.4.2.3 Binary Stochastic Correlated \( \mathcal{N}\mathcal{F}t \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Normalised Fractional-\( t \) Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. This Normalised Fractional-\( t \) Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.14** (\( \mathcal{M}_{N\mathcal{F}t}^t, J_{N\mathcal{F}t}^t, \) and \( X_{N\mathcal{F}t}^t \) with \( \mathcal{N}\mathcal{F}t \) Distribution Functions)

In this subsection, the parameters of \( \left( X_{N\mathcal{F}t}^t \right)_{t \in \mathbb{R}^+}, \mathcal{M}_{N\mathcal{F}t}^t, \) and \( J_{N\mathcal{F}t}^t \) are admitting Lemma 6.5 but structured as in Lemma 5.8 instead of Corollary 5.11.
Corollary 6.18 (Binary Stochastic Correlated $\mathcal{N}\mathcal{F}t$ Factor Copula Model)

Let $\left(\mathcal{X}_{\mathcal{N}\mathcal{F}t}^{t_t}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t$, and $\mathcal{J}_{\mathcal{N}\mathcal{F}t}^t$ be, respectively, a Lévy process specialised as a $\mathcal{N}\mathcal{F}t$, the $\mathcal{N}\mathcal{F}t$ systematic market risk factor, and the $\mathcal{N}\mathcal{F}t$ idiosyncratic risk factor those follow Lemma 6.5 and structured by the Binary stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i|\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t$. Then $p_{\xi_i|\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t}$ of the Binary Stochastic Correlated $\mathcal{N}\mathcal{F}t$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i|\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t} = (1 - q)F_{\mathcal{N}\mathcal{F}t}^{t_t} \left( \frac{F_{\mathcal{N}\mathcal{F}t}^{-1}\left(F_{\mathcal{N}\mathcal{F}t}(t) - \rho_1\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t\right)}{\sqrt{1 - \rho_1^2}} \right)$$

$$+ qF_{\mathcal{N}\mathcal{F}t}^{t_t} \left( \frac{F_{\mathcal{N}\mathcal{F}t}^{-1}\left(F_{\mathcal{N}\mathcal{F}t}(t) - \rho_2\mathcal{M}_{\mathcal{N}\mathcal{F}t}^t\right)}{\sqrt{1 - \rho_2^2}} \right)$$

6.4.2.4 Symmetric Stochastic Correlated $\mathcal{N}\mathcal{F}t$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.
In this subsection a direct consequence could be achieved once inserting the Normalised Fractional-\( t \) Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was noted in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. This Normalised Fractional-\( t \) Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.15** (\( \mathcal{M}^{t} \text{, } \mathcal{J}^{t} \text{, and } \mathcal{X}^{t} \text{ with } \mathcal{N}^{t} \text{ Distribution Functions} \))

In this subsection, the parameters of \((\mathcal{X}^{t}, \mathcal{M}^{t}, \mathcal{J}^{t}) \) are admitting Lemma 6.5 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.19** (Symmetric Stochastic Correlated \( \mathcal{N}^{t} \) Factor Copula Model)

Let \((\mathcal{X}^{t}, \mathcal{M}^{t}, \mathcal{J}^{t}) \) be, respectively, a Lévy process specialised as a \( \mathcal{N}^{t} \), the \( \mathcal{N}^{t} \) systematic market risk factor, and the \( \mathcal{N}^{t} \) idiosyncratic risk factor those follow Lemma 6.5 and structured by the symmetric stochastic correlation, \( \xi \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau \) admit Assumption 4.12 and Definition 4.15, and \( p_{\xi}^{\mathcal{M}^{t}} \) be the probability of \( \tau \) conditioned upon \( \mathcal{M}^{t} \). Then \( p_{\xi}^{\mathcal{M}^{t}} \) of the Symmetric Stochastic Correlated \( \mathcal{N}^{t} \) Factor Copula Model is given by the subsequent equality:

\[
p_{\xi}^{\mathcal{M}^{t}} = q F_{\mathcal{M}^{t}} F_{\mathcal{X}^{t}} (F_{\tau}(t)) + (1-q) \left[ (1-q) F_{\mathcal{J}^{t}} \left( \frac{F_{\mathcal{X}^{t}} (F_{\tau}(t)) - \rho \mathcal{M}^{t}}{\sqrt{1-\rho^{2}}} \right) \right] + q F_{\mathcal{J}^{t}} F_{\mathcal{X}^{t}} (F_{\tau}(t))
\]
6.4.2.5 NFFT Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Normalised Fractional-\(t\) Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Normalised Fractional-\(t\) Random Factor Loading Copula Model is introduced as a proposed model.

**Remark 6.16 (\(\mathcal{M}^t_{NFFT}, J^t_i, \mathcal{X}^t_i, S^t_i\) with NFFT Distribution Functions)**

In this subsection, the parameters of \((\mathcal{X}^t_i)_{t \in \mathbb{R}^+}, \mathcal{M}^t_{NFFT},\) and \(J^t_i\) are admitting Lemma 6.5 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

**Corollary 6.20 (NFFT Random Factor Loading Copula Model)**

Let \(\varphi_{N_i}\) be the unconditional number of default’s characteristic function that follows Theorem 5.5, \((\mathcal{X}^t_i)_{t \in \mathbb{R}^+}, \mathcal{M}^t_{NFFT},\) and \(J^t_i\) be, respectively, a Lévy process specialised as a NFFT, the NFFT systematic market risk factor, and the NFFT idiosyncratic risk factor those follow Lemma 6.5 and structured by the random factor loading. Then \(\varphi^\text{NFFT}_{N_i}\) of the NFFT Random Factor Loading Copula Model is given by the subsequent equality:
The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading
Copula Model”, and apply the Mixture Standard Gaussian and Normalised Fractional-\( t \) Distribution that admits the Lévy process.

This subsection starts by stating the Mixture Standard Gaussian and Normalised Fractional-\( t \) Distribution and its properties and subsequently applying it to the proposed models.

6.4.3.1 \( \mathbb{M}^{SG}_{NTt} \) Distribution and its Properties

This subsection states the Mixture Standard Gaussian and Normalised Fractional-\( t \) Distribution from its components. Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admit Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Mixture Standard Gaussian and Normalised Fractional-\( t \) Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

Mixture Case 6.2 (Mixture Standard Gaussian \& Normalised Fractional-\( t \) \( \mathbb{M}^{SG}_{NTt} \))

Let \( F_{ySG} \) be a Standard Gaussian Distribution that admits Definition 6.3\(^{33} \) and \( F_{\mathbb{W}_{NTt}} \) be a Normalised Fractional-\( t \) Distribution that admits Definition 6.10, where they are independent from each other, with \( p \in (0,1) \) as the probability of occurrence. Then the

\(^{33}\) It could be seen as limiting case of the Generalised Hyperbolic, i.e.

**Limiting Case (Generalized Hyperbolic with \( \alpha \to \infty, \delta \to \infty, \frac{\delta}{\alpha} \to \sigma^2 \)**

Let \( X \) be a generalised hyperbolic random variable that admits definition 2, \( \alpha \to \infty \) and \( \delta \to \infty \) in such a way that \( \frac{\delta}{\alpha} \to \sigma^2 \), then \( X_{G\mathcal{N}(\lambda,\alpha,\beta,\delta,\mu)} \) converge to a Normal or Gaussian, denoted by \( X_{G(\mu + \beta \sigma^2, \sigma^2)} \). For general representation it will be denoted by \( X_{G(\mu, \sigma)} \)
Mixture Standard Gaussian and Normalised Fractional-\( \mathbf{t} \) is structured by \( p \), and denoted by \( X_{M_{FG}(v,p)} \).

**Definition 6.11 (Mixture Standard Gaussian & Normalised Fractional-\( \mathbf{t} \) \( M_{FG}^{SG} \))**

A random variable \( X \) is said to be Mixture Standard Gaussian and Normalised Fractional-\( \mathbf{t} \) Distribution, denoted by \( X_{M_{FG}(v,p)} \), with \( v \in \mathbb{R}^+ \) as the fractional degree of freedom, \( \Gamma(x) \) as the Gamma Function, \( p \in (0,1) \) if its density is given by the subsequent equality:

\[
 f_X(x) = \frac{p}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + (1 - p) \left( \frac{v}{v-2} \right)^{\frac{1}{2}v+1} \left( \frac{\Gamma \left( \frac{1}{2}v+1 \right)}{\sqrt{\pi v^2}} \right) \left[ 1 + \frac{x^2}{v} \right] \left( \frac{1}{2}v-1 \right)
\]

**Property D6.11.1 (\( M_{FG}^{SG} \): Inheritance)**

Since \( X_{M_{FG}(v,p)} \) is a mixture of independent Standard Gaussian random variable and Normalised Fractional-\( \mathbf{t} \) random variable those follow, respectively, Definition 6.3 and Definition 6.10. Then each of them inherits its corresponding distribution properties.

**6.4.3.2 \( M_{FG}^{SG} \) Factor Copula Model**

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.
An immediate result could be achieved when applying the Mixture Standard Gaussian and Normalised Fractional-\(t\) Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Standard Gaussian and Normalised Fractional-\(t\) Factor Copula Model is introduced in (Wang et al., 2006) and (Wang et al., 2007).

By admitting Assumption 5.2 and the \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\) Distribution definition, \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\), \(\mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}\), and \(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) are given by the following Lemma.

**Lemma 6.6** (\(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{t}, \mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}, \) and \(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) with \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\) Distribution Functions)

Let \(\left(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t}\right)_{t \in \mathbb{R}^+}\) be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.2 as a \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\) that admits Definition 6.11, with \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) and \(\mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) as, respectively, the \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\) systematic market risk factor and the \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{SG}\) idiosyncratic risk factor, and \(v \in \mathbb{R}^+\) as the fractional degree of freedom of \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) and \(\mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}\). Then for \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{t}\), \(\mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}\), and \(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) to admit Assumption 5.2 conditions, their parameters has to be set as follow:

i. \(\mathcal{M}_{\mathbb{N}_{\mathcal{F}t}}^{t}(v,p)\), where \(v \in \mathbb{R}^+\) and \(p \in (0,1)\).

ii. \(\mathcal{J}_{\mathbb{N}_{\mathcal{F}t}}^{t}(v,p)\), where \(v \in \mathbb{R}^+\) and \(p \in (0,1)\).

iii. \(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t} = p\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t} + (1-p)\mathcal{X}_{\mathcal{N}_{\mathcal{F}t}}^{t}\), where \(\mathcal{X}_{\mathbb{N}_{\mathcal{F}t}}^{t}\) as in Lemma 6.1, \(\mathcal{X}_{\mathcal{N}_{\mathcal{F}t}}^{t}\) as in Lemma 6.5, and \(p \in (0,1)\).

**Proof:** (See Lemma 6.4).
Corollary 6.21 (\(\mathbb{M}_{N,F,t}^{SG}\) Factor Copula Model)

Let \(\left(\mathcal{X}_{M_{N,F,t}}^{t_i}, M_{M_{N,F,t}}^{t_i}\right)\), and \(\mathcal{J}_{M_{N,F,t}}^{t_i}\) be, respectively, a Lévy process specialised as a \(\mathbb{M}_{N,F,t}^{SG}\), the systematic market risk factor, and the \(\mathbb{M}_{N,F,t}^{SG}\) idiosyncratic risk factor those follow Lemma 6.6, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and \(p_{t_i}\) be the probability of \(\tau_i\) conditioned upon \(\mathcal{M}_{M_{N,F,t}}^{t_i}\). Then \(p_{t_i}\) of the \(\mathbb{M}_{N,F,t}^{SG}\) Factor Copula Model is given by the subsequent equality:

\[
p_{t_i} = \int_{\mathcal{M}_{M_{N,F,t}}^{t_i}} F_{t_i}^{-1} \left( F_{M_{N,F,t}} (t) - \rho_{t}^{M_{M_{N,F,t}}^{t_i}} \right) \frac{1}{\sqrt{1 - \rho_{t}^{2}}} \, d\mathcal{M}_{M_{N,F,t}}^{t_i}
\]

6.4.3.3 Binary Stochastic Correlated \(\mathbb{M}_{N,F,t}^{SG}\) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Mixture Standard Gaussian and Normalised Fractional- \(t\) Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1.
This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. This Mixture Standard Gaussian and Normalised Fractional- $\ell$ Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.17** ($M^t_{M_{N}}$, $J^t_{M_{N}}$, and $X^t_{M_{N}}$ with $M^{SG}_{N}$ Distribution Functions)

In this subsection, the parameters of $\left(X^t_{M_{N}}\right)_{t \in \mathbb{R}^+}$, $M^t_{M_{N}}$, and $J^t_{M_{N}}$ are admitting Lemma 6.6 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.22** (Binary Stochastic Correlated $M^{SG}_{N}$ Factor Copula Model)

Let $\left(X^t_{M_{N}}\right)_{t \in \mathbb{R}^+}$, $M^t_{M_{N}}$, and $J^t_{M_{N}}$ be, respectively, a Lévy process specialised as a $M^{SG}_{N}$, the $M^{SG}_{N}$ systematic market risk factor, and the $M^{SG}_{N}$ idiosyncratic risk factor those follow Lemma 6.6 and structured by the Binary stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\tau_i}$ be the probability of $\tau_i$ conditioned upon $M^t_{M_{N}}$. Then $p_{\tau_i}$ of the Binary Stochastic Correlated $M^{SG}_{N}$ Factor Copula Model is given by the subsequent equality:

$$p_{\tau_i} = (1 - q)F_{J^t_{M_{N}}} \left( F_{X^t_{M_{N}}}^{-1} \left( F_{\xi_i}(t) - \rho_1 M^t_{M_{N}} \right) \right) \frac{1}{\sqrt{1 - \rho_1^2}} + q F_{J^t_{M_{N}}} \left( F_{X^t_{M_{N}}}^{-1} \left( F_{\xi_i}(t) - \rho_2 M^t_{M_{N}} \right) \right) \frac{1}{\sqrt{1 - \rho_2^2}}$$
6.4.3.4 Symmetric Stochastic Correlated $M_{N,F_t}^{SG}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Standard Gaussian and Normalised Fractional-$t$ Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Standard Gaussian and Normalised Fractional-$t$ Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.18 ($M_{N,F_t}^{SG}$, $J_{N,F_t}^{SG}$, and $X_{N,F_t}^{SG}$ with $M_{N,F_t}^{SG}$ Distribution Functions)**

In this subsection, the parameters of $\left( X_{M_{N,F_t}^{SG}}^{t_i}, M_{M_{N,F_t}^{SG}}^{t_i}, J_{M_{N,F_t}^{SG}}^{t_i} \right)$ are admitting Lemma 6.6 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.23 (Symmetric Stochastic Correlated $M_{N,F_t}^{SG}$ Factor Copula Model)**

Let $\left( X_{M_{N,F_t}^{SG}}^{t_i}, M_{M_{N,F_t}^{SG}}^{t_i}, J_{M_{N,F_t}^{SG}}^{t_i} \right)$ be, respectively, a Lévy process specialised as a $M_{N,F_t}^{SG}$, the $M_{N,F_t}^{SG}$ systematic market risk factor, and the $M_{N,F_t}^{SG}$ idiosyncratic risk factor
those follow Lemma 6.6 and structured by the symmetric stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$

admit Assumption 4.12 and Definition 4.15, and $p_{\tau_i}^{M_{t}^{SG}}$ be the probability of $\tau_i$

conditioned upon $M_{t}^{SG}$ then $p_{\tau_i}^{M_{t}^{SG}}$ of the Symmetric Stochastic Correlated

$M_{t}^{SG}$ Factor Copula Model is given by the subsequent equality:

$$p_{\tau_i}^{M_{t}^{SG}} = \hat{q} F_{M_{t}^{SG}}^{M_{N,F}^{SG}} \left( F_{\tau_i}^{-1} \left( F_{\tau_i}(t) \right) \right)$$

$$+ (1 - \hat{q}) \left[ (1 - q) F_{\tau_i}^{M_{t}^{SG}}^{M_{N,F}^{SG}} \left( \frac{F_{\tau_i}^{-1} \left( F_{\tau_i}(t) \right) - \rho M_{t}^{SG}}{\sqrt{1 - \rho^2}} \right) \right]$$

$$+ q F_{\tau_i}^{M_{t}^{SG}}^{M_{N,F}^{SG}} \left( F_{\tau_i}^{-1} \left( F_{\tau_i}(t) \right) \right)$$

6.4.2.5 $M_{t}^{SG}$ Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Standard Gaussian and Normalised Fractional-$t$ Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8.
This model is proposed as better alternative than the based model. The Mixture Standard Gaussian and Normalised Fractional-$\mathbf{t}$ Random Factor Loading Copula Model is introduced as a proposed model.

**Remark 6.19** ($\mathcal{M}^{t_{MG},t_{MG}}, \mathcal{J}^{t_{MG},t_{MG}}, \text{ and } \mathcal{X}^{t_{MG},t_{MG}}$ with $\mathcal{M}^{SG}_{NFT}$ Distribution Functions)

In this subsection, the parameters of ($\mathcal{X}^{t_{MG},t_{MG}}_{NFT}$) are admitting Lemma 6.6 but structured as in Lemma 5.10 and Theorem 5.5 instead of Corollary 5.11.

**Corollary 6.24** ($\mathcal{M}^{SG}_{NFT}$ Random Factor Loading Copula Model)

Let $\varphi_{Nt}$ be the unconditional number of default’s characteristic function that follows Theorem 5.5, ($\mathcal{X}^{t_{MG},t_{MG}}_{NFT}$) be, respectively, a Lévy process specialised as a $\mathcal{M}^{SG}_{NFT}$, the $\mathcal{M}^{SG}_{NFT}$ systematic market risk factor, and the $\mathcal{M}^{SG}_{NFT}$ idiosyncratic risk factor those follow Lemma 6.6 and structured by the random factor loading. Then $\varphi^{SG}_{Nt}$ of the $\mathcal{M}^{SG}_{NFT}$ Random Factor Loading Copula Model is given by the subsequent equality:

$$
\varphi^{SG}_{Nt}(u) = \frac{1}{\sqrt{1 - \ell^2_1}} f^{t_{MG}}_{\mathcal{M}^{SG}_{NFT}}(m)dm \\
+ \frac{1}{\sqrt{1 - \ell^2_2}} f^{t_{MG}}_{\mathcal{M}^{SG}_{NFT}}(m)dm
$$

Where $\mathbf{K}_i = -\ell_1 \int_{-\infty}^{\mathbf{K}} mf^{t_{MG}}_{\mathcal{M}^{SG}_{NFT}}(m)dm - \ell_2 \int_{\mathbf{K}}^\infty mf^{t_{MG}}_{\mathcal{M}^{SG}_{NFT}}(m)dm$.

**6.4.4 A Variance Gamma Factor Copula**

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it
requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Variance Gamma Distribution that admits the Lévy process.

This Distribution is introduced as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6.

This subsection starts by stating the Variance Gamma Distribution and its properties and subsequently applying it to the proposed models.

6.4.4.1 $\mathcal{VG}$ Distribution and its Properties

This subsection states the Variance Gamma Distribution and introduces it as a limiting case of the Generalized Hyperbolic Distribution articulated in Definition 6.6. Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors
distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Variance Gamma Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Limiting Case 6.7 (Generalized Hyperbolic with \( \delta = 0 \))**

Let \( \mathcal{X} \) be Generalised Hyperbolic random variable that admits Definition 6.6, and \( \delta = 0 \), then \( \mathcal{X}_{\mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \beta, 0, \mu)} \) is said to be Variance Gamma, denoted by \( \mathcal{X}_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)} \).

**Definition 6.12 (Variance Gamma Distribution \( \mathcal{V}_{\mathcal{G}} \))**

A random variable \( \mathcal{X} \) is said to be Variance Gamma, denoted by \( \mathcal{X}_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)} \), with \(|\beta| \in [0, \alpha[, \Gamma(x) \) as the Gamma Function, and \( K_\lambda(\cdot) \) the modified Bessel function of the third kind of order \( \lambda \) that is given in Definition 6.6, if its density is given by the subsequent equality:

\[
\tilde{f}_{\mathcal{X}_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)}}(x) = \frac{(\omega)^\lambda |x - \mu|^{(\lambda-1)/2} e^{\beta(x - \mu)}}{\sqrt{\pi} \Gamma(\lambda)(2\alpha)^{1/2}} K_{\lambda-1/2}(\alpha |x - \mu|)
\]

And its moment generating function, denoted by \( \psi(u)_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)} \), when \(|\beta + u| < \alpha\), is given by the following equality:

\[
\psi(u)_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)} = e^{u\mu \left( \frac{\omega}{\xi(\omega \alpha)} \right)}^\lambda
\]

**Property D6.12.1 (\( \mathcal{V}_{\mathcal{G}} \): Mean, Variance, Skewness and Kurtosis)**

If \( \mathcal{X} \) is a Variance Gamma variable that follows Definition 6.12, then \( \mathcal{X}_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)} \) Mean, Variance, Skewness and Kurtosis, respectively, are given by the subsequent equality:

i. \[
\mathbb{E}[\mathcal{X}_{\mathcal{V}_{\mathcal{G}}(\lambda, \alpha, \beta, \mu)}] = \mu + \frac{2\beta \lambda}{\omega}
\]
ii. \( \mathbb{E}[X_{VG}(\lambda, \alpha, \beta, \mu)] = \frac{2\lambda \alpha}{\alpha^2} \)

iii. \( \mathbb{S}[X_{VG}(\lambda, \alpha, \beta, \mu)] = \frac{\sqrt{3} \beta (3\alpha^2 + \beta^2)}{\sqrt{\lambda \alpha^2}} \)

iv. \( \mathbb{K}[X_{VG}(\lambda, \alpha, \beta, \mu)] = 3 + 3\lambda^{-1} + \frac{12\alpha^2 \beta^2}{\lambda \alpha^2} \)

**Property D6.12.2 (VG: Convolution)**

If \( X \) and \( Y \) are Variance Gamma random variables that follow Definition 6.12, then they are stable under convolution, as shown in the subsequent equality:

\[
X_{VG}(\lambda_1, \alpha_1, \beta_1, \mu_1) + Y_{VG}(\lambda_2, \alpha_2, \beta_2, \mu_2) \sim Z_{VG}(\lambda_1 + \lambda_2, \alpha, \beta, \mu_1 + \mu_2)
\]

when

i. \( \alpha = \alpha_1 = \alpha_2 \)

ii. \( \beta = \beta_1 = \beta_2 \)

**Property D6.12.3 (VG: Scaling)**

If \( X \) is a Variance Gamma random variable that follows Definition 6.12, then \( X_{VG}(\lambda, \alpha, \beta, \mu) \) can be scaled by a constant \( c \), as shown in the subsequent equality:

\[
X_{VG}(\frac{\lambda}{c}, \frac{\alpha}{c}, \frac{\beta}{c}, \frac{\mu}{c})
\]

### 6.4.4.2 VG Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.
An immediate result could be achieved when applying the Variance Gamma Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Variance Gamma Factor Copula Model is introduced in (Brunlid, 2006) and (Moosbrucker, 2006).

By admitting Assumption 5.2 and the $\mathcal{VG}$ Distribution definition, $\mathcal{M}_\mathcal{VG}^{t_i}$, $\mathcal{J}_\mathcal{VG}^{t_i}$, and $\mathcal{X}_\mathcal{VG}^{t_i}$ are given by the following Lemma.

**Lemma 6.7 ($\mathcal{M}_\mathcal{VG}^{t_i}$, $\mathcal{J}_\mathcal{VG}^{t_i}$, and $\mathcal{X}_\mathcal{VG}^{t_i}$ with $\mathcal{VG}$ Distribution Functions)**

Let $(\mathcal{X}_\mathcal{VG}^{t_i})_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Limiting Case 6.7 as a $\mathcal{VG}$ that admits Definition 6.12, with $\mathcal{M}_\mathcal{VG}^{t_i}$, and $\mathcal{J}_\mathcal{VG}^{t_i}$ as, respectively, the $\mathcal{VG}$ systematic market risk factor, and the $\mathcal{VG}$ idiosyncratic risk factor. Then for $\mathcal{M}_\mathcal{VG}^{t_i}$, $\mathcal{J}_\mathcal{VG}^{t_i}$, and $\mathcal{X}_\mathcal{VG}^{t_i}$ to admits Assumption 5.2, their parameters has to be set as follow:

1. $\mathcal{M}_\mathcal{VG}^{t_i} = \mathcal{M}_\mathcal{VG}^{t_i} \left( \frac{\omega^2}{\sigma^2} \alpha_M, \beta_M, -\beta_M \right)$

2. $\mathcal{J}_\mathcal{VG}^{t_i} = \mathcal{J}_\mathcal{VG}^{t_i} \left( \frac{1 - \rho^2}{\rho^2}, \frac{\omega^2}{\sigma^2} \alpha_M, \beta_M, -\beta_M \right)$

3. $\mathcal{X}_\mathcal{VG}^{t_i} = \mathcal{X}_\mathcal{VG}^{t_i} \left( \frac{1}{\rho^2}, \frac{\omega^2}{\sigma^2} \alpha_M, \beta_M, -\beta_M \right)$

**Proof:**

Firstly, in case of $\mathcal{M}_\mathcal{VG}^{t_i}$ and $\mathcal{J}_\mathcal{VG}^{t_i}$, their $\delta$ and $\mu$ are set in order to satisfy Assumptions 5.2 Conditions, i.e. zero mean and unit variance. This could be achieved by the following statement:

1. Since $E[\mathcal{X}_\mathcal{VG}^{t_i}(\alpha, \alpha, \beta, \mu)] = \mu + \frac{2\beta \lambda}{\omega} = 0$, then $\mu = -\frac{2\beta \lambda}{\omega}$
ii. Since \( \mathcal{V}[\mathcal{X}_{\mathcal{V}(\lambda, \alpha, \beta, \mu)}] = \frac{2\lambda \sigma}{\omega^2} = 1 \), then \( \lambda = \frac{\omega^2}{2\sigma} \). Consequently, by replacing \( \lambda \) by \( \frac{\omega^2}{2\sigma} \) in \( \mu \), then it equals to \( \mu = -\beta \frac{\omega}{\sigma} \); i.e. \( \mathcal{V}\mathcal{G}\left( \frac{\omega^3}{2\sigma^2 M}, \alpha_M, \beta_M, -\beta_M \frac{\omega M}{\sigma M} \right) \)

\[
\mathcal{V}\mathcal{G}\left( \frac{\omega^3}{2\sigma^2 M}, \alpha_M, \beta_M, -\beta_M \frac{\omega M}{\sigma M} \right)
\]

Secondly, to find \( \mathcal{X}_{\mathcal{V}_G} \) three more steps are required:

**Step 1:** recall that \( \omega = (\alpha^2 - \beta^2) \) and \( \sigma = (\alpha^2 + \beta^2) \) and by setting \( \alpha_J = \alpha_M \sqrt{1 - \rho_i^2} \) and \( \beta_J = \beta_M \sqrt{1 - \rho_i^2} \), then \( \lambda_J \) is given by the subsequent equalities:

\[
\lambda_J = \frac{\omega_J}{2\sigma_J} = \frac{1}{2} \left( \frac{1 - \rho_i^2}{\rho_i} \right)^2 \left( \frac{1 - \rho_i^2}{\rho_i} \right)^2 \frac{\omega_J}{2\sigma_J} = \frac{1 - \rho_i^2}{\rho_i} \cdot \frac{\omega_J}{2\sigma_J}
\]

and \( \mu_J \) is given by the subsequent equalities:

\[
\mu_J = -\beta_J \frac{\omega_J}{\sigma_J} = -\frac{1}{2} \left( \frac{1 - \rho_i^2}{\rho_i} \right)^3 \left( \frac{1 - \rho_i^2}{\rho_i} \right)^2 \frac{\omega_J}{2\sigma_J} = -\frac{1 - \rho_i^2}{\rho_i} \cdot \beta_M \frac{\omega_J}{\sigma_J}
\]
Step 2: by Scaling $\mathcal{M}_{\mathcal{V}G}^t$ by $\rho_i^2$ the subsequent equality hold:

$$
\rho_i \mathcal{M}_{\mathcal{V}G}^t = \rho_i \mathcal{V}G \left( \frac{\omega_2^2}{\sigma_M}, \alpha_M, \beta_M, -\beta_M \frac{\omega_M}{\sigma_M} \right)
$$

$$
= \mathcal{V}G \left( \frac{\omega_2^2}{\sigma_M}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\rho_i \beta_M \frac{\omega_M}{\sigma_M} \right)
$$

and $J_{\mathcal{V}G}^t$ by $\sqrt{1 - \rho_i^2}$ the subsequent equality hold:

$$
\sqrt{1 - \rho_i^2} J_{\mathcal{V}G}^t = \sqrt{1 - \rho_i^2} \mathcal{V}G \left( \frac{1 - \rho_i^2}{\rho_i^2}, \frac{\omega_2^2}{\sigma_M}, \frac{1 - \rho_i^2}{\rho_i}, \alpha_M, \frac{1 - \rho_i^2}{\rho_i}, \beta_M, -\frac{1 - \rho_i^2}{\rho_i}, \frac{\omega_M}{\sigma_M} \right)
$$

$$
= \mathcal{V}G \left( \frac{1 - \rho_i^2}{\rho_i^2}, \frac{\omega_2^2}{\sigma_M}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\frac{1 - \rho_i^2}{\rho_i}, \frac{\beta_M \omega_M}{\sigma_M} \right)
$$

Step 3: the final step is computing $X_{\mathcal{V}G}^t$. This could be achieved as in the subsequent chain of equalities:

$$
X_{\mathcal{V}G}^t = \rho_i \mathcal{M}_{\mathcal{V}G}^t + \sqrt{1 - \rho_i^2} J_{\mathcal{V}G}^t
$$

$$
= \mathcal{V}G \left( \frac{\omega_2^2}{\sigma_M}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\rho_i \beta_M \frac{\omega_M}{\sigma_M} \right) + \mathcal{V}G \left( \frac{1 - \rho_i^2}{\rho_i^2}, \frac{\omega_2^2}{\sigma_M}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\rho_i \beta_M \frac{\omega_M}{\sigma_M} \right)
$$

$$
= \mathcal{V}G \left( \frac{\omega_2^2}{\sigma_M} + \frac{1 - \rho_i^2}{\rho_i}, \frac{\omega_2^2}{\sigma_M}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\rho_i \beta_M \frac{\omega_M}{\sigma_M} \right)
$$

$$
= \mathcal{V}G \left( \frac{\omega_2^2}{\sigma_M} \frac{1 - \rho_i^2}{\rho_i^2} \frac{\sigma_M}{\sigma_M^2} - \omega_2^2 \frac{\omega_2^2}{\sigma_M^2} \frac{1 - \rho_i^2}{\rho_i}, \frac{\alpha_M}{\rho_i}, \frac{\beta_M}{\rho_i}, -\rho_i \beta_M \frac{\omega_M}{\sigma_M} \right)
$$

$$
= \mathcal{V}G \left( \frac{1 - \rho_i^2}{\rho_i^2} \frac{\omega_2^2}{\sigma_M}, \frac{\sigma_M}{\sigma_M^2} \frac{1 - \rho_i^2}{\rho_i} \frac{\alpha_M}{\rho_i}, \frac{1 - \rho_i^2}{\rho_i} \frac{\beta_M}{\rho_i}, -\frac{1 - \rho_i^2}{\rho_i} \frac{\beta_M \omega_M}{\sigma_M} \right)
$$

Corollary 6.25 ($\mathcal{V}G$ Factor Copula Model)

Let $(X_{\mathcal{V}G}^t)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{V}G}^t$, and $J_{\mathcal{V}G}^t$ be, respectively, a Lévy process specialised as a $\mathcal{V}G$, the $\mathcal{V}G$ systematic market risk factor, and the $\mathcal{V}G$ idiosyncratic risk factor those follow Lemma 6.7, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i}^t | \mathcal{M}_{\mathcal{V}G}^t$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{V}G}^t$. Then $p_{\xi_i}^t | \mathcal{M}_{\mathcal{V}G}^t$ of the $\mathcal{V}G$ Factor Copula Model is given by the subsequent equality:
6.4.4.3 Binary Stochastic Correlated $\mathcal{VG}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Variance Gamma Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Variance Gamma Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

Remark 6.20 ($\mathcal{M}_{\mathcal{VG}}$, $\mathcal{J}_{\mathcal{VG}}^{t_i}$, and $\mathcal{X}_{\mathcal{VG}}^{t_i} \text{ with } \mathcal{VG}$ Distribution Functions)

In this subsection, the parameters of $\left(\mathcal{X}_{\mathcal{VG}}^{t_i}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{VG}}^{t_i}$, and $\mathcal{J}_{\mathcal{VG}}^{t_i}$ are admitting Lemma 6.7 but structured as in Lemma 5.8 instead of Corollary 5.11.

Corollary 6.26 (Binary Stochastic Correlated $\mathcal{VG}$ Factor Copula Model)

Let $\left(\mathcal{X}_{\mathcal{VG}}^{t_i}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{VG}}^{t_i}$, and $\mathcal{J}_{\mathcal{VG}}^{t_i}$ be, respectively, a Lévy process specialised as a $\mathcal{VG}$, the $\mathcal{VG}$ systematic market risk factor, and the $\mathcal{VG}$ idiosyncratic risk factor those follow...
Lemma 6.7 and structured by the Binary stochastic correlation, $\xi_t$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_t$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_t}^{\left\{\mathcal{M}^t_{vG}\right\}}$ be the probability of $\tau_t$ conditioned upon $\mathcal{M}^t_{vG}$. Then $p_{\xi_t}^{\left\{\mathcal{M}^t_{vG}\right\}}$ of the Binary Stochastic Correlated $\mathcal{V}\mathcal{G}$ Factor Copula Model is given by the subsequent equality:

$$
p_{\xi_t}^{\left\{\mathcal{M}^t_{vG}\right\}} = (1-q)F_{\mathcal{V}G}^{\tau_t} \left( \frac{F_{\mathcal{X}_{vG}}^{-1} \left( F_{\xi_t}(t) - \rho_1 \mathcal{M}^t_{vG} \right)}{\sqrt{1 - \rho_1^2}} \right) + qF_{\mathcal{V}G}^{\tau_t} \left( \frac{F_{\mathcal{X}_{vG}}^{-1} \left( F_{\xi_t}(t) - \rho_2 \mathcal{M}^t_{vG} \right)}{\sqrt{1 - \rho_2^2}} \right)
$$

6.4.4.4 Symmetric Stochastic Correlated $\mathcal{V}\mathcal{G}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Variance Gamma Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus
it is proposed as better alternative. The Variance Gamma Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.21 (\(M_{VG}^{t_i}, J_{VG}^{t_i}, \) and \(X_{VG}^{t_i}\) with \(VG\) Distribution Functions)**

In this subsection, the parameters of \((X_{VG}^{t_i})_{t \in \mathbb{R}^+}, M_{VG}^{t_i}, \) and \(J_{VG}^{t_i}\) are admitting Lemma 6.7 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.27 (Symmetric Stochastic Correlated \(VG\) Factor Copula Model)**

Let \((X_{VG}^{t_i})_{t \in \mathbb{R}^+}, M_{VG}^{t_i}, \) and \(J_{VG}^{t_i}\) be, respectively, a Lévy process specialised as a \(VG\), the \(VG\) systematic market risk factor, and the \(VG\) idiosyncratic risk factor those follow Lemma 6.7 and structured by the symmetric stochastic correlation, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and \(p_{t_i|M_{VG}}^{\xi_i}\) be the probability of \(\tau_i\) conditioned upon \(M_{VG}^{t_i}\). Then \(p_{t_i|M_{VG}}^{\xi_i}\) of the Symmetric Stochastic Correlated \(VG\) Factor Copula Model is given by the subsequent equality:

\[
p_{t_i|M_{VG}}^{\xi_i} = \hat{q}F_{M_{VG}^{t_i}}\left(\frac{F_{X_{VG}^{t_i}}^{-1}(F_{\tau_i}(t))}{\sqrt{1 - \rho^2}}\right) + (1 - \hat{q}) \left[ (1 - q)F_{J_{VG}^{t_i}}\left(\frac{F_{X_{VG}^{t_i}}^{-1}(F_{\tau_i}(t)) - \rho M_{VG}^{t_i}}{\sqrt{1 - \rho^2}}\right) + qF_{J_{VG}^{t_i}}\left(F_{X_{VG}^{t_i}}^{-1}(F_{\tau_i}(t))\right)\right]
\]

**6.4.4.5 \(VG\) Random Factor Loading Copula Model**

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also,
the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Variance Gamma Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Variance Gamma Random Factor Loading Copula Model is introduced as a proposed model.

In order to set the distributions, as in Lemma 6.7, the distributions will be broken on the two parts of this model, where \( \mathcal{M}_t < \kappa \) or \( \mathcal{M}_t \geq \kappa \). The following lemma will summarise this point.

**Lemma 6.8 (\( \mathcal{M}_{\mathcal{V}_G}^t, \mathcal{J}_{\mathcal{V}_G}^{t_i}, \text{ and } \mathcal{X}_{\mathcal{V}_G}^{t_i} \) with \( \mathcal{V}_G \) Distribution Functions)**

Let \( \left( \mathcal{X}_{\mathcal{V}_G}^{t_i} \right)_{t \in \mathbb{R}^+} \) be a Lévy process that follows Lemma 5.10 and specialised upon Limiting Case 6.7 as a \( \mathcal{V}_G \) that admits Definition 6.12, with \( \mathcal{M}_{\mathcal{V}_G}^t \), \( \mathcal{J}_{\mathcal{V}_G}^{t_i} \), and \( \mathcal{X}_{\mathcal{V}_G}^{t_i} \) as, respectively, the \( \mathcal{V}_G \) systematic market risk factor, and the \( \mathcal{V}_G \) idiosyncratic risk factor.

Then for \( \mathcal{M}_{\mathcal{V}_G}^t, \mathcal{M}_{\mathcal{V}_G}^{t^1}, \mathcal{M}_{\mathcal{V}_G}^{t^2}, \mathcal{J}_{\mathcal{V}_G}^{t^1}, \mathcal{J}_{\mathcal{V}_G}^{t^2}, \mathcal{X}_{\mathcal{V}_G}^{t^1}, \) and \( \mathcal{X}_{\mathcal{V}_G}^{t^2} \) to admits Assumption 5.2, their parameters has to be set as following:

i. \( \mathcal{M}_{\mathcal{V}_G}^t \left( \frac{\alpha^2}{\sigma^2}, \beta, -\frac{\beta^2}{\sigma^2} \right) \)

ii. \( \mathcal{M}_{\mathcal{V}_G}^t \left( \frac{\alpha^2}{\sigma^2}, \beta, -\frac{\beta^2}{\sigma^2} \right) \)

iii. \( \mathcal{M}_{\mathcal{V}_G}^t \left( \frac{\alpha^2}{\sigma^2}, \beta, -\frac{\beta^2}{\sigma^2} \right) \)
of the

Corollary 6.28 (VG Random Factor Loading Copula Model)

Let \( \varphi_{Xt} \) be the unconditional number of default’s characteristic function that follows

Theorem 5.5, \( \left( X_{t \in \mathbb{R}^t} \right)_{t \in \mathbb{R}^+, i \in \mathbb{K}}, M_{VG}^t, \) and \( J_{VG}^t \) be, respectively, a Lévy process specialised as a VG, the VG systematic market risk factor, and the VG idiosyncratic risk factor those follow Lemma 6.8 and structured by the random factor loading. Then \( \varphi_{Xt}^{VG} \)

of the VG Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi_{Xt}^{VG}(u) = \int \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu}) F_{J_t^{v_i}}^{i - 1} (F_{X_t^{v_i}} (F_{t_1} (t)) - k_i - \ell_1 m) \right) f_{M_{VG}^t} (m) dm + \int \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu}) F_{J_t^{v_i}}^{i - 2} (F_{X_t^{v_i}} (F_{t_2} (t)) - k_i - \ell_2 m) \right) f_{M_{VG}^t} (m) dm
\]

Where \( k_i = -\ell_1 \int_{-\infty}^{\kappa} mf_{M_{VG}^t} (m) dm - \ell_2 \int_{\kappa}^{\infty} mf_{M_{VG}^t} (m) dm. \)

6.4.5 A Mixture Gaussian and Variance Gamma Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it
requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply the Mixture Gaussian and Variance Gamma Distribution that admits the Lévy process.

This subsection starts by stating the Mixture Gaussian and Variance Gamma Distribution and its properties and subsequently applying it to the proposed models.

6.4.5.1 $\mathcal{MG}_V$ Distribution and its Properties

This subsection states the Mixture Gaussian and Variance Gamma Distribution and introduces it from its components. Subsequently, this distribution is formalised by stating its definition and followed by it properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be
infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Mixture Gaussian and Variance Gamma Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

Mixture Case 6.3 (Mixture Gaussian and Variance Gamma Distribution $\mathcal{M}_{\text{VG}}^{SG}$)

Let $F_{y_{SG}}$ be a Standard Gaussian Distribution that admits Definition 6.3 and $F_{W_{VG}}$ be a Variance Gamma Distribution that admits Definition 6.12, where they are independent from each other, with $p \in (0,1)$ as the probability of occurrence. Then the Mixture Standard Gaussian and Variance Gamma is structured by $p$, and denoted by $\mathcal{X}_{\mathcal{M}_{\text{VG}}^{SG}}(\lambda, \alpha, \beta, \mu, p)$.

Definition 6.13 (Mixture Gaussian and Variance Gamma Distribution $\mathcal{M}_{\text{VG}}^{SG}$)

A random variable $X$ is said to be Mixture Standard Gaussian and Variance Gamma Distribution, denoted by $\mathcal{X}_{\mathcal{M}_{\text{VG}}^{SG}}(\lambda, \alpha, \beta, \mu, p)$, with $\Gamma(x)$ as the Gamma Function, $p \in (0,1)$ if its density is given by the subsequent equality:

$$f_{\mathcal{X}_{\mathcal{M}_{\text{VG}}^{SG}}(\lambda, \alpha, \beta, \mu, p)}(x) = \frac{pe^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} + \frac{(1-p)(\omega)^\lambda |x - \mu|^{\lambda\frac{1}{2}}e^{\beta(x - \mu)}}{\sqrt{\pi\Gamma(\lambda)(2\alpha)^{\lambda\frac{1}{2}}}}K_{\lambda\frac{1}{2}}(\alpha|x - \mu|)$$

Property D6.13.1 ($\mathcal{M}_{\text{VG}}^{SG}$: Inheritance)

Since $\mathcal{M}_{\text{VG}}^{SG}(\lambda, \alpha, \beta, \mu, p)$ is a mixture of independent Standard Gaussian random variable and Variance Gamma random variable those follow, respectively, Definition 6.3 and Definition 6.13. Then each of them inherits its corresponding distribution properties.
6.4.5.2 $\mathcal{M}^{SG}_{VG}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Mixture Gaussian and Variance Gamma Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and Variance Gamma Factor Copula Model is introduced as a proposed model.

By admitting Assumption 5.2 and the $\mathcal{M}^{SG}_{VG}$ Distribution definition, $\mathcal{M}^{t}_{M_{VG}}, J^{t_{i}}_{M_{VG}},$ and $X^{t_{i}}_{M_{VG}}$ are given by the following Lemma.

**Lemma 6.9 ($\mathcal{M}^{t}_{M_{VG}}, J^{t_{i}}_{M_{VG}}, and X^{t_{i}}_{M_{VG}}$ with $\mathcal{M}^{SG}_{VG}$ Distribution Functions)**

Let $\left(X^{t_{i}}_{M^{SG}_{VG}}\right)_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.3 as a $\mathcal{M}^{SG}_{VG}$ that admits Definition 6.13, with $\mathcal{M}^{t}_{M^{SG}_{VG}}$ and $J^{t_{i}}_{M^{SG}_{VG}}$ as, respectively, the $\mathcal{M}^{SG}_{VG}$ systematic market risk factor, and the $\mathcal{M}^{SG}_{VG}$ idiosyncratic risk...
factor. Then for $\mathcal{M}_t^{\mathbb{M}}$, $\mathcal{J}_t^{\mathbb{M}}$, and $X_t^{\mathbb{M}}$ to admits Assumption 5.2, their parameters has to be set as following:

\[
\begin{align*}
\text{i.} & \quad \mathcal{M}_t^{\mathbb{M}} \left( \frac{\sigma_M^2}{2 \sigma_M^2} \beta_M, \frac{\sigma_M^2}{\sigma_M^2} \right) \\
\text{ii.} & \quad \mathcal{J}_t^{\mathbb{M}} \left( \left( 1 - \frac{\rho_t^2}{\rho^2} \right)^{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2} \right) \beta_M, \frac{1}{2} \right) \\
\text{iii.} & \quad X_t^{\mathbb{M}} \left( \left( 1 - \frac{\rho_t^2}{\rho^2} \right)^{\frac{1}{2}}, \frac{1}{2}, \frac{1}{2} \right) \beta_M, \frac{1}{2} \right) \beta_M, \frac{1}{2} \right)
\end{align*}
\]

Corollary 6.29 ($\mathbb{M}^{\mathbb{S}}$ Factor Copula Model)

Let $\left( X_t^{\mathbb{M}}, \mathcal{M}_t^{\mathbb{M}}, \mathcal{J}_t^{\mathbb{M}} \right)_{t \in \mathbb{R}^+}$ be, respectively, a Lévy process specialised as a $\mathbb{M}^{\mathbb{S}}$, the $\mathbb{M}^{\mathbb{S}}$ systematic market risk factor, and the $\mathbb{M}^{\mathbb{S}}$ idiosyncratic risk factor those follow Lemma 6.9, $\xi_t$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{t_i}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_t^{\mathbb{M}}$. Then $p_{t_i}$ of the $\mathbb{M}^{\mathbb{S}}$ Factor Copula Model is given by the subsequent equality:

\[
\frac{\xi_t}{p_{t_i}} = F_{\mathcal{J}_t^{\mathbb{M}}} \left( \mathcal{M}_t^{\mathbb{M}} \left( \frac{F_{t_i}^{-1} \left( F_{t_i} \left( t \right) \right) - \rho_i \mathcal{M}_t^{\mathbb{M}}}{\sqrt{1 - \rho_t^2}} \right) \right)
\]

6.4.5.3 Binary Stochastic Correlated $\mathbb{M}^{\mathbb{S}}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the
models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Mixture Gaussian and Variance Gamma Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and Variance Gamma Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.22 (\(M^{t, SG}_{M^{t, VG}}\), \(J^{t, SG}_{M^{t, VG}}\), and \(X^{t, SG}_{M^{t, VG}}\) with \(M^{SG}_{VG}\) Distribution Functions)**

In this subsection, the parameters of \(X^{t, SG}_{M^{t, VG}}\), \(M^{t, SG}_{M^{t, VG}}\), and \(J^{t, SG}_{M^{t, VG}}\) are admitting Lemma 6.9 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.30 (Binary Stochastic Correlated \(M^{SG}_{VG}\) Factor Copula Model)**

Let \(X^{t, SG}_{M^{t, VG}}\), \(M^{t, SG}_{M^{t, VG}}\), and \(J^{t, SG}_{M^{t, VG}}\) be, respectively, a Lévy process specialised as a \(M^{SG}_{VG}\), the \(M^{SG}_{VG}\) systematic market risk factor, and the \(M^{SG}_{VG}\) idiosyncratic risk factor those follow Lemma 6.9 and structured by the Binary stochastic correlation, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and \(p_{\xi_i}^{M^{t, SG}_{M^{t, VG}}}\) be the probability of \(\tau_i\).
conditioned upon $\mathcal{M}_{t^{SG}}$. Then $p_{t^{i}}|_{\mathcal{M}_{t^{SG}}}$ of the Binary Stochastic Correlated $\mathbb{M}_{t^{SG}}$ Factor Copula Model is given by the subsequent equality:

$$
\begin{aligned}
    p_{t^{i}}|_{\mathcal{M}_{t^{SG}}^{t,SG}} &= (1-q)F_{\mathcal{M}_{V}^{t,SG}}^{t,SG}
    \left(\frac{F_{X^{t,SG}}^{-1}\left(F_{t^{i}}(t)\right) - \rho_{1}\mathcal{M}_{t^{SG}}}{\sqrt{1 - \rho_{1}^{2}}}\right)
    + qF_{\mathcal{M}_{V}^{t,SG}}^{t,SG}
    \left(\frac{F_{X^{t,SG}}^{-1}\left(F_{t^{i}}(t)\right) - \rho_{2}\mathcal{M}_{t^{SG}}}{\sqrt{1 - \rho_{2}^{2}}}\right)
\end{aligned}
$$

6.4.5.4 Symmetric Stochastic Correlated $\mathbb{M}_{V}^{SG}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Gaussian and Variance Gamma Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and
Variance Gamma Symmetric Stochastic Correlated Factor Copula Model was proposed by ______ or introduced as a proposed model.

**Remark 6.23 (\(M_{M_{Vg}}^t, \theta_{t_i}^{M_{Vg}^S}, \text{ and } \chi_{t_i}^{M_{Vg}^S} \) with \(M_{Vg}^S\) Distribution Functions)**

In this subsection, the parameters of \(\chi_{t_i}^{M_{Vg}^S}, M_{M_{Vg}}^t, \text{ and } \theta_{t_i}^{M_{Vg}^S}\) are admitting Lemma 6.9 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.31 (Symmetric Stochastic Correlated \(M_{Vg}^S\) Factor Copula Model)**

Let \(\chi_{t_i}^{M_{Vg}^S}, M_{M_{Vg}}^t, \text{ and } \theta_{t_i}^{M_{Vg}^S}\) be, respectively, a Lévy process specialised as a \(M_{Vg}^S\), the \(M_{Vg}^S\) systematic market risk factor, and the \(M_{Vg}^S\) idiosyncratic risk factor those follow Lemma 6.9 and structured by the symmetric stochastic correlation, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and \(p_{t_i}\) be the probability of \(\tau_i\) conditioned upon \(M_{M_{Vg}}^t\). Then \(p_{t_i}\) of the Symmetric Stochastic Correlated \(M_{Vg}^S\) Factor Copula Model is given by the subsequent equality:

\[
p_{t_i} | M_{M_{Vg}}^t = \tilde{q} F_{\chi_{t_i}^M_{Vg}} \left( F_{t_i}^{-1} \left( F_{t_i}(t) \right) \right) 
+ (1 - \tilde{q}) \left( 1 - q \right) F_{\theta_{t_i}^t} \left( \frac{F_{t_i}^{-1} \left( F_{t_i}(t) \right) - \rho M_{M_{Vg}}^t}{\sqrt{1 - \rho^2}} \right) 
+ q F_{\theta_{t_i}^t} \left( F_{t_i}^{-1} \left( F_{t_i}(t) \right) \right)
\]
6.4.5.5 \( \mathcal{M}^{SG}_{\mathcal{V}_G} \) Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Gaussian and Variance Gamma Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Mixture Gaussian and Variance Gamma Random Factor Loading Copula Model is introduced as a proposed model.

In order to set the distributions, as in Lemma 6.9, the distributions will be broken on the two parts of this model, where \( \mathcal{M}_t < \kappa \) or \( \mathcal{M}_t \geq \kappa \). The following Lemma will summarise this point.

**Lemma 6.10 (\( \mathcal{M}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \), \( \mathcal{J}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \), and \( \mathcal{X}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \) with \( \mathcal{M}^{SG}_{\mathcal{V}_G} \) Distribution Functions)**

Let \( \left( \mathcal{X}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \right)_{t \in \mathbb{R}^+} \) be a Lévy process that follows Lemma 5.10 and Theorem 5.5 and specialised upon Mixture Case 6.3 as a \( \mathcal{M}^{SG}_{\mathcal{V}_G} \) that admits Definition 6.13, with \( \mathcal{M}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \), and \( \mathcal{J}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}} \) as, respectively, the \( \mathcal{M}^{SG}_{\mathcal{V}_G} \) systematic market risk factor, and the \( \mathcal{M}^{SG}_{\mathcal{V}_G} \) idiosyncratic risk factor. Then for \( \mathcal{M}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}}, \mathcal{M}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^1}, \mathcal{M}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^2}, \mathcal{J}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^1}, \mathcal{J}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^2}, \mathcal{X}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^1}, \) and \( \mathcal{X}^{t}_{\mathcal{M}^{SG}_{\mathcal{V}_G}^2} \) to admits Assumption 5.2, their parameters has to be set as following:
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

\[ \mathcal{M}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{\omega^2}{2m^2}, \alpha, -\beta \right) \]

\[ \mathcal{M}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_1}, \alpha \left( \frac{1}{\epsilon_1} \right) \right) \]

\[ \mathcal{M}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_2}, \alpha \left( \frac{1}{\epsilon_2} \right) \right) \]

\[ \mathcal{J}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_1}, \alpha \left( \frac{1}{\epsilon_1} \right) \right) \]

\[ \mathcal{J}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_2}, \alpha \left( \frac{1}{\epsilon_2} \right) \right) \]

\[ \mathcal{X}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_1}, \alpha \left( \frac{1}{\epsilon_1} \right) \right) \]

\[ \mathcal{X}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \left( \frac{1}{\rho^2}, \frac{1}{\epsilon_2}, \alpha \left( \frac{1}{\epsilon_2} \right) \right) \]

where \( \omega, \bar{\omega}, \alpha, \) and \( \beta \) are related to \( \mathcal{M} \)

**Corollary 6.32 (\( \mathbb{M}_{VG}^{\mathbb{S}G} \) Random Factor Loading Copula Model)**

Let \( \varphi_{N_t} \) be the unconditional number of default’s characteristic function that follows

**Theorem 5.5.** \( \left( \mathcal{X}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}}, \mathcal{M}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}}, \right) \) and \( \mathcal{J}^t_{\mathbb{M}_{VG}^{\mathbb{S}G}} \) be, respectively, a Lévy process specialised as a \( \mathbb{M}_{VG}^{\mathbb{S}G} \), the systematic market risk factor, and the \( \mathbb{M}_{VG}^{\mathbb{S}G} \) idiosyncratic risk factor those follow Lemma 6.10 and structured by the random factor loading. Then \( \varphi_{N_t} \) of the \( \mathbb{M}_{VG}^{\mathbb{S}G} \) Random Factor Loading Copula Model is given by the subsequent equality:
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

Chapter Six: Lévy Factor Copula and its Skewed Version from Theory to Application

6.4.6 A Mixture Normalised Fractional-\(t\) and Variance Gamma Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading

\[
\begin{align*}
q_{Nt}^{SG}(u) &= \frac{1}{\mathcal{N}(u)} \left( 1 - \left(1 - e^{iu}\right) \right) \left( \frac{1 - \ell_1 m}{1 - \ell_1^2} \right) f_{M_t}^{\ell_1}(m)dm \\
&+ \frac{1}{\mathcal{N}(u)} \left( 1 - \left(1 - e^{iu}\right) \right) \left( \frac{1 - \ell_2 m}{1 - \ell_2^2} \right) f_{M_t}^{\ell_2}(m)dm
\end{align*}
\]

Where \( \ell_i = -\ell_1 \int_{-\infty}^{K} m f_{M_t}^{\ell_1}(m)dm - \ell_2 \int_{K}^{\infty} m f_{M_t}^{\ell_2}(m)dm. \)
Copula Model”, and apply the Mixture Normalised Fractional-$t$ and Variance Gamma Distribution that admits the Lévy process.

This subsection starts by stating the Mixture Normalised Fractional-$t$ and Variance Gamma Distribution and its properties and subsequently applying it to the proposed models.

**6.4.6.1 $M^{NFT}_G$ Distribution and its Properties**

This subsection states the Mixture Normalised Fractional-$t$ and Variance Gamma Distribution and introduces it from its components. Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Mixture Normalised Fractional-$t$ and Variance Gamma Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Mixture Case 6.4 (Mixture Normalised Fractional-$t$ and Variance Gamma Distribution $M^{NFT}_G$)**

Let $F_{g_{NFT}}$ be a Normalised Fractional-$t$ Distribution that admits Definition 6.10 and $F_{W_G}$ be a Variance Gamma Distribution that admits Definition 6.12, where they are independent from each other, with $p \in (0,1)$ as the probability of occurrence. Then the Mixture Normalised Fractional-$t$ and Variance Gamma is structured by $p$, and denoted by $X_{M^{NFT}_G(v,\lambda,\alpha,\beta,\mu,\rho)}$. 
Definition 6.14 (Mixture Normalised Fractional-$t$ and Variance Gamma Distribution $\mathbb{M}_{NFG}^{NFT}$)

A random variable $\mathcal{X}$ is said to be Mixture Normalised Fractional-$t$ and Variance Gamma Distribution, denoted by $\mathcal{X}_{\mathbb{M}_{NFG}^{NFT}(v,\lambda,\alpha,\beta,\mu,p)}$, with $v \in \mathbb{R}^+$ as the fractional degree of freedom, and $\Gamma(x)$ as the Gamma Function, $p \in (0,1)$ if its density is given by the subsequent equality:

$$f_{\mathcal{X}_{\mathbb{M}_{NFG}^{NFT}(v,\lambda,\alpha,\beta,\mu,p)}}(x) = p \left( \frac{\sqrt{\frac{v}{v-2}}}{\sqrt{\pi v}} \frac{\Gamma \left( \frac{1}{2} v + 1 \right)}{\sqrt{\pi v} \Gamma \left( \frac{1}{2} v \right)} \left[ 1 + \frac{x^2}{v} \right]^{\frac{1}{2} v - 1} \right) + (1-p)(\omega)^\lambda |x - \mu|^{\left(\lambda - \frac{1}{2}\right)} e^{\beta(x - \mu)} \frac{K_{\frac{\lambda - 1}{2}}(\alpha |x - \mu|)}{\sqrt{\pi} \Gamma(\lambda) (2\alpha)^{\left(\lambda - \frac{1}{2}\right)}}$$

Property D6.14.1 ($\mathbb{M}_{NFG}^{NFT}$: Inheritance)

Since $\mathbb{M}_{NFG}^{NFT}(v,\lambda,\alpha,\beta,\mu,p)$ is a mixture of independent Normalised Fractional-$t$ random variable and Variance Gamma random variable those follow, respectively, Definition 6.10 and Definition 6.13. Then each of them inherits its corresponding distribution properties.

6.4.6.2 $\mathbb{M}_{NFG}^{NFT}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.
An immediate result could be achieved when applying the Mixture Normalised Fractional-$t$ and Variance Gamma Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Normalised Fractional-$t$ and Variance Gamma Factor Copula Model is introduced as a proposed model.

By admitting Assumption 5.2 and the $\mathcal{M}^{N_{\mathcal{V}_G}}_{\mathcal{V}_G}$ Distribution definition, $\mathcal{M}^{t}_{\mathcal{V}_G}$, $\mathcal{J}^{t_i}_{\mathcal{V}_G}$, and $\mathcal{X}^{t_i}_{\mathcal{V}_G}$ are given by the following Lemma.

**Lemma 6.11 ($\mathcal{M}^{t}_{\mathcal{V}_G}$, $\mathcal{J}^{t_i}_{\mathcal{V}_G}$, and $\mathcal{X}^{t_i}_{\mathcal{V}_G}$ with $\mathcal{M}^{N_{\mathcal{V}_G}}_{\mathcal{V}_G}$ Distribution Functions)**

Let $\left(\mathcal{X}^{t_i}_{\mathcal{V}_G}\right)_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.4 as a $\mathcal{M}^{N_{\mathcal{V}_G}}_{\mathcal{V}_G}$ that admits Definition 6.14, with $\mathcal{M}^{t}_{\mathcal{V}_G}$, and $\mathcal{J}^{t_i}_{\mathcal{V}_G}$ as, respectively, the $\mathcal{M}^{N_{\mathcal{V}_G}}_{\mathcal{V}_G}$ systematic market risk factor and the $\mathcal{M}^{N_{\mathcal{V}_G}}_{\mathcal{V}_G}$ idiosyncratic risk factor. Then for $\mathcal{M}^{t}_{\mathcal{V}_G}$, $\mathcal{J}^{t_i}_{\mathcal{V}_G}$, and $\mathcal{X}^{t_i}_{\mathcal{V}_G}$ to admits Assumption 5.2, their parameters has to be set as following:

1. $\mathcal{M}^{t}_{\mathcal{V}_G} = \left(\frac{\beta M - \beta M^t}{\frac{\beta M^t}{\alpha M^t}}\right)$
2. $\mathcal{J}^{t_i}_{\mathcal{V}_G} = \left(\frac{\left(1 - \frac{\beta M}{\beta M^t}\right)^2}{\frac{\beta M}{\beta M^t}}\right)$
3. $\mathcal{X}^{t_i}_{\mathcal{V}_G} = p\mathcal{X}^{t_i}_{\mathcal{V}_G} + (1-p)\mathcal{X}^{t_i}_{\mathcal{V}_G}$, where $\mathcal{X}^{t_i}_{\mathcal{V}_G}$ as in Lemma 6.5, and $\mathcal{X}^{t_i}_{\mathcal{V}_G}$ as in Lemma 6.7, and $p \in (0,1)$.  

**Proof:** (See Lemma 6.4 and Lemma 6.7).
Corollary 6.33 (\( \mathbb{M}^{NFT}_{VG} \) Factor Copula Model)

Let \( \mathcal{X}^{t}_{M^{NFT}_{VG}} \), \( M^{NFT}_{VG} \), and \( J^{t}_{M^{NFT}_{VG}} \) be, respectively, a Lévy process specialised as a \( \mathbb{M}^{NFT}_{VG} \), the \( \mathbb{M}^{NFT}_{VG} \) systematic market risk factor, and the \( \mathbb{M}^{NFT}_{VG} \) idiosyncratic risk factor those follow Lemma 6.11, \( \xi \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau \) admit Assumption 4.12 and Definition 4.15, \( p^{t}_{\xi} \mid M^{NFT}_{VG} \) be the probability of \( \tau \) conditioned upon \( M^{NFT}_{VG} \). Then \( p^{t}_{\xi} \mid M^{NFT}_{VG} \) of the \( \mathbb{M}^{NFT}_{VG} \) Factor Copula Model is given by the subsequent equality:

\[
p^{t}_{\xi} \mid M^{NFT}_{VG} = F_{\mathcal{X}^{t}_{M^{NFT}_{VG}}} \left( F^{-1}_{M^{NFT}_{VG}} \left( p^{t}_{\xi} \right) - \rho_{i} M^{t}_{M^{NFT}_{VG}} \right) \frac{1}{\sqrt{1 - \rho_{i}^{2}}}
\]

6.4.6.3 Binary Stochastic Correlated \( \mathbb{M}^{NFT}_{VG} \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Mixture Normalised Fractional-\( t \) and Variance Gamma Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection
5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Normalised Fractional-$t$ and Variance Gamma Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.24** ($\mathcal{M}_{\mathcal{M}^{NFT}_V}, \mathcal{J}_{\mathcal{M}^{NFT}_V}$, and $\chi_{\mathcal{M}^{NFT}_V}$ with $\mathcal{M}^{NFT}_V$ Distribution Functions)

In this subsection, the parameters of $\left(\chi_{\mathcal{M}^{NFT}_V}^{t_1}, \mathcal{M}_{\mathcal{M}^{NFT}_V}, \mathcal{J}_{\mathcal{M}^{NFT}_V}\right)$ are admitting Lemma 6.11 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.34** (Binary Stochastic Correlated $\mathcal{M}^{NFT}_V$ Factor Copula Model)

Let $\left(\chi_{\mathcal{M}^{NFT}_V}^{t_1}, \mathcal{M}_{\mathcal{M}^{NFT}_V}, \mathcal{J}_{\mathcal{M}^{NFT}_V}\right)$ be, respectively, a Lévy process specialised as a $\mathcal{M}^{NFT}_V$, the $\mathcal{M}^{NFT}_V$ systematic market risk factor, and the $\mathcal{M}^{NFT}_V$ idiosyncratic risk factor those follow Lemma 6.11 and structured by the Binary stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i}^{t_1} \mid \mathcal{M}_{\mathcal{M}^{NFT}_V}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{M}^{NFT}_V}$. Then $p_{\xi_i}^{t_1} \mid \mathcal{M}_{\mathcal{M}^{NFT}_V}$ of the Binary Stochastic Correlated $\mathcal{M}^{NFT}_V$ Factor Copula Model is given by the subsequent equality:

$$
p_{\xi_i}^{t_1} \mid \mathcal{M}_{\mathcal{M}^{NFT}_V} = (1-q)F_{\mathcal{M}^{NFT}_V}^1 \left( \frac{\left( F_{\mathcal{M}^{NFT}_V}^{\tau_1} - \rho_1 \mathcal{M}_{\mathcal{M}^{NFT}_V} \right)}{\sqrt{1 - \rho_1^2}} \right) + qF_{\mathcal{M}^{NFT}_V}^1 \left( \frac{\left( F_{\mathcal{M}^{NFT}_V}^{\tau_1} - \rho_2 \mathcal{M}_{\mathcal{M}^{NFT}_V} \right)}{\sqrt{1 - \rho_2^2}} \right)
$$
6.4.6.4 Symmetric Stochastic Correlated $M^{N_{I}}_{Vg}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Normalised Fractional-$t$ and Variance Gamma in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Normalised Fractional-$t$ and Variance Gamma Symmetric Stochastic Correlated Factor Copula Model is introduced as a proposed model.

**Remark 6.25** ($M^{t}_{M_{Vg}}, J^{t}_{M_{Vg}},$ and $X^{t}_{M_{Vg}}$ with $M^{N_{I}}_{Vg}$ Distribution Functions)

In this subsection, the parameters of $(X^{t}_{M^{N_{I}}_{Vg}})_{t \in \mathbb{R}^{+}}, M^{t}_{M^{N_{I}}_{Vg}},$ and $J^{t}_{M^{N_{I}}_{Vg}}$ are admitting Lemma 6.11 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.35** (Symmetric Stochastic Correlated $M^{N_{I}}_{Vg}$ Factor Copula Model)

Let $(X^{t}_{M^{N_{I}}_{Vg}})_{t \in \mathbb{R}^{+}}, M^{t}_{M^{N_{I}}_{Vg}},$ and $J^{t}_{M^{N_{I}}_{Vg}}$ be, respectively, a Lévy process specialised as a $M^{N_{I}}_{Vg}$, the $M^{N_{I}}_{Vg}$ systematic market risk factor, and the $M^{N_{I}}_{Vg}$ idiosyncratic risk
factor those follow Lemma 6.11 and structured by the symmetric stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit Assumption 4.12 and Definition 4.15, and \( p_{t_i}^{M_t^{NFFt}} \) be the probability of \( \tau_i \) conditioned upon \( M_t^{NFFt} \). Then \( p_{t_i}^{M_t^{NFFt}} \) of the Symmetric Stochastic Correlated Factor Copula Model is given by the subsequent equality:

\[
\begin{align*}
\xi_i \left| M_t^{NFFt} \right. & \equiv q F_{M_t^{NFFt}} \left( F_{\xi_i}^{-1} \left( F_{\xi_i}(t) \right) \right) \\
& + (1-q) \left[ (1-q) F_{M_t^{NFFt}} \left( F_{\xi_i}^{-1} \left( F_{\xi_i}(t) - \rho M_t^{NFFt} \right) \frac{\sqrt{1 - \rho^2}}{1 - \rho^2} \right) \right] \\
& + q F_{M_t^{NFFt}} \left( F_{\xi_i}^{-1} \left( F_{\xi_i}(t) \right) \right) 
\end{align*}
\]

6.4.6.5 \( M_t^{NFFt} \) Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Normalised Fractional-\( t \) and Variance Gamma Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8.
This model is proposed as better alternative than the based model. The Mixture Normalised Fractional-\(t\) and Variance Gamma Random Factor Loading Copula Model is introduced as a proposed model.

In order to set the distributions, as in Lemma 6.11, the distributions will be broken on the two parts of this model, where \(M_t < \kappa\) or \(M_t \geq \kappa\). The following Lemma will summarise this point.

**Lemma 6.12 (\(M_{tM_{NFT}^t}^t, J_{M_{NFT}^t}^t, \text{ and } X_{M_{NFT}^t}^t\text{ with } \mathbb{M}_{NFT}^t\text{ Distribution Functions})**

Let \(\mathcal{X}_{M_{NFT}^t}^t\) be a Lévy process that follows Lemma 5.10 and Theorem 5.5 and specialised upon Mixture Case 6.4 as a \(\mathbb{M}_{NFT}^t\) that admits Definition 6.14, with \(M_{tM_{NFT}^t}^t\), and \(J_{M_{NFT}^t}^t\) as, respectively, the \(\mathbb{M}_{NFT}^t\) systematic market risk factor, and the \(\mathbb{M}_{NFT}^t\) idiosyncratic risk factor. Then for \(M_{tM_{NFT}^t}^t, \ M_{tM_{NFT}^t1}^t, \ M_{tM_{NFT}^t2}^t, \ J_{M_{NFT}^t1}^t, J_{M_{NFT}^t2}^t, X_{M_{NFT}^t1}^t\) and \(X_{M_{NFT}^t2}^t\) to admits Assumption 5.2, their parameters has to be set as following:

i. \[ M_{tM_{NFT}^t}^t \left( v, \omega, \alpha, \beta, -\beta \omega \right) \]

ii. \[ M_{tM_{NFT}^t1}^t \left( v, \frac{1-t_1}{t_2}, \omega, \frac{1-t_1}{t_1}, \alpha, \frac{1-t_1}{t_1}, \beta, -\beta \omega \right) \]

iii. \[ M_{tM_{NFT}^t2}^t \left( v, \frac{1-t_2}{t_2}, \omega, \frac{1-t_2}{t_2}, \alpha, \frac{1-t_2}{t_2}, \beta, -\beta \omega \right) \]

iv. \[ J_{M_{NFT}^t1}^t \left( v, \frac{1-t_1}{t_1}, \omega, \frac{1-t_1}{t_1}, \alpha, \frac{1-t_1}{t_1}, \beta, -\beta \omega \right) \]

v. \[ J_{M_{NFT}^t2}^t \left( v, \frac{1-t_2}{t_2}, \omega, \frac{1-t_2}{t_2}, \alpha, \frac{1-t_2}{t_2}, \beta, -\beta \omega \right) \]
vi. \( X^t_{\text{M}_{\text{N}t}^{\text{F}t}} = pX^t_{\text{N}_{\text{F}t}^{\text{t}}} + (1-p)X^t_{\text{V}_{\text{G}}} \), where \( X^t_{\text{N}_{\text{F}t}^{\text{t}}} \) as in Lemma 6.12, and

\[
X^t_{\text{V}_{\text{G}}} \left( \frac{1}{r_1^2} \frac{a^2}{2\sigma} \frac{1}{r_2} \alpha \left( \frac{1}{r_1} \right) \beta \left( \frac{1}{r_2} \right) \frac{\omega}{m} \right), \quad \text{and} \quad p \in (0,1).
\]

vii. \( X^t_{\text{M}_{\text{N}t}^{\text{F}t}^2} = pX^t_{\text{N}_{\text{F}t}^{\text{t}}} + (1-p)X^t_{\text{V}_{\text{G}}} \), where \( X^t_{\text{N}_{\text{F}t}^{\text{t}}} \) as in Lemma 6.12, and

\[
X^t_{\text{V}_{\text{G}}} \left( \frac{1}{r_1^2} \frac{a^2}{2\sigma} \frac{1}{r_2} \alpha \left( \frac{1}{r_1} \right) \beta \left( \frac{1}{r_2} \right) \frac{\omega}{m} \right), \quad \text{and} \quad p \in (0,1).
\]

where \( \omega, \sigma, \alpha, \) and \( \beta \) are related to \( \mathcal{M} \)

**Corollary 6.36 (\( \text{M}_{\text{N}t}^{\text{F}t} \) Random Factor Loading Copula Model)**

Let \( \varphi_{N_t} \) be the unconditional number of default’s characteristic function that follows

**Theorem 5.5, \( \left( X^t_{\text{M}_{\text{N}t}^{\text{F}t}} \right)_{t \in \mathbb{R}^+} \), \( \mathcal{M}_{\text{N}t}^{\text{F}t} \), and \( \mathcal{J}_{\text{N}t}^{\text{F}t} \) be, respectively, a Lévy process specialised as a \( \text{M}_{\text{N}t}^{\text{F}t} \), the \( \text{M}_{\text{N}t}^{\text{F}t} \) systematic market risk factor, and the \( \text{M}_{\text{N}t}^{\text{F}t} \) idiosyncratic risk factor those follow Lemma 6.12 and structured by the random factor loading. Then \( \varphi_{N_t} \) of the \( \text{M}_{\text{N}t}^{\text{F}t} \) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi_{\text{N}_t} (u) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{\mathcal{J}_{\text{N}t}^{\text{F}t}} \left( \frac{F_{\mathcal{X}_{\text{M}_{\text{N}t}^{\text{F}t}}}^{-1} \left( F_{\mathcal{X}_{\text{M}_{\text{N}t}^{\text{F}t}}} \left( t \right) - \kappa_i - \ell_1 m \right)}{\sqrt{1 - \ell_1^2}} \right) \right) f_{\mathcal{M}_{\text{M}_{\text{N}t}^{\text{F}t}}} (m) dm \\
+ \int_{\kappa}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{\mathcal{J}_{\text{N}t}^{\text{F}t}} \left( \frac{F_{\mathcal{X}_{\text{M}_{\text{N}t}^{\text{F}t}}}^{-1} \left( F_{\mathcal{X}_{\text{M}_{\text{N}t}^{\text{F}t}}} \left( t \right) - \kappa_i - \ell_2 m \right)}{\sqrt{1 - \ell_2^2}} \right) \right) f_{\mathcal{M}_{\text{M}_{\text{N}t}^{\text{F}t}}} (m) dm
\]

Where \( \kappa_i = -\ell_1 \int_{-\infty}^{\kappa} m f_{\mathcal{M}_{\text{M}_{\text{N}t}^{\text{F}t}}} (m) dm - \ell_2 \int_{\kappa}^{\infty} m f_{\mathcal{M}_{\text{M}_{\text{N}t}^{\text{F}t}}} (m) dm. \)
6.4.7 A Normal Inverse Gaussian Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply Normal Inverse Gaussian Distribution that admits the Lévy process. This Distribution is introduced as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6.

This subsection starts by stating the Normal Inverse Gaussian Distribution and its properties and subsequently applying it to the proposed models.

6.4.7.1 $\mathcal{NIG}$ Distribution and its Properties

This subsection states the Normal Inverse Gaussian Distribution and introduces it as a limiting case of Generalized Hyperbolic Distribution articulated in Definition 6.6.
Subsequently, this distribution is formalised by stating its definition and followed by its properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admit Levy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Normal Inverse Gaussian distribution to the Levy Factor Copula Model, the Binary Stochastic Correlated Levy Factor Copula Model, the Symmetric Stochastic Correlated Levy Factor Copula Model and the Levy Random Factor Loading Copula Model.

**Limiting Case 6.8 (Generalized Hyperbolic with \( \lambda = -\frac{1}{2} \))**

Let \( \mathcal{X} \) be Generalised Hyperbolic random variable that admits Definition 6.6, and \( \lambda = -\frac{1}{2} \), then \( \mathcal{X}_{\mathcal{G}H(-\frac{1}{2}, \alpha, \beta, \delta, \mu)} \) is said to be Normal Inverse Gaussian, denoted by \( \mathcal{X}_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)} \).

**Definition 6.15 (Normal Inverse Gaussian Distribution \( \mathcal{NIG} \))**

A random variable \( \mathcal{X} \) is said to be Normal Inverse Gaussian, denoted by \( \mathcal{X}_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)} \), with \( |\beta| \in [0, \alpha] \) and \( K_\lambda(\cdot) \) as the modified Bessel function of the third kind of order 1, as given in Definition 6.6. If its density is given by the subsequent equality:

\[
f_{\mathcal{X}_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)}}(x) = \frac{\delta \alpha \, e^{(\delta y + \beta (x-u))}}{\pi \sqrt{\delta^2 + (x - \mu)^2}} K_1 \left( \frac{\alpha \sqrt{\delta^2 + (x - \mu)^2}}{\sqrt{\delta^2 + (x - \mu)^2}} \right)
\]

And its moment generating function, denoted by \( \psi(u)_{\mathcal{NIG}(\lambda, \alpha, \beta, \delta, \mu)} \), is given by the following equality:

\[
\psi(u)_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)} = e^{\mu u} \frac{e^{\delta \sqrt{\omega}}}{e^{\delta \sqrt{\xi_{(\omega, u)}}}}
\]
Property D.6.15.1 ($N\mathcal{IG}$: Mean, Variance, Skewness and Kurtosis)

If $X$ is a Normal Inverse Gaussian random variable that follows Definition 6.15, then $X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}$ Mean, Variance, Skewness and Kurtosis, respectively, are given by the subsequent equality:

\begin{align*}
i. \quad & \mathbb{E}[X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}] = \mu + \frac{\beta\delta}{\omega^2} \\
ii. \quad & \mathbb{V}[X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}] = \frac{\delta\alpha^2}{\omega^2} \\
iii. \quad & \mathbb{S}[X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}] = \frac{3\beta}{\omega^2\alpha\sqrt{\delta}} \\
iv. \quad & \mathbb{K}[X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}] = \frac{3(1+4\left(\frac{\beta}{\delta}\right)^2)}{\delta\sqrt{\omega}}
\end{align*}

A strong argument for using the $N\mathcal{IG}$ distribution with the Lévy Factor copula is its closeness under convolution property.

Property D.6.15.2 ($N\mathcal{IG}$: Convolution)

If $X$ and $Y$ are Normal Inverse Gaussian random variables those follow Definition 6.15, then they are stable under convolution, as shown in the subsequent equality:

\[ X_{N\mathcal{IG}(\alpha_1,\beta_1,\delta_1,\mu_1)} + Y_{N\mathcal{IG}(\alpha_2,\beta_2,\delta_2,\mu_2)} \sim Z_{N\mathcal{IG}(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)} \]

when

\begin{align*}
i. \quad & \alpha = \alpha_1 = \alpha_2 \\
ii. \quad & \beta = \beta_1 = \beta_2
\end{align*}

Property D.6.15.3 ($N\mathcal{IG}$: Scaling)

If $X$ is a Normal Inverse Gaussian random variable that follows Definition 6.15, then $X_{N\mathcal{IG}(\alpha,\beta,\delta,\mu)}$ can be scaled by a constant $c$, as shown in the subsequent equality:

\[ X_{\mathcal{N}\mathcal{IG}\left(\frac{c\alpha}{c\delta},\frac{c\beta}{c\delta},\frac{c\delta}{c\delta},\frac{c\mu}{c}\right)} \]
6.4.7.2 $\mathcal{N}\mathcal{I}\mathcal{G}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Normal Inverse Gaussian Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Normal Inverse Gaussian Factor copula model is introduced in (Guegan and Houdain, 2005), where the Double Normal Inverse Gaussian Factor copula model is introduced in (Brunlid, 2006), (Ferrarese, 2006), and (Kalemanova et al., 2007).

By admitting Assumption 5.2 and the Gaussian Distribution, $\mathcal{M}_{t_N \mathcal{I} \mathcal{G}}$, $\mathcal{J}_{t_N \mathcal{I} \mathcal{G}}$, and $\mathcal{X}_{t_N \mathcal{I} \mathcal{G}}$ with $\mathcal{N}\mathcal{I}\mathcal{G}$ are given by the following Lemma.

**Lemma 6.13 ($\mathcal{M}_{t_N \mathcal{I} \mathcal{G}}$, $\mathcal{J}_{t_N \mathcal{I} \mathcal{G}}$, and $\mathcal{X}_{t_N \mathcal{I} \mathcal{G}}$ with $\mathcal{N}\mathcal{I}\mathcal{G}$ Distribution Functions)**

Let $(\mathcal{X}_{t_N \mathcal{I} \mathcal{G}}^t)_{t \in \mathbb{R}^+}$ be a Lévy process that follows Corollary 5.11 and specialised upon Limiting Case 6.8 as a $\mathcal{N}\mathcal{I}\mathcal{G}$ that admits Definition 6.15, with $\mathcal{M}_{t_N \mathcal{I} \mathcal{G}}$, and $\mathcal{J}_{t_N \mathcal{I} \mathcal{G}}$ as, respectively, the $\mathcal{N}\mathcal{I}\mathcal{G}$ systematic market risk factor, and the $\mathcal{N}\mathcal{I}\mathcal{G}$ idiosyncratic risk
factor. Then for $\mathcal{M}^t_{\mathcal{N}JG}$, $\mathcal{J}^t_{\mathcal{N}JG}$, and $\mathcal{X}^t_{\mathcal{N}JG}$ to admits Assumption 5.2, their parameters has to be set as following:

\[
\begin{align*}
\text{i. } & \quad \mathcal{M}^t_{\mathcal{N}JG} \left( \alpha, \beta \frac{\omega}{\alpha^2} - \frac{\beta \omega}{\alpha^2} \right) \\
\text{ii. } & \quad \mathcal{J}^t_{\mathcal{N}JG} \left( \frac{1}{\rho_i} \alpha \left( \frac{1}{\rho_i} \right) \beta \left( \frac{1}{\rho_i} \right) \frac{3}{\alpha^2} \left( \frac{1}{\rho_i} \right) \frac{\beta \omega}{\alpha^2} \right) \\
\text{iii. } & \quad \mathcal{X}^t_{\mathcal{N}JG} \left( \frac{1}{\rho_i} \alpha \left( \frac{1}{\rho_i} \right) \beta \left( \frac{1}{\rho_i} \right) \frac{3}{\alpha^2} \left( \frac{1}{\rho_i} \right) \frac{\beta \omega}{\alpha^2} \right)
\end{align*}
\]

where $\omega$, $\rho$, $\alpha$, and $\beta$ are related to $\mathcal{M}$.

**Proof:** (Similar to Lemma 6.7)

**Corollary 6.37 (\(\mathcal{N}JG\) Factor Copula Model)**

Let \( \left( \mathcal{X}^t_{\mathcal{N}JG} \right)_{t \in \mathbb{R}^+} \), $\mathcal{M}^t_{\mathcal{N}JG}$, and $\mathcal{J}^t_{\mathcal{N}JG}$ be, respectively, a Lévy process specialised as a $\mathcal{N}JG$, the $\mathcal{N}JG$ systematic market risk factor, and the $\mathcal{N}JG$ idiosyncratic risk factor those follow Lemma 6.13, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i|\mathcal{M}^t_{\mathcal{N}JG}}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}^t_{\mathcal{N}JG}$. Then $p_{\xi_i|\mathcal{M}^t_{\mathcal{N}JG}}$ of the $\mathcal{N}JG$ Factor Copula Model is given by the subsequent equality:

\[
p_{\xi_i|\mathcal{M}^t_{\mathcal{N}JG}} = F_{\mathcal{X}^t_{\mathcal{N}JG}} \left( \frac{F^{-1}_{\mathcal{X}^t_{\mathcal{N}JG}} \left( F_{\tau_i}(t) \right) - \rho_i \mathcal{M}^t_{\mathcal{N}JG}}{\sqrt{1 - \rho_i^2}} \right)
\]

### 6.4.7.3 Binary Stochastic Correlated $\mathcal{N}JG$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the
models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Normal Inverse Gaussian Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Normal Inverse Gaussian Binary Stochastic Correlated Factor Copula Model was proposed in (Kalemanova et al., 2007).

**Remark 6.26 (\(M_{\mathcal{J}3g}^{t_i}, \mathcal{J}_{\mathcal{J}3g}^{t_i}, \text{and } X_{\mathcal{J}3g}^{t_i} \text{ with } \mathcal{N}\mathcal{J}\mathcal{G} \text{ Distribution Functions})**

In this subsection, the parameters of \((X_{\mathcal{J}3g}^{t_i})_{t \in \mathbb{R}^+}, M_{\mathcal{J}3g}^{t_i}, \text{ and } \mathcal{J}_{\mathcal{J}3g}^{t_i}\) are admitting Lemma 6.13 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.38 (Binary Stochastic Correlated \(\mathcal{N}\mathcal{J}\mathcal{G}\) Factor Copula Model)**

Let \((X_{\mathcal{J}3g}^{t_i})_{t \in \mathbb{R}^+}, M_{\mathcal{J}3g}^{t_i}, \text{ and } \mathcal{J}_{\mathcal{J}3g}^{t_i}\) be, respectively, a Lévy process specialised as a \(\mathcal{N}\mathcal{J}\mathcal{G}\), the \(\mathcal{N}\mathcal{J}\mathcal{G}\) systematic market risk factor, and the \(\mathcal{N}\mathcal{J}\mathcal{G}\) idiosyncratic risk factor those follow Lemma 6.13 and structured by the Binary stochastic correlation, \(\xi_i\) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i\) admit Assumption 4.12 and Definition 4.15, and \(p_{\tau_i | M_{\mathcal{J}3g}^{t_i}}\) be the probability of \(\tau_i\) conditioned upon \(M_{\mathcal{J}3g}^{t_i}\). Then \(p_{\tau_i | M_{\mathcal{J}3g}^{t_i}}\) of the Binary Stochastic Correlated \(\mathcal{N}\mathcal{J}\mathcal{G}\) Factor Copula Model is given by the subsequent equality:
As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Normal Inverse Gaussian Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Normal Inverse Gaussian Symmetric Stochastic Correlated Factor Copula Model was proposed in (Kalemanova et al., 2007).

**Remark 6.27** ($\mathcal{M}^t_{\mathcal{N}^2g}$, $\mathcal{J}^t_{\mathcal{N}^2g}$, and $\mathcal{X}^t_{\mathcal{N}^2g}$ with $\mathcal{N}^2G$ Distribution Functions)

In this subsection, the parameters of $\left(\mathcal{X}^t_{\mathcal{N}^2g}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}^t_{\mathcal{N}^2g}$, and $\mathcal{J}^t_{\mathcal{N}^2g}$ are admitting Lemma 6.13 but structured as in Lemma 5.9 instead of Corollary 5.11.
Corollary 6.39 (Symmetric Stochastic Correlated $\mathcal{N}\mathcal{J}\mathcal{G}$ Factor Copula Model)

Let $\left( X_{\mathcal{N}\mathcal{J}\mathcal{G}}^{t_i} \right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t$, and $\mathcal{J}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t$ be, respectively, a Lévy process specialised as a $\mathcal{N}\mathcal{J}\mathcal{G}$, the $\mathcal{N}\mathcal{J}\mathcal{G}$ systematic market risk factor, and the $\mathcal{N}\mathcal{J}\mathcal{G}$ idiosyncratic risk factor those follow Lemma 6.13 and structured by the symmetric stochastic correlation, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i \mid \mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t$. Then $p_{\xi_i \mid \mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t}$ of the Symmetric Stochastic Correlated $\mathcal{N}\mathcal{J}\mathcal{G}$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i \mid \mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} = q F_{\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} \left( F_{\mathcal{X}_{\mathcal{N}\mathcal{J}\mathcal{G}}^{t_i}}^{-1} \left( F_{\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} (t) \right) \right)$$

$$+ (1 - q) \left[ (1 - q) F_{\mathcal{J}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} \left( \frac{F_{\mathcal{X}_{\mathcal{N}\mathcal{J}\mathcal{G}}^{t_i}}^{-1} \left( F_{\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} (t) \right) - \rho \mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t}{\sqrt{1 - \rho^2}} \right) \right]$$

$$+ q F_{\mathcal{J}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} \left( F_{\mathcal{X}_{\mathcal{N}\mathcal{J}\mathcal{G}}^{t_i}}^{-1} \left( F_{\mathcal{M}_{\mathcal{N}\mathcal{J}\mathcal{G}}^t} (t_i) \right) \right)$$

6.4.7.5 $\mathcal{N}\mathcal{J}\mathcal{G}$ Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Normal Inverse Gaussian Distribution in the proposed Lévy Random Factor Loading
Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Normal Inverse Gaussian Random Factor Loading Copula Model was proposed in (Kalemanova et al., 2007).

In order to set the distributions, as in Lemma 6.13, the distributions will be broken on the two parts of this model, where \( \mathcal{M}_t < \kappa \) or \( \mathcal{M}_t \geq \kappa \). The following Lemma will summarise this point.

**Lemma 6.14 (\( \mathcal{M}_{NG}^t, \mathcal{J}_{NG}^t, \text{and } X_{NG}^t \) with \( NG \) Distribution Functions)**

Let \( \left( X_{NG}^{t_i} \right)_{t \in \mathbb{R}^+} \) be a Lévy process that follows Lemma 5.10 and Theorem 5.5 and specialised upon Limiting Case 6.13 as a \( NG \) that admits Definition 6.15, with \( \mathcal{M}_{NG}^t, \mathcal{J}_{NG}^t \), and \( X_{NG}^t \) as, respectively, the \( NG \) systematic market risk factor, and the \( NG \) idiosyncratic risk factor. Then for \( \mathcal{M}_{NG}^t, \mathcal{M}_{NG}^t, \mathcal{M}_{NG}^t, \mathcal{J}_{NG}^t, \mathcal{J}_{NG}^t, \mathcal{X}_{NG}^t \), and \( \mathcal{X}_{NG}^t \) to admits Assumption 5.2, their parameters has to be set as following:

1. \[
\mathcal{M}_{NG}^t \left( \alpha, \beta, \frac{3}{2} \frac{\omega^2}{\alpha^2} - \frac{\omega}{\alpha^2} \right)
\]
2. \[
\mathcal{M}_{NG}^t \left( \left( \frac{1-t_1}{\ell_1} \right) \alpha, \beta \left( \frac{1-t_1^2}{\ell_1^2} \right) \frac{3}{2} \frac{\omega^2}{\alpha^2} - \frac{\omega}{\alpha^2} \right)
\]
3. \[
\mathcal{M}_{NG}^t \left( \left( \frac{1-t_2}{\ell_2} \right) \frac{\omega^2}{2 \alpha}, \alpha \left( \frac{1-t_2^2}{\ell_2} \right) \beta, \beta \left( \frac{1-t_2^2}{\ell_2^2} \right) \frac{3}{2} \frac{\omega^2}{\alpha^2} - \frac{\omega}{\alpha^2} \right)
\]
4. \[
\mathcal{J}_{NG}^t \left( \left( \frac{1-t_1}{\ell_1} \right) \alpha, \beta \left( \frac{1-t_1^2}{\ell_1^2} \right) \frac{3}{2} \frac{\omega^2}{\alpha^2} - \frac{\omega}{\alpha^2} \right)
\]
5. \[
\mathcal{J}_{NG}^t \left( \left( \frac{1-t_2}{\ell_2} \right) \frac{\omega^2}{\alpha}, \beta \left( \frac{1-t_2^2}{\ell_2^2} \right) \frac{3}{2} \frac{\omega^2}{\alpha^2} - \frac{\omega}{\alpha^2} \right)
\]
where \( \omega, \sigma, \alpha, \) and \( \beta \) are related to \( \mathcal{M} \)

**Proof:** (Similar to Lemma 6.7)

**Corollary 6.40 \((\mathcal{N}\mathcal{J}\mathcal{G} \text{ Random Factor Loading Copula Model})\)**

Let \( \varphi_{\mathcal{N}i} \) be the unconditional number of default’s characteristic function that follows Theorem 5.5, \((\mathcal{X}^{t_i}_{\mathcal{N}G})_{t\in \mathbb{R}^+}, \mathcal{M}^t_{\mathcal{N}G}, \) and \( \mathcal{J}^{t_i}_{\mathcal{N}G} \) be, respectively, a Lévy process specialised as a \( \mathcal{N}\mathcal{J}\mathcal{G} \), the \( \mathcal{N}\mathcal{J}\mathcal{G} \) systematic market risk factor, and the \( \mathcal{N}\mathcal{J}\mathcal{G} \) idiosyncratic risk factor those follow Lemma 6.14 and structured by the random factor loading. Then \( \varphi^{\mathcal{N}\mathcal{G}}_{\mathcal{N}t} \) of the \( \mathcal{N}\mathcal{J}\mathcal{G} \) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi^{\mathcal{N}\mathcal{G}}_{\mathcal{N}t}(u) = \int_{-\infty}^{\infty} \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu})f_{\mathcal{J}^{t_i}_{\mathcal{N}G}^1}^{\mathcal{N}\mathcal{G}^2}(F^{\mathcal{N}\mathcal{G}^4}_{\mathcal{X}^{t_i}_{\mathcal{N}G}}(F_{\mathcal{N}G}^{t_i}(t)) - k_i - \ell_1 m) \left( 1 - \frac{\ell_1^2}{2} \right) \right) f_{\mathcal{M}^{\mathcal{N}\mathcal{G}}}(m)dm \\
+ \int_{\kappa}^{\infty} \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{iu})f_{\mathcal{J}^{t_i}_{\mathcal{N}G}^2}^{\mathcal{N}\mathcal{G}^2}(F^{\mathcal{N}\mathcal{G}^4}_{\mathcal{X}^{t_i}_{\mathcal{N}G}}(F_{\mathcal{N}G}^{t_i}(t)) - k_i - \ell_2 m) \left( 1 - \frac{\ell_2^2}{2} \right) \right) f_{\mathcal{M}^{\mathcal{N}\mathcal{G}}}(m)dm
\]

Where \( k_i = -\ell_1 \int_{-\infty}^{\kappa} mf_{\mathcal{M}^{\mathcal{N}\mathcal{G}}^1}(m)dm - \ell_2 \int_{\kappa}^{\infty} mf_{\mathcal{M}^{\mathcal{N}\mathcal{G}}^2}(m)dm. \)

**6.4.8 A Mixture Gaussian and Normal Inverse Gaussian Factor Copula**

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the
model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply the Mixture Gaussian and Normal Inverse Gaussian Distribution that admits the Lévy process.

This subsection starts by stating the Mixture Gaussian and Normal Inverse Gaussian Distribution and its properties and subsequently applying it to the proposed models.

6.4.8.1 $\mathcal{M}^{\mathcal{G}}_{\mathcal{N}_{\mathcal{I}}}^\mathcal{G}$ Distribution and its Properties

This subsection states the Mixture Gaussian and Normal Inverse Gaussian Distribution and introduces it from it components. Subsequently, this distribution is formalised by stating its definition and followed by it properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Mixture Gaussian and Normal Inverse Gaussian Distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy
Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.

**Mixture Case 6.5 (Mixture: Gaussian & Normal Inverse Gaussian Distribution \( \mathbb{M}_{\text{SG\,NIG}} \))**

Let \( F_{Y_{SG}} \) be a Standard Gaussian Distribution that admits Definition 6.3 and \( F_{W_{NIG}} \) be a Normal Inverse Gaussian Distribution that admits Definition 6.15, where they are independent from each other, with \( p \in (0,1) \) as the probability of occurrence. Then the Mixture Standard Gaussian and Normal Inverse Gaussian is structured by \( p \), and denoted by \( X_{\mathbb{M}_{\text{SG\,NIG}}(\alpha,\beta,\delta,\mu,p)} \).

**Definition 6.16 (Mixture: Gaussian & Normal Inverse Gaussian Distribution \( \mathbb{M}_{\text{SG\,NIG}} \))**

A random variable \( X \) is said to be Mixture Standard Gaussian and Normal Inverse Gaussian Distribution, denoted by \( X_{\mathbb{M}_{\text{SG\,NIG}}(\alpha,\beta,\delta,\mu,p)} \), with \( |\beta| \in ]0,\alpha[ \), \( K_\lambda(\cdot) \) as the modified Bessel function of the third kind of order 1 that is given in Definition 6.6, and \( p \in (0,1) \), if its density is given by the subsequent equality:

\[
f_X_{\mathbb{M}_{\text{SG\,NIG}}(\alpha,\beta,\delta,\mu,p)}(x) = \frac{p}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + (1-p) \frac{\delta\alpha e^{(\delta\gamma+\beta(x-u))}}{\pi \sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)
\]

**Property D6.16.1 (\( \mathbb{M}_{\text{SG\,NIG}} \): Inheritance)**

Since \( \mathbb{M}_{\text{SG\,NIG}}(\alpha,\beta,\delta,\mu,p) \) is a mixture of independent Standard Gaussian random variable and Normal Inverse Gaussian random variable those follow, respectively, Definition 6.3 and Definition 6.15. Then each of them inherits its corresponding distribution properties.
6.4.8.2 $\mathbb{M}_{N\mathcal{IG}}^{SG}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Mixture Gaussian and Normal Inverse Gaussian Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and Normal Inverse Gaussian Factor Copula Model was proposed in (Yang et al., 2009).

By admitting Assumption 5.2 and the $\mathbb{M}_{N\mathcal{IG}}^{SG}$ Distribution definition, $(\mathcal{M}_{\mathcal{M}_{NG}}^{t}, \mathcal{J}_{\mathcal{M}_{NG}}^{t}, \mathcal{X}_{\mathcal{M}_{NG}}^{t})$, and the $\mathbb{M}_{N\mathcal{IG}}^{SG}$ by the following Lemma.

**Lemma 6.15 (\(\mathcal{M}_{\mathcal{M}_{NG}}^{t}, \mathcal{J}_{\mathcal{M}_{NG}}^{t}, \mathcal{X}_{\mathcal{M}_{NG}}^{t}\) with \(\mathbb{M}_{N\mathcal{IG}}^{SG}\) Distribution Functions)**

Let \(\mathcal{X}_{\mathcal{M}_{NG}}^{t}\) be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.5 as a $\mathbb{M}_{N\mathcal{IG}}^{SG}$ that admits Definition 6.16, with $\mathcal{M}_{\mathcal{M}_{NG}}^{t}$, and $\mathcal{J}_{\mathcal{M}_{NG}}^{t}$ as, respectively, the $\mathbb{M}_{N\mathcal{IG}}^{SG}$ systematic market risk factor, and the $\mathbb{M}_{N\mathcal{IG}}^{SG}$ idiosyncratic risk.
factor. Then for $\mathcal{M}^{t, SG}_{N^3 g}$, $\mathcal{S}^{t, SG}_{N^3 g}$, and $\mathcal{X}^{t, SG}_{N^3 g}$ to admits Assumption 5.2, their parameters has to be set as following:

i. $\mathcal{M}^{t, SG}_{N^3 g} \left( \alpha, \beta \frac{\omega \pi}{\alpha \pi^2} \frac{\beta \omega}{\alpha \pi} \right)$

ii. $\mathcal{S}^{t, SG}_{N^3 g} \left( \left( \frac{1 - \rho^2}{\rho_1} \right), \alpha \left( \frac{1 - \rho^2}{\rho_1} \right), \beta \left( \frac{1 - \rho^2}{\rho_1} \right) \frac{\omega^2}{\alpha \pi^2}, \frac{\beta \omega}{\alpha \pi} \right)$

iii. $\mathcal{X}^{t, SG}_{N^3 g} \left( \frac{1}{\rho_2}, \beta \left( \frac{1}{\rho_1} \right) \frac{\omega^2}{\alpha \pi^2}, \frac{\beta \omega}{\alpha \pi} \right)$

where $\omega$, $\bar{\omega}$, $\alpha$, and $\beta$ are related to $\mathcal{M}$

**Proof:** (Similar to Lemma 6.7)

**Corollary 6.41 ($\mathcal{M}^{SG}_{N^3 g}$ Factor Copula Model)**

Let $(\mathcal{X}^{t, SG}_{N^3 g})_{t \in \mathbb{R}^+}$, $\mathcal{M}^{t, SG}_{N^3 g}$, and $\mathcal{S}^{t, SG}_{N^3 g}$ be, respectively, a Lévy process specialised as a $\mathcal{M}^{SG}_{N^3 g}$, the $\mathcal{M}^{SG}_{N^3 g}$ systematic market risk factor, and the $\mathcal{M}^{SG}_{N^3 g}$ idiosyncratic risk factor those follow Lemma 6.15, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $\xi_i^{t, SG}_{N^3 g}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}^{t, SG}_{N^3 g}$. Then $\xi_i^{t, SG}_{N^3 g}$ of the $\mathcal{M}^{SG}_{N^3 g}$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i^{t, SG}_{N^3 g}} = F_{\mathcal{X}^{t, SG}_{N^3 g}} \left( \frac{F_{\xi_i^{t, SG}_{N^3 g}} \left( F_{\xi_i^{t, SG}_{N^3 g}}(t) \right) - \rho_i \mathcal{M}^{t, SG}_{N^3 g}}{\sqrt{1 - \rho_i^2}} \right)$$
6.4.8.3 Binary Stochastic Correlated $M_{N}^{Sg}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Mixture Gaussian and Normal Inverse Gaussian Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and Normal Inverse Gaussian Binary Stochastic Correlated Factor Copula Model was proposed in (Yang et al., 2009).

**Remark 6.28 ($M_{N}^{Sg}$, $J_{N}^{i}$, and $X_{N}^{t}$ with $M_{N}^{Sg}$ Distribution Functions)**

In this subsection, the parameters of $\left( X_{N}^{t} \right)_{t \in \mathbb{R}^{+}}$, $M_{N}^{Sg}$, and $J_{N}^{i}$ are admitting Lemma 6.15 but structured as in Lemma 5.8 instead of Corollary 5.11.

**Corollary 6.42 (Binary Stochastic Correlated $M_{N}^{Sg}$ Factor Copula Model)**

Let $\left( X_{N}^{t} \right)_{t \in \mathbb{R}^{+}}$, $M_{N}^{Sg}$, and $J_{N}^{i}$ be, respectively, a Lévy process specialised as a $M_{N}^{Sg}$, the $M_{N}^{Sg}$ systematic market risk factor, and the $M_{N}^{Sg}$ idiosyncratic risk factor
those follow Lemma 6.15 and structured by the Binary stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit Assumption 4.12 and Definition 4.15, and \( p_{t_i}^{(\mathcal{M}^{t, SG}_{N^3 G})} \) be the probability of \( \tau_i \) conditioned upon \( \mathcal{M}_t^{t, SG}_{N^3 G} \). Then \( p_{t_i}^{(\mathcal{M}^{t, SG}_{N^3 G})} \) of the Binary Stochastic Correlated \( \mathcal{M}^{t, SG}_{N^3 G} \) Factor Copula Model is given by the subsequent equality:

\[
\xi_i \bigg| \mathcal{M}_t^{t, SG}_{N^3 G} = (1 - q)F_{t_i}^{t, SG}_{N^3 G} \left( \frac{F^{-1}_{t_i} \left( F_{t_i}(t) - \rho_1 \mathcal{M}_t^{t, SG}_{N^3 G} \right)}{\sqrt{1 - \rho_1^2}} \right) + qF_{t_i}^{t, SG}_{N^3 G} \left( \frac{F^{-1}_{t_i} \left( F_{t_i}(t) - \rho_2 \mathcal{M}_t^{t, SG}_{N^3 G} \right)}{\sqrt{1 - \rho_2^2}} \right)
\]

### 6.4.8.4 Symmetric Stochastic Correlated \( \mathcal{M}^{SG}_{N^3 G} \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Gaussian and Normal Inverse Gaussian Distribution in the proposed Lévy
Symmetric Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.2. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Gaussian and Normal Inverse Gaussian Symmetric Stochastic Correlated Factor Copula Model was proposed in (Yang et al., 2009).

**Remark 6.29 (M⁰_{\mathcal{M}^SG}, \mathcal{J}^t_{\mathcal{M}^SG}, \text{ and } \mathcal{X}^{t_i}_{\mathcal{M}^SG} \text{ with } \mathcal{M}^SG_N Distribution Functions)**

In this subsection, the parameters of \(\mathcal{X}^{t_i}_{\mathcal{M}^SG} \) and \(\mathcal{M}^t_{\mathcal{M}^SG} \), \(\mathcal{J}^{t_i}_{\mathcal{M}^SG} \) are admitting Lemma 6.15 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.43 (Symmetric Stochastic Correlated \(\mathcal{M}^SG_N \) Factor Copula Model)**

Let \(\mathcal{X}^{t_i}_{\mathcal{M}^SG_N}, \mathcal{M}^t_{\mathcal{M}^SG_N}, \text{ and } \mathcal{J}^{t_i}_{\mathcal{M}^SG_N} \) be, respectively, a Lévy process specialised as a \(\mathcal{M}^SG_N \), the \(\mathcal{M}^SG_N \) systematic market risk factor, and the \(\mathcal{M}^SG_N \) idiosyncratic risk factor those follow Lemma 6.15 and structured by the symmetric stochastic correlation, \(\xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \(\tau_i \) admit Assumption 4.12 and Definition 4.15, and \(\mathcal{P}_{\tau_i}^{\mathcal{M}^t_{\mathcal{M}^SG_N}} \) be the probability of \(\tau_i \) conditioned upon \(\mathcal{M}^t_{\mathcal{M}^SG_N} \). Then \(\mathcal{P}_{\tau_i}^{\mathcal{M}^t_{\mathcal{M}^SG_N}} \) of the Symmetric Stochastic Correlated \(\mathcal{M}^SG_N \) Factor Copula Model is given by the subsequent equality:
The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Gaussian and Normal Inverse Gaussian Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Mixture Gaussian and Normal Inverse Gaussian Random Factor Loading Copula Model was proposed in (Yang et al., 2009).

In order to set the distributions, as in Lemma 6.15, the distributions will be broken on the two parts of this model, where $\mathcal{M}_t < \kappa$ or $\mathcal{M}_t \geq \kappa$. The following Lemma will summarise this point.
Lemma 6.16 ($\mathcal{M}_N^N$, $\mathcal{J}_N^N$, and $\mathcal{X}_N^N$ with $\mathcal{M}_N^N$ Distribution Functions)

Let $\left(\mathcal{X}_N^N\right)_{t\in \mathbb{R}^+}$ be a Lévy process that follows Lemma 5.10 and specialised upon Mixture Case 6.5 as a $\mathcal{M}_N^N$ that admits Definition 6.16, with $\mathcal{M}_N^N$, $\mathcal{J}_N^N$, and $\mathcal{X}_N^N$ as, respectively, the $\mathcal{M}_N^N$ systematic market risk factor, and the $\mathcal{M}_N^N$ idiosyncratic risk factor. Then for $\mathcal{M}_N^N$, $\mathcal{M}_N^N$, $\mathcal{J}_N^N$, $\mathcal{J}_N^N$, $\mathcal{X}_N^N$, and $\mathcal{X}_N^N$ to admits Assumption 5.2, their parameters has to be set as following:

i. $\mathcal{M}_N^N \left(\alpha, \beta \frac{3}{\sigma^2}, \frac{\beta \omega}{\sigma^2} \right)$

ii. $\mathcal{M}_N^N \left(\frac{1}{r_1} \frac{1}{r_1}, \alpha \frac{1}{r_2} \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

iii. $\mathcal{M}_N^N \left(\frac{1}{r_1} \frac{1}{r_1}, \alpha \frac{1}{r_2} \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

iv. $\mathcal{J}_N^N \left(\frac{1}{r_1} \frac{1}{r_1}, \alpha \frac{1}{r_2} \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

v. $\mathcal{J}_N^N \left(\frac{1}{r_1} \frac{1}{r_1}, \alpha \frac{1}{r_2} \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

vi. $\mathcal{X}_N^N \left(\frac{1}{r_1} \frac{1}{r_1}, \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

vii. $\mathcal{X}_N^N \left(\frac{1}{r_2} \frac{1}{r_2}, \beta \frac{3}{\sigma^2} \frac{\beta \omega}{\sigma^2} \right)$

where $\omega, \sigma, \alpha, \beta$ are related to $\mathcal{M}$.
Corollary 6.44 (\(M_{N/2}^{SG}\) Random Factor Loading Copula Model)

Let \(\varphi_{N_t}\) be the unconditional number of default’s characteristic function that follows Theorem 5.5, \(X_{t_i}^{t_i} \equiv M_{t_i}^{SG},\) and \(J_{t_i}^{t_i}\) be, respectively, a Lévy process specialised as a \(M_{N/2}^{SG}\), the \(M_{N/2}^{SG}\) systematic market risk factor, and the \(M_{N/2}^{SG}\) idiosyncratic risk factor those follow Lemma 6.16 and structured by the random factor loading. Then \(\varphi_{N_t}\) of the \(M_{N/2}^{SG}\) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi_{N_t}^{M_{N/2}^{SG}}(u) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(1 - (1 - e^{iu}) \right) F_{J_{t_i}^{t_i}}^{t_i} \left( \frac{F_{X_{t_i}^{t_i}}^{t_i} (F_{t_i} (t)) - \kappa_i - \ell_1 m}{\sqrt{1 - \ell_1^2}} \right) f_{M_t}^{M_{N/2}^{SG}} (m) dm + \int_{\infty}^{\infty} \prod_{i=1}^{n} \left(1 - (1 - e^{iu}) \right) F_{J_{t_i}^{t_i}}^{t_i} \left( \frac{F_{X_{t_i}^{t_i}}^{t_i} (F_{t_i} (t)) - \kappa_i - \ell_2 m}{\sqrt{1 - \ell_2^2}} \right) f_{M_t}^{M_{N/2}^{SG}} (m) dm
\]

Where \(\kappa_i = -\ell_1 \int_{-\infty}^{\infty} m f_{M_t}^{M_{N/2}^{SG}} \left( m \right) dm - \ell_2 \int_{\infty}^{\infty} m f_{M_t}^{M_{N/2}^{SG}} \left( m \right) dm.

6.4.9 A Mixture Fractional-\(t\) and Normal Inverse Gaussian Factor Copula

The Gaussian Factor Copula model, which was introduced in (Li, 2000), became the market’s standard model although it has various well-known drawbacks, for instance, it requires more tail dependence and thus it does not fit the market quotes. Then, various authors have proposed different approaches to bring more tail dependence into the model. Therefore, as mentioned in Chapter 5, there were two directions to integrate extra tail dependence into the model and overcome this problem.

As in the first direction many authors tried to overcome this drawback by extending the Gaussian Copula model by skewing its correlation by replacing the Gaussian
distribution by another distribution that contains more skewness. Conversely, the other direction of incorporating extra tail dependence into the Gaussian Factor model and overcoming its limitation was to stochastic its correlation or to stochastic its risk exposure by loading its factor.

This subsection inherits Chapter 5’s proposed models, i.e. “Lévy Factor Copula Model”, “Binary Stochastic Correlated Lévy Factor Copula Model”, “Symmetric Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model”, and apply the Mixture Fractional-\(t\) and Normal Inverse Gaussian Distribution that admits the Lévy process.

This subsection starts by stating the Mixture Fractional-\(t\) and Normal Inverse Gaussian Distribution and its properties and subsequently applying it to the proposed models.

**6.4.9.1 \(\mathbb{M}^N_{\mathcal{F}t} \text{ Distribution and its Properties}**

This subsection states the Mixture Fractional- \(t\) and Normal Inverse Gaussian Distribution and introduces it from its components. Subsequently, this distribution is formalised by stating its definition and followed by it properties.

The definition and its properties are essential when choosing the distribution parameters. Since the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance. This subsection is critical when applying the Mixture Fractional-\(t\) and Normal Inverse Gaussian distribution to the Lévy Factor Copula Model, the Binary Stochastic Correlated Lévy Factor Copula Model, the Symmetric Stochastic Correlated Lévy Factor Copula Model and the Lévy Random Factor Loading Copula Model.
Mixture Case 6.6 (Mixture: Normalised Fractional-\(t\) & Normal Inverse Gaussian Distribution \(\mathbb{M}^{{NFT}}_{N3g}\))

Let \(F_{y_{NFT}}\) be a Normalised Fractional-\(t\) Distribution that admits Definition 6.10 and \(F_{W_{N3g}}\) be a Normal Inverse Gaussian Distribution that admits Definition 6.15, where they are independent from each other, with \(p \in (0,1)\) as the probability of occurrence.

Then the Mixture Normalised Fractional-\(t\) and Normal Inverse Gaussian is structured by \(p\), and denoted by \(X_{\mathbb{M}^{{NFT}}_{N3g}(\nu,\alpha,\beta,\delta,\mu,p)}\).

Definition 6.17 (Mixture: Normalised Fractional-\(t\) & Normal Inverse Gaussian Distribution \(\mathbb{M}^{{NFT}}_{N3g}\))

A random variable \(X\) is said to be Mixture Normalised Fractional-\(t\) and Normal Inverse Gaussian Distribution, denoted by \(X_{\mathbb{M}^{{NFT}}_{N3g}(\nu,\alpha,\beta,\delta,\mu,p)}\), with \(|\beta| \in [0, \alpha]\), \(K_{\lambda}(\cdot)\) as the modified Bessel function of the third kind of order 1 that is given in Definition 6.6, \(\nu \in \mathbb{R}^+\) as the fractional degree of freedom, \(\Gamma(x)\) as the Gamma Function, and \(p \in (0,1)\). If its density is given by the subsequent equality:

\[
 f_{X_{\mathbb{M}^{{NFT}}_{N3g}(\nu,\alpha,\beta,\delta,\mu,p)}}(x) = p \left( \frac{\sqrt{\nu}}{\sqrt{v-2}} \right) \frac{\Gamma\left(\frac{1}{2}v+1\right)}{\Gamma\left(\frac{1}{2}v\right)} \left[ 1 + \frac{x^2}{v} \right]^{\left(\frac{v}{2}-1\right)} + (1-p) \frac{\delta \alpha e^{(\delta \gamma+\beta(x-u))}}{\pi \sqrt{\delta^2 + (x-\mu)^2}} K_1 \left( \alpha \sqrt{\delta^2 + (x-\mu)^2} \right)
\]

Property D6.17.1 (\(\mathbb{M}^{{NFT}}_{N3g}\): Inheritance)

Since \(\mathbb{M}^{{NFT}}_{N3g}(\nu,\alpha,\beta,\delta,\mu,p)\) is a mixture of independent Normalised Fractional-\(t\) random variable and Normal Inverse Gaussian random variable those follow, respectively, Definition 6.10 and Definition 6.15. Then each of them inherits its corresponding distribution properties.
6.4.9.2 $\mathcal{M}_{N}^{\mathcal{C}}$ Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the model by replacing the Gaussian distribution by other distributions those contain more skewness feature is essential. This could be achieved by replacing the Lévy process of the proposed model, i.e. the Lévy Factor Copula Model, with a distribution that admits it; the Systematic Market Risk Factor and Idiosyncratic Risk Factors distributions’ must admits Lévy process definition and its properties and being be infinitely divisible distributed and have zero with equal finite variance.

An immediate result could be achieved when applying the Mixture Fractional-$t$ and Normal Inverse Gaussian Distribution to the Lévy Factor Copula Model that was articulated in Subsection 5.6. This model overcomes, partially, the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Fractional-$t$ and Normal Inverse Gaussian Factor Copula Model is introduced as a proposed model.

By admitting Assumption 5.2 and the $\mathcal{M}_{N}^{\mathcal{C}}$ Distribution, $\mathcal{M}_{N}^{\mathcal{C}}$, $\mathcal{J}_{N}^{\mathcal{C}}$, and $\mathcal{X}_{N}^{\mathcal{C}}$ are given by the following Lemma.

**Lemma 6.17 ($\mathcal{M}_{N}^{\mathcal{C}}$, $\mathcal{J}_{N}^{\mathcal{C}}$, and $\mathcal{X}_{N}^{\mathcal{C}}$ with $\mathcal{M}_{N}^{\mathcal{C}}$ Distribution Functions)**

Let $\left(\mathcal{X}_{N}^{\mathcal{C}}\right)$ be a Lévy process that follows Corollary 5.11 and specialised upon Mixture Case 6.6 as a $\mathcal{M}_{N}^{\mathcal{C}}$ that admits Definition 6.17, with $\mathcal{M}_{N}^{\mathcal{C}}$, $\mathcal{J}_{N}^{\mathcal{C}}$, and $\mathcal{X}_{N}^{\mathcal{C}}$ as, respectively, the $\mathcal{M}_{N}^{\mathcal{C}}$ systematic market risk factor and the $\mathcal{M}_{N}^{\mathcal{C}}$ idiosyncratic risk.
factor. Then for $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$, $\mathcal{D}_{\mathcal{N}_{\mathcal{F}_t}}$, and $\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}}$ to admits Assumption 5.2, their parameters has to be set as following:

i. $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}(v, \alpha, \beta, \frac{\omega^2}{\alpha^2} - \frac{\beta \omega}{\alpha^2} p)$

ii. $\mathcal{D}_{\mathcal{N}_{\mathcal{F}_t}}(v, \left(\frac{1 - \rho_i^2}{\rho_i}\right) \alpha, \left(\frac{1 - \rho_i^2}{\rho_i}\right) \beta, \frac{3}{\alpha^2} - \left(\frac{1 - \rho_i^2}{\rho_i}\right) \frac{\beta \omega}{\alpha^2} p)$

iii. $\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}} = p \mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}} + (1 - p) \mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}}$, where $\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}}$ as in Lemma 6.5, and

$\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}} \left(\frac{1}{\rho_i} \alpha, \frac{1}{\rho_i} \beta, \frac{3}{\alpha^2} - \frac{1}{\rho_i} \frac{\beta \omega}{\alpha^2}\right)$, and $p \in (0, 1)$.

where $\omega$, $\sigma$, $\alpha$, and $\beta$ are related to $\mathcal{M}$

**Proof:** (Similar to Lemma 6.4 and 6.7)

**Corollary 6.45 ($\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$ Factor Copula Model)**

Let $\left(\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}}\right)_{t \in \mathbb{R}^+}$, $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$, and $\mathcal{D}_{\mathcal{N}_{\mathcal{F}_t}}$ be, respectively, a Lévy process specialised as a $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$, the $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$ systematic market risk factor, and the $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$ idiosyncratic risk factor those follow Lemma 6.17, $\xi_i$ be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time $\tau_i$ admit Assumption 4.12 and Definition 4.15, and $p_{\xi_i}$ be the probability of $\tau_i$ conditioned upon $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$. Then $p_{\xi_i}$ of the $\mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}$ Factor Copula Model is given by the subsequent equality:

$$p_{\xi_i} = F_{\mathcal{D}_{\mathcal{N}_{\mathcal{F}_t}}} \left(\frac{F_{\mathcal{X}_{\mathcal{N}_{\mathcal{F}_t}}}(t) - \rho_i \mathcal{M}_{\mathcal{N}_{\mathcal{F}_t}}}{\sqrt{1 - \rho_i^2}}\right)$$
6.4.9.3 Binary Stochastic Correlated \( M_{NFG}^{NT} \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a direct result could be obtained when injecting the Mixture Fractional-\( t \) and Normal Inverse Gaussian Distribution in the proposed Lévy Binary Stochastic Correlated Factor Copula Model that was articulated in Subsection 5.7.1. This model overcomes the limitation of the standard model; it contains more tail dependence and thus it is proposed as better alternative. The Mixture Fractional-\( t \) and Normal Inverse Gaussian Binary Stochastic Correlated Factor Copula Model is introduced as a proposed model.

Remark 6.30 (\( M_{NFG}^{NT} \), \( J_{NFG}^{NT} \), and \( X_{NFG}^{NT} \) with \( M_{NFG}^{NT} \) Distribution Functions)

In this subsection, the parameters of \( \left( X_{NFG}^{NT}, M_{NFG}^{NT}, J_{NFG}^{NT} \right) \) are admitting Lemma 6.17 but structured as in Lemma 5.8 instead of Corollary 5.11.

Corollary 6.46 (Binary Stochastic Correlated \( M_{NFG}^{NT} \) Factor Copula Model)

Let \( \left( X_{NFG}^{NT}, M_{NFG}^{NT}, J_{NFG}^{NT} \right) \) be, respectively, a Lévy process specialised as a \( M_{NFG}^{NT} \), the \( M_{NFG}^{NT} \) systematic market risk factor, and the \( M_{NFG}^{NT} \) idiosyncratic risk
factor those follow Lemma 6.17 and structured by the Binary stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit Assumption 4.12 and Definition 4.15, and \( p_{t_i} \) be the probability of \( \tau_i \) conditioned upon \( \mathcal{M}^{t}_{\mathcal{W}_{N3g}} \). Then \( p_{t_i} \) of the Binary Stochastic Correlated \( \mathcal{M}^{t}_{\mathcal{W}_{N3g}} \) Factor Copula Model is given by the subsequent equality:

\[
\xi_i \bigg| \mathcal{M}^{t}_{\mathcal{W}_{N3g}} = (1 - q)F_{\mathcal{M}^{t}_{\mathcal{W}_{N3g}}}^{-1} \left( F_{t_i}(t) - \rho_1 \mathcal{M}^{t}_{\mathcal{W}_{N3g}} \right) \frac{1}{\sqrt{1 - \rho_1^2}} + qF_{\mathcal{M}^{t}_{\mathcal{W}_{N3g}}}^{-1} \left( F_{t_i}(t) - \rho_2 \mathcal{M}^{t}_{\mathcal{W}_{N3g}} \right) \frac{1}{\sqrt{1 - \rho_2^2}}
\]

6.4.9.4 Symmetric Stochastic Correlated \( \mathcal{M}^{t}_{\mathcal{W}_{N3g}} \) Factor Copula Model

As stated previously, the market standard Model “Gaussian Factor Copula Model”, which was introduced in (Li, 2000), is unqualified to fit the market tranches, where it over-prices the mezzanine and under-prices the equity and senior. Thus, skewing the models’ correlation by a stochastic correlation is necessary. Gaussian Factor Copula model is extended to incorporate stochastic correlation in (Burtschell et al., 2005), where it was updated in (Burtschell et al., 2009), and (Schloegl, 2005). In (Yang et al., 2009) the Gaussian Stochastic Correlated Factor Copula model has been extended by replacing the Gaussian Distributions by a mixture of a Gaussian and Normal Inverse Distributions.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Fractional-\( t \) and Normal Inverse Gaussian Distribution in the proposed Lévy Symmetric Stochastic Correlated Factor Copula Model that was articulated in
Subsection 5.7.2. The Mixture Fractional-\( \tau \) and Normal Inverse Gaussian Symmetric Stochastic Correlated Factor Copula Model overcomes, partially, the limitation of the standard model, and thus it is introduced as a proposed model.

**Remark 6.31** \( (\mathcal{M}^{t}_{N^{PF}} , \mathcal{J}^{t}_{N^{PF}}, \text{ and } \mathcal{X}^{t}_{N^{PF}} ) \) with \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \) Distribution Functions

In this subsection, the parameters of \( (\mathcal{X}^{t}_{N^{PF}_{\mathcal{N}_{3}}}) \), \( \mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \), and \( \mathcal{J}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \) are admitting Lemma 6.17 but structured as in Lemma 5.9 instead of Corollary 5.11.

**Corollary 6.47** (Symmetric Stochastic Correlated \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \) Factor Copula Model)

Let \( (\mathcal{X}^{t}_{N^{PF}_{\mathcal{N}_{3}}}) \), \( \mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \), and \( \mathcal{J}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \) be, respectively, a Lévy process specialised as a \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \), the \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \) systematic market risk factor, and the \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \) idiosyncratic risk factor those follow Lemma 6.17 and structured by the symmetric stochastic correlation, \( \xi_i \) be as supposed in Assumption 4.14 and admits Definition 4.20, the random default time \( \tau_i \) admit Assumption 4.12 and Definition 4.15, and \( p_{\tau_i} \) be the probability of \( \tau_i \) conditioned upon \( \mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \). Then \( p_{\tau_i} \) of the Symmetric Stochastic Correlated \( \mathcal{M}^{N^{PF}_{\mathcal{N}_{3}}} \) Factor Copula Model is given by the subsequent equality:

\[
p_{\tau_i} \bigg|_{\mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}}} = \hat{q} F_{\mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}}} \left( F_{\chi_{N^{PF}_{\mathcal{N}_{3}}}}^{-1} \left( F_{\xi_i}(t) \right) \right) + (1 - \hat{q}) \left[ (1 - q) F_{\mathcal{J}^{t}_{N^{PF}_{\mathcal{N}_{3}}}} \left( F_{\chi_{N^{PF}_{\mathcal{N}_{3}}}}^{-1} \left( F_{\xi_i}(t) - \rho \mathcal{M}^{t}_{N^{PF}_{\mathcal{N}_{3}}} \right) \right) \right] \\
+ q F_{\mathcal{J}^{t}_{N^{PF}_{\mathcal{N}_{3}}}} \left( F_{\chi_{N^{PF}_{\mathcal{N}_{3}}}}^{-1} \left( F_{\xi_i}(t) \right) \right)
\]
Random Factor Loading Copula Model

The Gaussian Random Loading Factor Model proposed by Andersen & Sidenius [2005] has, partially, overcome dissimilarity between the markets’ and based Gaussian Factor Models’ loss distributions, where the prior curve is curved more than the latter. Also, the problem of generating zero losses is prevented. Accordingly, tracking, empirically, the markets’ direction could be done through this implementation method, where the credit entities’ correlation could be set to be higher in bull markets than bearish ones.

In this subsection a straightforward consequence could be achieved once inserting the Mixture Fractional-$\tau$ and Normal Inverse Gaussian Distribution in the proposed Lévy Random Factor Loading Copula Model that was articulated in Subsection 5.8. This model is proposed as better alternative than the based model. The Mixture Fractional-$\tau$ and Normal Inverse Gaussian Random Factor Loading Copula Model is introduced as a proposed model.

In order to set the distributions, as in Lemma 6.17, the distributions will be broken on the two parts of this model, where $M_t < \kappa$ or $M_t \geq \kappa$. The following Lemma will summarise this point.

**Lemma 6.18 (\(M^N_{\mathbb{N}^g} \), \(J^N_{\mathbb{N}^g} \), and \(X^N_{\mathbb{N}^g} \) with \(\mathbb{M}^N_{\mathbb{N}^g} \) Distribution Functions)**

Let \(X^t_{\mathbb{M}^N_{\mathbb{N}^g}}\) be a Lévy process that follows Lemma 5.10 and Theorem 5.5 and specialised upon Mixture Case 6.6 as a \(M^N_{\mathbb{N}^g} \) that admits Definition 6.17, with \(M^t_{\mathbb{M}^N_{\mathbb{N}^g}} \)
and \(J^t_{\mathbb{M}^N_{\mathbb{N}^g}} \) as, respectively, the \(M^N_{\mathbb{N}^g} \) systematic market risk factor, and the \(M^N_{\mathbb{N}^g} \)
idosyncratic risk factor. Then for \(M^t_{\mathbb{M}^N_{\mathbb{N}^g}} \), \(M^t_{\mathbb{M}^N_{\mathbb{N}^g}^1} \), \(M^t_{\mathbb{M}^N_{\mathbb{N}^g}^2} \),
and \(\mathbb{M}^N_{\mathbb{N}^g} \) to admits Assumption 5.2, their parameters has to be set as following:
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

Chapter Six: Lévy Factor Copula and its Skewed Version from Theory to Application

Page 256

1. \( M^t_\mathbb{N} \left( \nu, \alpha, \beta \frac{\omega}{\sigma^2}, \frac{\beta \omega}{\sigma^2}, \rho \right) \)

2. \( M^t_\mathbb{N} \left( \nu, \alpha, \beta \frac{\omega}{\sigma^2}, \frac{\beta \omega}{\sigma^2}, \rho \right) \)

3. \( M^t_\mathbb{N} \left( \nu, \alpha, \beta \frac{\omega}{\sigma^2}, \frac{\beta \omega}{\sigma^2}, \rho \right) \)

4. \( J^t_\mathbb{N} \left( \nu, \alpha, \beta \frac{\omega}{\sigma^2}, \frac{\beta \omega}{\sigma^2}, \rho \right) \)

5. \( J^t_\mathbb{N} \left( \nu, \alpha, \beta \frac{\omega}{\sigma^2}, \frac{\beta \omega}{\sigma^2}, \rho \right) \)

6. \( X^t_\mathbb{N} = p X^t_\mathbb{N} + (1 - p) X^t_\mathbb{N} \), where \( X^t_\mathbb{N} \) as in Lemma 6.5, and \( p \in (0,1) \).

7. \( X^t_\mathbb{N} = p X^t_\mathbb{N} + (1 - p) X^t_\mathbb{N} \), where \( X^t_\mathbb{N} \) as in Lemma 6.5, and \( p \in (0,1) \).

where \( \omega, \sigma, \alpha, \beta \) are related to \( M \)

**Proof:** (Similar to Lemma 6.4 and 6.7)

**Corollary 6.48** (\( M^t_\mathbb{N} \) Random Factor Loading Copula Model)

Let \( \psi_{N_t} \) be the unconditional number of default’s characteristic function that follows

**Theorem 5.5,** \( X^t_\mathbb{N} \) \( \in \mathbb{R}^+ \), \( M^t_\mathbb{N} \), and \( J^t_\mathbb{N} \) be, respectively, a Lévy process specialised as a \( M^t_\mathbb{N} \), the \( M^t_\mathbb{N} \) systematic market risk factor, and the \( M^t_\mathbb{N} \) idiosyncratic risk factor those follow Lemma 6.18 and structured by the random factor.
loading. Then \( \varphi_{N_t} \) of the \( M^{N,F}_N \) Random Factor Loading Copula Model is given by the subsequent equality:

\[
\varphi_{N_t}(u) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{x_i}^{-1} \left( \frac{F_{x_i}(t_i) - \kappa_i - \ell_1 m}{\sqrt{1 - \ell_1^2}} \right) \right) \varphi_{M}^t(m) dm + \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) F_{y_i}^{-1} \left( \frac{F_{y_i}(t_i) - \kappa_i - \ell_2 m}{\sqrt{1 - \ell_2^2}} \right) \right) \varphi_{M}^t(m) dm
\]

Where \( \kappa_i = -\ell_1 \int_{-\infty}^{\infty} \varphi_{M}^t(m) dm - \ell_2 \int_{-\infty}^{\infty} \varphi_{M}^t(m) dm \).

### 6.5 Summary

In this chapter, the theory of Lévy Factor Copula models and its skewed versions, which were proposed in chapter 5, have been implemented by many limiting and mixture cases of the Lévy Skew Alpha-Stable Distribution and the Generalized Hyperbolic Distribution those admits the Lévy process. Most of these skewed versions are proposed as new dynamic alternative frameworks those could capture the actual credit risk derivatives quotes. The proposed models are presented in Table 6.1.
Chapter Seven

Pricing m\textsuperscript{th} to Default and Ranked m out of n of the Basket Default Swap and Collateralised Debt Obligation

7.1 Outline

- Introduction
- Pricing Homogeneous m\textsuperscript{th} to Default CDS
- Pricing Homogeneous Ranked m out of n CDS
- Pricing Non-Homogeneous Ranked m out of n CDS
- Pricing CDO
- Mathematical Summery
7.2 Introduction

Numerous exotic CR derivatives are structured upon the standard Credit Derivatives Swap (CDS). A CDS is a contract that its payoff is valued upon the default losses of a credit entity in response for an agreed premium. Principally, the CDS is a bi-contract, i.e. the buyer and seller of the protection. A premium or spread is paid periodically by the protection buyer as an insurance payment until the maturity of the contract or default event occurs. In case of default, the protection seller compensates the fractional loss or called the unrecovered fraction of the credit entity’s value times the CDS notional amount. To clarify it more, the protection seller will not make any payment if there is no default. Payments flow made from the protection buyer to the protection seller is called the Protection Leg or “Premium Leg”. In contrary, the payments flow from the protection seller to the protection buyer is called the “Default Leg”.

In the case of default, the settlement of the CDS contract could be either a cash settlement or a physical settlement. In the former, the protection seller will make an immediate cash payment to the protection buyer at the time of default. In contrast, when a default event occurs in the case of the physical settlement, the protection buyer transfers the credit entity to the protection seller in return for the CDS National or equally for the notional of the credit entity. Subsequently, the protection seller can directly sell the credit entity in return for its value after default. The credit entity’s market value after default is equal to its recovered fraction’s value times the CDS notional amount. Again the protection seller’s loss is equal to the unrecovered fraction of the credit entity’s value times the CDS notional amount, which is equal to the cash settlement. As a consequence of their resemblance, a cash settlement always will be assumed in the subsequent when modelling a CDS contract. Note that the same assumption will hold in the \( n^{th} \) to default CDS, \( \left( \frac{n}{m} \right)^{th} \) to default CDS, an CDO.
Oppositely to cash settlement promised by the protection seller, the protection buyer pays a periodic premium, called also “\(\text{CDS Spread}\)”. This periodic premium is fixed at the start of the \(\text{CDS}\) contract. Subsequent to the credit entity’s default, the protection buyer is obligate to pay the fractional amount of the premium payment that has accrued since the preceding periodic payment to the credit entity default, which is called the accrued premium or the “Accrued Leg”. When the credit entity default, the protection buyer payment stream “Premium Leg” will be terminated with the fractional payment “Accrued Leg”.

As declared previously, the periodic premiums are fixed at the start of the \(\text{CDS}\) contract, but it is done so that the expected value of the \(\text{CDS}\) contract is equal for both the protection seller and the protection buyer. This is achieved by equalising the expectation value of the premium leg to the expectation value of the default leg. The periodic premiums are characteristically made once, twice, or four time a year.

Various credit derivatives products have been described intuitively and illustrated with some cash flow examples in chapter two. This chapter scrutinises these credit derivatives products from a modelling point of view. To be precise, in this chapter the pricing of the homogenous \(m^{th}\) to default \(\text{CDS}\), the homogenous \(\left(\frac{n}{m\cap R}\right)^{th}\) to default basket \(\text{CDS}\), the non-homogenous \(\left(\frac{n}{m\cap R}\right)^{th}\) to default basket \(\text{CDS}\), and the \(\text{CDO’s}\) are illustrated mathematically.

### 7.3 Pricing of the Homogenous \(m^{th}\) to Default CDS

Unlike the standard \(\text{CDS}\) contract, the Basket Credit Default Swaps are based on referencing a number of credit entities. Instead of valuing each \(\text{CDS}\) contract alone, they will be considered as a one pool that is valued upon the number of defaults protected
against. Since the proposed valuation is based on the event of a $m^{th}$ to default credit entities, it is referred as the $m^{th}$ to default CDS.

In the case of the 1$^{st}$ to default CDS, the protection buyer will continue paying the periodic premium leg regularly until either the 1$^{st}$ to default event occur for any credit entity out of the basket or the maturity of the contract arrives, where the protection seller is obligated in the case of 1$^{st}$ to default event to pay the default leg. In general, the protection buyer of a $m^{th}$ to default CDS pays the periodic premium leg regularly until either the $m^{th}$ to default events occur for any $m$ credit entities out of the basket or the maturity of the contract arrives, where the protection seller is obligated in the case of $m^{th}$ to default events to pay the default leg, where it is valued by the same approach as the standard CDS. Subsequent to the occurrence of the default, there is a settlement and the contract is concluded with no more payments by either party are required.

Matching up the valuation of $m^{th}$ to default CDS with the standard CDS is noted in the following to standardise the concept. There are scheduled periodic premiums at the beginning of the $m^{th}$ to default CDS contract that are calculated so that the expected payments by the protection buyer and the expected payment by the protection seller in the $m^{th}$ to default CDS contract are equivalent. Similarly to the standard CDS contract, the expectation value of the premium leg and the expectation value of the default leg are equalised in order to achieve this task. Likewise the standard CDS, the periodic premiums are routinely made every quarter of the year, half of the year, or annually.

In order to price the $m^{th}$ to default CDS, it is important to drive the default and premium legs in sequence to equalise their expectations. In this section it is assumed that there are no accrued payments and thus it includes three main parts after the subsequent assumptions and definitions, i.e. pricing its default leg, pricing its premium leg, and pricing its fair periodic payments.
Definition 7.1 (Premium Payments Dates $t_\ell$)

The premium payment dates, denoted by $t_\ell$, where $\ell \in [1, L]$ and $t_L = T$ represents the maturity payment date of the basket default swap.

Definition 7.2 (Payment Length $\Delta t_\ell$)

The payment length, denoted by $\Delta t_\ell$, is the length between two sequential premium payments dates $[t_{\ell-1}, t_\ell]$.

The above two definitions declare the predefined premium payment leg dates. These definitions are two important blocks in modelling the credit risk derivatives fair price.

In the following, an important assumption, i.e. the time of default is independent from the interest rate, is hold unless explicitly mentioned.

Assumption 7.1 (Independency of Default Times and Interest Rate)

It is assumed that the default times and interest rates are independent of each other.

This assumption prevents more complicated structure between the interest rate and the time to default, such as stochastic dependence. The same assumption, in the following, is hold unless explicitly mentioned between the nominal and the time to default. To introduce this assumption, the nominal definition is articulated. Nevertheless, this assumption could straightforwardly be relaxed by taking into account a time dependent nominal when modelling the fair price of a credit risk derivative product.

Definition 7.3 (Nominal $\mathcal{K}_i$)

The nominal, denoted by $\mathcal{K}_i$, is an agreed amount of the principal unites used to calculate the exchanged payments between the derivatives contactors’ “counterparties”.

Assumption 7.2 (Independence of Default Times and Nominal $\mathcal{K}_i$)

It is assumed that the nominal $\mathcal{K}_i$ that follows Definition 7.3 and the default times $\tau_i$ are independent.
Another important block when modelling the credit risk derivatives is to specify the recovered percentage of the default entity, i.e. recovery rate. In the following, it is assumed that the recovery rate is, as well, independent from the time of default and the interest rate. To propose this postulation, the recovery rate definition is expressed.

**Definition 7.4 (Recovery Rate $\delta_i$)**

The Recovery Rate, denoted by $\delta_i$ of credit reference $i$, is an efficient measure of foreclosure procedures of a specified credit reference $i$. It is expressed by the percentage that claimants could recover from the bankrupt firm (World-Bank, 2005). In contrast, the unrecovered rate is given by $(1 - \delta_i)$.

**Assumption 7.3 (Unrecovered Payment)**

It is assumed that the payment of the unrecovered rate of company $(1 - \delta_i)$ that admits Definition 7.4 is:

- Based only on the nominal $\mathcal{K}_i$ that admits Definition 7.3 and follows Assumption 7.2.
- Independent from the default times and interest rate.

After defining the nominal and assuming its dependency from the time of default and, on the other hand, defining the recovery rate and assuming its dependency from the interest rate and the time of default, the loss given default could be defined as follow.

**Definition 7.5 (Loss Given Default $\mathcal{D}_i$)**

For a specific company $i$, let $\mathcal{K}_i$ be the nominal amount that admits Definition 7.3 and follows Assumption 7.2 and $(1 - \delta_i)$ be the unrecovered rate that admits Definition 7.4. If Assumption 7.3 being hold, then the loss given default, denoted by $\mathcal{D}_i$, is given by the nominal’s fraction of the unrecovered rate, i.e. $\mathcal{D}_i = \mathcal{K}_i(1 - \delta_i)$.

The loss given default definition, stated previously, is built upon the above definitions and assumptions, where it could be defined differently when its base is different.
The purpose of modelling any type of credit risk derivatives is to find its fair price. This fair value contains two sides: the premium and the default legs. However, the fair premium leg valuation is usually divided to be paid periodically. The next definition will be the core to define all credit risk derivatives periodic premium payments.

**Definition 7.6 (Periodic Premium $\mathcal{P}_\Delta$)**

The periodic premium, denoted by $\mathcal{P}_\Delta$, is a fair rate paid periodically on a notional that simultaneously decreases with each default of a credit reference or decreases of its corresponding amount.

**Assumption 7.4 ($\mathcal{CD}$s Periodic Premium $\mathcal{P}_\Delta$)**

Let $\mathcal{P}_\Delta$ be the $\mathcal{CD}$s periodic premium that admits Definition 7.6, then $\mathcal{P}_\Delta$ is valued to equalise the risk neutral expectation of the default and premium legs.

The other part of the fair valuation is the default leg, where on the $m^{th}$ to default $\mathcal{CD}$ it depends on the $m^{th}$ default.

**Assumption 7.5 ($m^{th}$ to Default Payment)**

Let the $n$ be the number of credit references that follows Assumption 4.1, and $m$, where $m \leq n$, be the pre-specified default times those require payment; i.e. in a $m^{th}$ to default $\mathcal{CD}$, a default payment is required on the non-recovered part of $m$'s defaulted referenced entities.

To assume and model a homogeneous basket of $\mathcal{CD}$s, equalising the credit entities weights is a requirement. This is achieved throw equalising its nominal and recovery rate components.

**Assumption 7.6 (Equal Nominal $\mathcal{K}$)**

The nominals amount $\mathcal{K}_i$ of company $i$, which admits Definition 7.3, are assumed to be equal and given by the subsequent quality, unless explicitly stated otherwise:

$$\mathcal{K} = \mathcal{K}_i = 1$$
Assumption 7.7 (Equal Recovery Rate $\delta$)

The recovery rate $\delta_i$ of company $i$ that admits Definition 7.4, are assumed to be equal and given by the subsequent quality:

$$\delta = \delta_i$$

Beside the previous two assumptions, assuming the independence between the nominal and the recovery rate is an essential assumption to fulfil the homogeneity of the basket of CDS. This assumption is expressed in the following.

Assumption 7.8 (Homogeneous CDS)

The CDS is assumed to be Homogeneous by assuming the independency between the nominal amount $K$, which admits Definition 7.3 and follow Assumption 7.6, and the recovery rate $\delta$, which admits Definition 7.4 and follows Assumption 7.6.

With the loss given default defined in Definition 7.5 and the homogeneity of the basket of CDS supposition in Assumption 7.8, modelling the homogeneous loss given default is a direct result.

Lemma 7.1 (Homogeneous Loss Given Default $D$)

For a specific company $i$, let $K_i$ be the nominal amount that admits Definition 7.3 and follows Assumption 7.2 and Assumption 7.6, $(1 - \delta_i)$ be the unrecovered rate that admits Definition 7.4 and follows Assumption 7.3 and Assumption 7.7, and $D_i$ be the loss given default that admits Definition 7.5, then the homogeneous loss given default, denoted by $D$, is given by the nominal’s fraction of the unrecovered rate, i.e. $D_i = D = (1 - \delta)$.

Even with the assumptions of the independency between the nominal amount $K$ and the recovery rate $\delta$ and the equality of each of them across the credit references, the marginal default probabilities may differ. As a consequence, the only part which is required to compute the CDS default payment leg $DL$ is the existence of the $k^{th}$ to
default times distribution.

**Definition 7.7 (Discount Factor $B_{t_{\ell}}$)**

Let $(B_t)_{t \in [0,T]}$ be the price process, and $t_{\ell}$ be the premium payment dates that admits Definition 7.1, where $\ell \in [1,L] \cup 0$ and $t_L = T$ represents the maturity payment date of the basket default swap. Then the discount factor at time=0, denoted by $B_{t_L}$, is the factor that transform the expected value of the contract at maturity $t_L$, i.e. given that $B_{t_0}$ and $B_{t_L}$ are, respectively, the price at time 0 and the expected price at maturity, $B_{t_{\ell}} = E\left[\frac{B_{t_0}}{B_{t_{\ell^*}}}\right]$.

The discounted factor ratio is formulated to evaluation the price of any credit risk derivative product at any time $t_{\ell}$, where $t_{\ell} < t_L$. The following assumption, i.e. spot forward rate assumption, is the other angle needed to evaluation the price of any credit risk derivative product.

**Assumption 7.9 (Spot Forward Rate $f_t^w$)**

Let the spot forward rate, denoted by $f_t^w$, and $B_t$ be the discount factor that admits Definition 7.7, then it is assumed that the subsequent smoothness assumption hold:

$$f_t^w B_t = - \frac{dB_t}{dt}$$

The subsequent assumption could be relaxed easily when needed. This assumption will hold when modelling the basket of $CDS$, while relaxed in the $CDO$.

**Assumption 7.10 (Accrued Leg $AL$)**

Accrued premium payments are assumed to be equal to zero, unless explicitly stated otherwise, i.e. $AL = 0$. Or equivalently there is no default between premium payments dates $t_{AL}$.

At this point the elements needed to model the expected premium payment leg are completed.
Lemma 7.2 (Premium Payment Leg $\mathcal{P}L$)

Let $\mathcal{N}_t$ be a default counter process, which counts the number of default until time $t$, that admits Definition 4.13, $t_\Delta$ be the payment length that follows Definition 7.2, $\mathcal{P}_\Delta$ be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, $\mathcal{B}_t$ be the discount factor that follows Definition 7.7, $\mathcal{K}_t$ be the nominal amount that admits Definition 7.3 and follows Assumption 7.6. Then the premium payment leg, denoted by $\mathcal{P}L$, is given by summing over all possible premium payment dates as stated in the subsequent equality:

$$\mathcal{P}L = \sum_{t=1}^{L} [t_\Delta \mathcal{P}_\Delta \mathcal{B}_t \mathcal{K}_t \mathcal{Q}^*(\mathcal{N}_t = k)]$$

Proof:

Since it is obvious that the expected discounted price of the premium leg is managed by the number of defaults, i.e. $[t_\Delta \mathcal{P}_\Delta \mathcal{B}_t \mathcal{K}_t \mathcal{J}_{k^{th} \Delta t}]$, and the $\mathbb{E}^Q [J_{x^{th} \Delta t}] = \mathcal{Q}^*(\mathcal{N}_t < k)$. Then the premium leg could be given by the subsequent chain of equalities:

$$\mathcal{P}L = \sum_{t=1}^{L} \mathbb{E}^Q [t_\Delta \mathcal{P}_\Delta \mathcal{B}_t \mathcal{K}_t \mathcal{J}_{k^{th} \Delta t}]$$

$$= \sum_{t=1}^{L} [t_\Delta \mathcal{P}_\Delta \mathcal{B}_t \mathcal{K} \mathbb{E}^Q [J_{x^{th} \Delta t}]]$$

$$= \sum_{t=1}^{L} [t_\Delta \mathcal{P}_\Delta \mathcal{B}_t \mathcal{K} \mathcal{Q}^*(\mathcal{N}_t = k)]$$

The above lemma stated that: from the distribution function of $\mathcal{N}_t$, the expected $k^{th}$ to default of the basket of CDS premium payment leg is equal to the discounted summation of all possible premium payment dates. This payment is modelled through the probabilities of $k$ credit entities being in default at time $t$, $\mathcal{Q}^*(\mathcal{N}_t = k)$. This leg only involves the semi explicit probabilities of $\mathcal{Q}^*(\mathcal{N}_t = k)$.
To compute the $k^{th}$ to default of the basket of CDS default payment leg, the only part which is required to compute the CDS default payment leg $\mathcal{D}L$ is the existence of the $k^{th}$ to default times distribution, but to be more precise it depends on the survival until the $k^{th}$ to default times.

**Lemma 7.3 (Default Payment Leg $\mathcal{D}L$)**

Let $\mathcal{B}_t$ be the discount factor that follows Definition 7.7, $f^w_t$ be the spot forward rate that admits Assumption 7.9, $\mathcal{D}$ be homogeneous loss given default that follows Lemma 7.1, $G^k_t$ be the survival function of $k^{th}$ to default time that admits Corollary 5.11, and homogeneity assumption of the credit default payment as in Assumption 7.8 is hold, then the price of the $k^{th}$ to default payment leg, denoted by $\mathcal{D}L$, is given by the subsequent equality:

$$\mathcal{D}L = \mathcal{D} \left( 1 - G^k_T \mathcal{B}_T + \int_0^T f^w_t \mathcal{B}_t G^k_t dt \right)$$

**Proof:**

In view of Lemma 4.9, the discounted payoff of the default payment leg could be written as:

$$\mathcal{D}(\tau^{k^{th}}) \mathcal{B}_{(\tau^{k^{th}})}$$

Then by the independence between the interest rates and recovery rates of Assumption 7.1, by the transfer theorem, by integrating by parts, and finally by the substituting the spot forward rate $f^w_t$ that follows Assumption 7.9, the chain of equalities hold:

$$\mathcal{D}L = \mathbb{E}^Q \left[ \mathcal{D}(\tau^{k^{th}}) \mathcal{B}_{(\tau^{k^{th}})} \right]$$

$$= -\mathcal{D} \int_0^T \mathcal{B}_t dG^k_t$$

$$= \mathcal{D} \left( 1 - G^k_T \mathcal{B}_T + \int_0^T G^k_t d\mathcal{B}_t \right)$$

$$= \mathcal{D} \left( 1 - G^k_T \mathcal{B}_T + \int_0^T f^w_t \mathcal{B}_t G^k_t dt \right)$$
With Lemma 7.2 and Lemma 7.3, the \( k^{th} \) to default of the basket of CDS periodic premium or called the \( k^{th} \) to default of the basket of CDS fair price is given by equalising its expected premium payment leg to its expected default payment leg. The following theorem summarise this subsection and articulated \( k^{th} \) to default of the basket of CDS periodic premium.

**Theorem 7.1 (The \( k^{th} \) to Default CDS Periodic Premium \( \mathcal{P}_\Delta \))**

Let \( \mathcal{N}_{t_\ell} \) be a default counter process, which counts the number of default until time \( t \), that admits Definition 4.13, \( t_{d_\ell} \) be the payment length that follows Definition 7.2, \( \mathcal{P}_\Delta \) be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, \( \mathcal{B}_{t_\ell} \) be the discount factor that follows Definition 7.7, \( f^{w}_t \) be the spot forward rate that admits Assumption 7.9, \( \mathcal{D} \) be homogeneous loss given default that follows Lemma 7.1, \( \mathcal{K}_\ell \) be the nominal amount that admits Definition 7.3 and follows Assumption 7.6, \( G^k_T \) be the survival function of \( k^{th} \) to default time that admits Corollary 5.11, and homogeneity assumption of the credit default payment as in Assumption 7.8 is hold, \( \mathcal{P}_L \) be the premium payment leg that admits Lemma 7.2, and \( \mathcal{D}_L \) the \( k^{th} \) to default payment leg that admits Lemma 7.3. Then the \( \mathcal{P}_\Delta \) is given by the subsequent equality:

\[
\mathcal{P}_\Delta = \frac{1 - \delta}{\sum_{\ell=1}^L \left[ t_{d_\ell} \mathcal{B}_{t_\ell} \mathcal{K}_\ell^* (\mathcal{N}_{t_\ell} = k) \right]} \left( 1 - G^k_T \mathcal{B}_T + \int_0^T f^{w}_t \mathcal{B}_t G^k_t dt \right)
\]

**Proof:**

Since the \( k^{th} \) to default CDS is built upon the assumption of non-arbitrage opportunities, i.e. \( \mathbb{E}^Q [\mathcal{P}_L] = \mathbb{E}^Q [\mathcal{D}_L] \), then by simple algebra \( \mathcal{P}_\Delta \) is proved from Lemma 7.2 and 7.3.

By this theorem, the \( k^{th} \) to default of the basket of CDS components are modelled with an assumption of the inexistence of any default in-between the premium payments dates.
### 7.4 Pricing of the Homogenous ranked m out of n CDS

m out of n Basket Credit Default Swaps, which is referred as the \( \binom{n}{m}^{th} \) to default CDS, is another type of the Basket Credit Default Swaps contracts, where the protection seller is obliged to pay the default leg to the protection buyer, when a \( m^{th} \) to default event occur. The default leg cash-flow and valuation method is proceeded in the same manner that the \( m^{th} \) to default CDS is carried out. Complementary to the default leg of \( m^{th} \) to default CDS, the premium leg of the \( \binom{n}{m}^{th} \) to default CDS is paid upon the un-defaulted credit entities of the whole basket until \( m^{th} \) to default events occur.

In this subsection, particular product of the \( \binom{n}{m}^{th} \) to default CDS will be considered called the ranked credit entities of the \( \binom{n}{m}^{th} \) to default CDS. This kind of contracts supplies protection for ranked defaults in the basket, where it covers the defaults of the credit entities ranked between \( \mathcal{R}_m \) and \( \mathcal{R}_M \) where the later is excluded, i.e. \( 1 \leq \mathcal{R}_m \leq m \leq \mathcal{R}_M \leq n \). The ranked \( \binom{n}{m}^{th} \) to default CDS is referred as \( \binom{n}{[\mathcal{R}_m, \mathcal{R}_M]}^{th} \) to default CDS. Indeed, this means that instead of protecting against a particular number of defaults, the protection is covering a range of defaults.

The default payments of the \( \binom{n}{[\mathcal{R}_m, \mathcal{R}_M]}^{th} \) to default CDS could occur in one or more of the subsequent dates: \( \tau^{(\mathcal{R}_m+1)^{th}}, \cdots, \tau^{m^{th}}, \cdots, \tau^{(\mathcal{R}_M)^{th}} \) if the \( m \in [\mathcal{R}_m, \mathcal{R}_M] \) and the \( (m^{th} \text{ to default}) < t_e \). Each default occur in between of the ranking barriers obligates a payment equals to the unrecovered fraction of the defaulted credit entity times its nominal. However, looking at the default payment from mathematical point of view and trying to fulfil the equilibrium the expected value of the default leg with the expected value of the premium leg is essential in the \( \binom{n}{[\mathcal{R}_m, \mathcal{R}_M]}^{th} \) to default CDS. Indeed, a basic algebra confirms that the default leg is equivalent to the sum of the default legs paying
the credit entities’ unrecovered fraction over all possible $m^{th}$ to default events, but this part is not covered in this subsection, where it will be detailed in Chapter 8.

Analogously to the $m^{th}$ to default CDS, the premium payments dates of the \((\frac{n}{\mathcal{R}_{m,\mathcal{R}_{M}}})^{th}\) to default CDS are predetermined. In contrary, the premium legs are not always equal, where it depends on the number of defaulted credit entities and their ranking at the prearranged payment dates. Thus, the premium payments amounts could be one of three cases. The first is when the number of defaults are less than the lower credit entities’ ranking $\mathcal{R}_{m}$, the premium is equivalent to entire protected credit entities between $\mathcal{R}_{m}$ and $\mathcal{R}_{M}$. The second is when the number of default has exceeded the lower credit entities’ ranking $\mathcal{R}_{m}$ but it did not rise above the higher credit entities’ ranking $\mathcal{R}_{M}$, the premium is equivalent to the remaining protected credit entities between $\mathcal{R}_{m}$ and $\mathcal{R}_{M}$.

The last case is when the number of default has exceeded the higher credit entities’ ranking $\mathcal{R}_{M}$, the premium leg is terminated and it is said that the basket of CDS is exhausted. At this point it is worth mentions that the number of defaults are sorted in order, as it is in the case of all basket credit default products, so that the defaults are classified to be in three ranges: below, in between, or above the ranking barriers.

With the aim of pricing the basket of \(\left(\frac{n}{\mathcal{R}_{m,\mathcal{R}_{M}}}\right)^{th}\) to default CDS, it is significant to model the default and premium legs sequentially to equalise their expectations. As in the previous subsection, in this subsection it is assumed that there are no accrued payments and thus it contains three key elements, after the following assumptions and definitions, i.e. pricing its default leg, pricing its premium leg, and pricing its fair periodic payments.

In order to standardise the notation, the \(\left(\frac{n}{m}\right)^{th}\) to default CDS payment assumption is stated in the following assumption.
Assumption 7.11 \((\binom{n}{m})^{th}\) to Default CDS Payment

Let the \(n\) be the number of credit references, as assumed in Assumption 4.1, and \(m\), where \(m \leq n\), be the pre-specified default time that requires payment; i.e. in a \((\binom{n}{m})^{th}\) to default CDS a default payment is required on the non-recovered part of \(m\)'s defaulted referenced entities.

Unlike the basket of \(k^{th}\) to default CDS, the basket of \((\binom{n}{\mathcal{R}_m,\mathcal{R}_M})^{th}\) to default CDS have lower and upper barriers those control payment legs. The next two assumptions formalise this terminology.

Assumption 7.12 (Ranking Barrier \(\mathcal{R}\))

Let \(\mathcal{R}_m\) (included) be a lower protection ranking barrier, \(\mathcal{R}_M\) (excluded) be a higher protection ranking barrier, \(n\) be the number of credit references, as assumed in Assumption 4.1, and \(m\) be the pre-specified default times that requires payment that follow Assumption 7.11, then a \(m\) out of \(n\) CDS default payment is arisen between \(1 \leq \mathcal{R}_m \leq m \leq \mathcal{R}_M \leq n\).

For example if we have \(n = 16\), \(m = 6\) and it starts from the \(4^{th}\) to default, then \(\mathcal{R}_m = 4^{th}\) to default, \(\mathcal{R}_M = 10^{th}\), and denoted by \((\binom{16}{4,10})^{th}\) to default CDS.

Assumption 7.13 (Ranking Different \(\mathcal{R}_\Delta\))

Let \(\mathcal{R}_m\) (included) be a lower protection ranking barrier, \(\mathcal{R}_M\) (excluded) be a higher protection ranking barrier those follows Assumption 7.12, then the ranking different, denoted by \(\mathcal{R}_\Delta\), is given by the difference between the lower and higher protection ranking barriers, i.e. \(\mathcal{R}_\Delta = \mathcal{R}_M - \mathcal{R}_m\).

Consequently, with Assumption 7.12 and Assumption 7.13, the basket of \((\binom{n}{\mathcal{R}_m,\mathcal{R}_M})^{th}\) to default CDS premium payment leg could be classified in three parts: below, in between,
or above the ranking barriers. The next two lemmas show how modelling the premium payment leg is controlled by the nominal amount.

**Lemma 7.4 (Nominal Amount $\mathcal{K}_i$ of Premium Payment Leg)**

Let $\mathcal{R}_m$ and $\mathcal{R}_M$ be, respectively, the lower and the higher protection ranking barriers those follow Assumption 7.12, $\mathcal{R}_\Delta$ be ranking difference that follows Assumption 7.13, and $\mathcal{N}_{t\varepsilon}$ be a default counter process, which counts the number of default until time $t$, that admits Definition 4.13, then the remaining of the nominal amount $\mathcal{K}_i$ that admits Definition 7.3, is given by the triplet if equalities:

$$
\mathcal{K}_i = \begin{cases} 
0 & \text{if } \mathcal{N}_{t\varepsilon} \geq \mathcal{R}_M \\
\mathcal{R}_m - \mathcal{N}_{t\varepsilon} & \text{if } \mathcal{R}_m \leq \mathcal{N}_{t\varepsilon} < \mathcal{R}_M \\
\mathcal{R}_\Delta & \text{if } \mathcal{N}_{t\varepsilon} < \mathcal{R}_m 
\end{cases}
$$

**Lemma 7.5 (Discounted Expectation of Premium Payment Leg $\mathcal{P}\mathcal{L}_{t\varepsilon}$)**

Let $\mathcal{R}_m$ and $\mathcal{R}_M$ be, respectively, the lower and the higher protection ranking barriers those follow Assumption 7.12, $\mathcal{R}_\Delta$ be ranking difference that follows Assumption 7.13, $\mathcal{N}_{t\varepsilon}$ be a default counter process, which counts the number of default until time $t$, that admits Definition 4.13, $t_{\Delta\varepsilon}$ be the payment length that follows Definition 7.2, $\mathcal{P}_\Delta$ be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, $\mathcal{B}_{t\varepsilon}$ be the discount factor that follows Definition 7.7, and $\mathcal{K}_i$ be the nominal amount that admits Definition 7.3 and follows Lemma 7.4, then the discounted expectation of premium payment at time $t_{\varepsilon}$, denoted by $\mathcal{P}\mathcal{L}_{t_{\varepsilon}}$, is given by the following equality:

$$
\mathcal{P}\mathcal{L}_{t_{\varepsilon}} = t_{\Delta\varepsilon}\mathcal{P}_\Delta \mathcal{B}_{t_{\varepsilon}} \times \left( \mathcal{R}_\Delta \mathcal{Q}^* (\mathcal{N}_{t_{\varepsilon}} < \mathcal{R}_m) + \sum_{k=\mathcal{R}_m}^{\mathcal{R}_M} (\mathcal{R}_M - k) \mathcal{Q}^* (\mathcal{N}_{t\varepsilon} = k) \right),
$$

With the structure given in Lemma 7.4 and the distribution function of $\mathcal{N}_{t\varepsilon}$, Lemma 7.5 states that the expected basket of $\left(\frac{n}{\mathcal{R}_m, \mathcal{R}_M}\right)_k^{th}$ to default CDS premium payment leg at

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34 The proof of this lemma is straightforward
time $t_\ell$ is equal to the of periodic premium payment multiplied by the discounting factor depending on the number of credit entities being in default at time $t_\ell$ and where do these credit entities are located in the classification articulated in the above Lemma 7.4.

The previous lemma could be finally written by summing over all possible premium payment dates. This payment is modelled through the probabilities of $k$ credit entities being in default at time $t_\ell$, $\mathcal{Q}^*\left(\mathcal{N}_{t_\ell} = k\right)$. This leg only involves the semi explicit probabilities of $\mathcal{Q}^*\left(\mathcal{N}_{t_\ell} = k\right)$, as it is illustrated in the next corollary.

**Corollary 7.1 (Premium Payment Leg $\mathcal{PL}$)**

Let $\mathcal{R}_m$ and $\mathcal{R}_M$ be, respectively, the lower and the higher protection ranking barriers those follow Assumption 7.12, $\mathcal{R}_\Delta$ be ranking difference that follows Assumption 7.13, $\mathcal{N}_{t_\ell}$ be a default counter process, which counts the number of default until time $t$, that admits Definition 4.13, $t_{\Delta \ell}$ be the payment length that follows Definition 7.2, $\mathcal{P}_\Delta$ be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, $\mathcal{B}_{t_\ell}$ be the discount factor that follows Definition 7.7, $\mathcal{K}_i$ be the nominal amount that admits Definition 7.3 and follows Lemma 7.4, and $\mathcal{PL}_{t_\ell}$ be the discounted expectation of premium payment at time $t_\ell$ that admit Lemma 7.5, then the premium payment leg, denoted by $\mathcal{PL}$, is given by summing over all possible premium payment dates as stated in the subsequent equality:

$$\mathcal{PL} = \sum_{\ell=1}^\ell t_{\Delta \ell} \mathcal{P}_\Delta \mathcal{B}_{t_\ell} \times \left(\mathcal{R}_\Delta \sum_{k=0}^{\mathcal{R}_m-1} \mathcal{Q}^*\left(\mathcal{N}_{t_\ell} = k\right) + \sum_{k=\mathcal{R}_m}^{\mathcal{R}_M} (\mathcal{R}_M - k) \mathcal{Q}^*\left(\mathcal{N}_{t_\ell} = k\right)\right)$$

Headed for computing the expected basket of $\binom{n}{\mathcal{R}_m, \mathcal{R}_M}^{th}$ to default CDS default payment leg, an assumption of default payments dates is important. This supposition assumes that the default dates are earlier than the maturity date and are randomised sequence. Each element in this sequence represent the $k^{th}$, where $\mathcal{R}_m \leq k^{th} < \mathcal{R}_M$. 
Assumption 7.14 (Default Payments Dates $\tau^{k}\text{th}$)

It is assumed that the default payments, are

i. Prior to the maturity date of the $m^{th}$ to default CDS, i.e. $t_L$

ii. At dates $\tau^{(R_m+1)th}, \ldots, \tau^{kth}, \ldots, \tau^{(R_M)th}$

Where $R_m$ is the lower protection ranking barriers and $R_M$ is the higher protection ranking barriers those follow Assumption 7.12, i.e. $R_m \leq k^{th} < R_M$.

Calculating the $k^{th}$ to default current price of the payoff of the default payment leg is given by summing over $k$ possible defaults of the recover part of the referenced credit entities, as it is shown in the next Lemma.

Lemma 7.6 (Default Payment Leg $DL$)

Let $B_t$ be the discount factor that follows Definition 7.7, $f_t^{w}$ be the spot forward rate that admits Assumption 7.9, $D$ be homogeneous loss given default that follows Lemma 7.1, $G_T^k$ be the survival function of $k^{th}$ to default time that admits Corollary 5.11, and homogeneity assumption of the credit default payment as in Assumption 7.8 is hold, then the price of the $k^{th}$ to default payment leg, denoted by $DL$, is given by the subsequent equality:

$$DL = D \left( 1 - G_T^k B_T + \int_0^T f_t^{w} B_t G_T^k dt \right)$$

Proof:

In view of Lemma 4.9, the discounted payoff of the default payment leg could be written as:

$$DL[0,T] \left( \tau^{kth} \right) B(\tau^{kth})$$

Then by the independence between the interest rates and recovery rates of Assumption 7.1, by the transfer theorem, by integrating by parts, and finally by the substituting the
spot forward rate $f_t^w$ that follows Assumption 7.9, the chain of equalities hold:

$$
\mathcal{D}\mathcal{L} = \mathbb{E}^{\mathbb{Q}^T}\left[D\left[\eta_{[0,T]}(\tau^k)\right]B_{(\tau^k)}\right] \\
= -\mathcal{D}\int_0^T B_t dG^k_t \\
= \mathcal{D}\left(1 - G^k_T B_T + \int_0^T G^k_t dB_t\right) \\
= \mathcal{D}\left(1 - G^k_T B_T + \int_0^T f_t^w B_t G^k_t dt\right)
$$

The subsequent corollary is an immediate result when applying the default payment leg to model the first to default payment leg.

**Corollary 7.2 (1st to Default Default Payment Leg $\mathcal{D}\mathcal{L}_{1st}$)**

Let $\mathcal{B}_{t_i}$ be the discount factor that follows Definition 7.7, $f_t^w$ be the spot forward rate that admits Assumption 7.9, $\mathcal{K}$ be the nominal amount that admits Definition 7.3 and follows Assumption 7.6, $\delta$ be the recovery rate that admits Definition 7.4 and follows Assumption 7.3, $G^1_{1st}$ be the survival function of 1st to default time that admits Corollary 5.11, and the homogeneity assumption of the credit default payment as in Assumption 7.8 is hold, then the price of the 1st to default payment leg, denoted by $\mathcal{D}\mathcal{L}_{1st}$, is given by the subsequent equality:

$$
\mathcal{D}\mathcal{L}_{1st} = 1 - G^1_{1st} B_T + \int_0^T f_t^w B_t G^1_{1st} dt
$$

where $G^1_{1st} = \int \prod_{i=1}^n q_{t_i}^\xi |m| f_{\mathcal{M}_t}(m) dm$.

With Corollary 7.1 and Lemma 7.6, the homogeneous $^{th} \left(\mathcal{M}_{m,R,M}\right)$ to default CDS periodic premium or called the homogeneous $^{th} \left(\mathcal{M}_{m,R,M}\right)$ to default of the basket of CDS fair price is given by equalising its expected premium payment leg to its expected default payment leg. The following theorem summarise this subsection and expressed the homogeneous $^{th} \left(\mathcal{M}_{m,R,M}\right)$ to default of the basket of CDS periodic premium.
Theorem 7.2 (The Homogeneous \( \ell^{th} \) to Default CDS Periodic Premium \( P_\Delta \))

Let \( R_m \) and \( R_M \) be, respectively, the lower and the higher protection ranking barriers those follow Assumption 7.12, \( R_\Delta \) be ranking difference that follows Assumption 7.13, \( N_{t_\ell} \) be a default counter process, which counts the number of default until time \( t \), that admits Definition 4.13, \( t_{\Delta t} \) be the payment length that follows Definition 7.2, \( P_\Delta \) be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, \( B_{t_\ell} \) be the discount factor that follows Definition 7.7, \( K_i \) be the nominal amount that admits Definition 7.3 and follows Lemma 7.4, \( f_t^w \) be the spot forward rate that admits Assumption 7.9, \( D \) be homogeneous loss given default that follows Lemma 7.1, \( G_k^T \) be the survival function of \( k^{th} \) default time that admits Corollary 5.11, the homogeneity assumption of the credit default payment as in Assumption 7.8 is hold, \( P_L \) the premium payment leg that admits Corollary 7.1, and \( D_L \) be the \( k^{th} \) default payment leg that admits Lemma 7.6. Then the \( P_\Delta \) is given by the subsequent equality:

\[
P_\Delta = \frac{D \left( 1 - G_T^k B_T + \int_0^T f_t^w B_t G_k^T dt \right)}{\sum_{t=1}^{t_{\Delta t}} t_{\Delta t} B_{t_\ell} \times \left( R_\Delta \sum_{k=0}^{R_m-1} Q^*(N_{t_\ell} = k) + \sum_{k=R_m}^{R_M} (R_M - k) Q^*(N_{t_\ell} = k) \right)}
\]

By this theorem, the \( \left( \ell^{th} \right) \) to default of the basket of CDS components are modelled with an assumption of the inexistence of any default in-between the premium payments dates.

7.5 Pricing of the Non-Homogenous ranked \( m \) out of \( n \) CDS

In the universal case, where the credit entities in the basket of CDS are not equally weighted, the computations are a bit more involved when modelling the default leg. The general case is called the basket of non-homogeneous \( \left( \ell^{th} \right) \) to default of CDS.
With the aim of pricing the non-homogenous basket of \( \left( \ell_{m, M} \right)^{th} \) to default CDS, it is significant to model the default and premium legs consecutively to equalise their expectations. The non-homogenous basket of \( \left( \ell_{m, M} \right)^{th} \) to default CDS premium leg is equal to the homogenous one articulated in the previous subsection.

As in the previous subsection, in this subsection it is assumed that there are no accrued payments and thus it contains three key elements, after the following assumptions and definitions, i.e. pricing its default leg, pricing its premium leg, and pricing its fair periodic payments.

In this subsection, the order followed in the previous subsection is flipped, where the first to default payment leg is modelled before the more general case of \( \left( \ell_{m, M} \right)^{th} \) to default CDS default payment leg.

**Lemma 7.7 (1st to Default Payment Leg DL)**

Let \( B_{t, t} \) be the discount factor that follows Definition 7.7, \( f_t^{w} \) be the spot forward rate that admits Assumption 7.9, \( D_t \) be loss given default that admits Definition 7.5, \( G_t^{1st} \) be the survival function of 1st to default time that admits Corollary 5.11, \( \Gamma_t^{1st} \) be the hazard process that follows Assumption 4.13, then the price of the 1st to default payment leg, denoted by \( DL_{1st} \), is given by the subsequent equality:

\[
DL_{1st} = \sum_{t=1}^{n} D_t \int_0^T \Gamma_t^{1st} G_t^{1st} B_t dt
\]

**Proof**

With \( \tau^{1st} \) as random time that follows Assumption 4.5 and in view of the proof of Lemma 7.6 the discounted payoff of the 1st to default payment leg could be written as:

\[
\sum_{t=1}^{n} D_t B_{t, t} \int_{\tau^{1st}}^{T} \tau^{1st} G_{\tau^{1st}} B_{t, \tau^{1st}} dt
\]
Then, sequentially, by employing the independence between the interest rates and recovery rates of Assumption 7.1, applying the iterated expectations theorem, utilising the transfer theorem, differentiating with respect to $t_i$, and concluding by the inspiration of the conditional hazard rates definition, the chain of equalities hold:

$$D\mathcal{L}_{1st} = \mathbb{E}^{Q^*} \left[ \sum_{i=1}^{n} D_i B_{\tau_i}^{t_{1st}} \mathcal{I}_{t_{1st} \geq \tau_i \leq \tau_i} \right]$$

$$= \sum_{i=1}^{n} \mathbb{E}^{Q^*} \left[ \mathbb{E}^{Q^*} \left[ D_i B_{\tau_i}^{t_{1st}} \mathcal{I}_{t_{1st} \geq \tau_i \leq \tau_i} \left| \tau_i \right. \right] \right]$$

$$= \sum_{i=1}^{n} D_i \mathbb{E}^{Q^*} \left[ B_{\tau_i}^{t_{1st}} \mathbb{Q}^* \left( t_{1st} \geq \tau_i \mid \tau_i \right) \mathcal{I}_{\tau_i \leq \tau_i} \right]$$

$$= \sum_{i=1}^{n} D_i \int_0^\tau \mathbb{Q}^* (t_{1st} \geq \tau_i \mid \tau_i = t) f_{t_i} B_t dt$$

$$= - \sum_{i=1}^{n} D_i \int_0^\tau \frac{\partial G(t, ..., t)}{\partial t_i} B_t dt$$

$$= \sum_{i=1}^{n} D_i \int_0^\tau t_i^{G(t_i^{1st})} B_t dt$$

Where $\mathcal{F}$ signify the marginal density of $\tau_i$, and $G(t) = \int \prod_{i=1}^{n} Q_t^{lim} f_{t_i}(m) dm$.

As mentioned in the introduction, in the universal case the credit entities in the basket of CDS are not equally weighted. Therefore, the computations of the expectation of default leg need to compute the expected number of defaults excluding a specific credit entity at a time. To introduce this concept, the counter process of excluding a specific credit entity, which was introduced in Definition 4.14, is rephrased and proved in the following lemma.

**Lemma 7.8 (Counter Process of Excluded $\mathcal{N}_{\tau_i}^{(-i)^{th}}$)**

Let $\mathcal{N}_{\tau_i}^{(-i)^{th}}$ be Counter Process of Excluded $i$ that follows Definition 4.14, and the $m^{th-to-default}$ is associated with the name $i$, then the subsequent equality hold:

$$\mathcal{N}_{\tau_i}^{(-i)^{th}} = m - 1$$
Proof:

In view of Assumption 4.6, Definition 4.14, and Lemma 7.4, this lemma is straightforward result.

In this subsection, the default payment leg is assumed to be as a single payment, where more general cases could be handled straightforwardly. Therefore, by taking into account the proof of Lemma 7.7 and given Lemma 7.8, the discounted payoff of the default payment leg could be treated as in the following lemma.

Lemma 7.9 \((\text{m}^{\text{th}}\text{ to Default Payment Leg } \mathcal{DL})\)

Let \(B_t\) be the discount factor that follows Definition 7.7, \(f_t^w\) be the spot forward rate that admits Assumption 7.9, \(D_t\) be loss given default that admits Definition 7.5, \(N_{t_i}^{(-i)^{th}}\) be Counter Process of Excluded \(i\) that follows Definition 4.14 and Lemma 7.8, \(p_{t_i}^{\xi_i|M_t}\) be the probability of default conditioned upon the Systematic Market Risk Factor \(M_t\) that follows Corollary 5.11. Then the price of the \(k^{\text{th}}\) default payment leg, denoted by \(\mathcal{DL}\), is given by the subsequent equality:

\[
\mathcal{DL} = \mathbb{E}^{Q^*}\left[\sum_{i=1}^{n} B_t D_t Q^* N_{t_i}^{(-i)^{th}} = m - 1 \middle| M_t\right] p_{t_i}^{\xi_i|M_t}
\]

Proof:

In view of the proof of Lemma 7.7 and given Lemma 7.8 the discounted payoff of the default payment leg could be written as:

\[
\sum_{i=1}^{n} D_t B_t \xi_i^{T_i t \leq T_j} N_{t_i}^{(-i)^{th}} = m - 1
\]

Then by the independence between the interest rates and recovery rates of Assumption 7.1, by the iterated expectation theorem on the random time \(\tau_i\) being conditionally independent on \(M_t\) from Assumption 5.5, and finally by using the transfer theorem, i.e. integrating over the conditional distribution of \(\tau_i\), the chain of equalities hold:
\[
D_L = \mathbb{E}^{Q^*}\left[ \sum_{i=1}^{n} D_i \mathcal{B}_{t_i} J_{t_i \in \mathcal{T}} J_{\mathcal{N}_{t_i}^{(-i)^{th}} = m-1} \right] \\
= \mathbb{E}^{Q^*}\left[ \sum_{i=1}^{n} D_i \mathbb{E}^{Q^*}\left[ \mathcal{B}_{t_i} J_{t_i \in \mathcal{T}} J_{\mathcal{N}_{t_i}^{(-i)^{th}} = m-1} \mid \mathcal{M}_t, \tau_i = t \right] \right] \\
= \mathbb{E}^{Q^*}\left[ \sum_{i=1}^{n} D_i \int_0^T \mathcal{B}_t Q^* \left( \mathcal{N}_t^{(-i)^{th}} = m-1 \right \mid \mathcal{M}_t \right] dp_{\xi_i|M_t}^{M_t} \right]
\]

The subsequent theorem encapsulates this subsection and articulates the non-homogeneous \((\mathcal{R}_m, \mathcal{R}_m^n)^{th}\) to default of the basket of CDS periodic premium.

**Theorem 7.3 (The Non-Homogeneous \((\mathcal{R}_m, \mathcal{R}_m^n)^{th}\) to Default CDS Periodic Premium \(\mathcal{P}_\Delta\))**

Let \(\mathcal{R}_m\) and \(\mathcal{R}_M\) be, respectively, the lower and the higher protection ranking barriers those follow Assumption 7.12, \(\mathcal{R}_\Delta\) be ranking difference that follows Assumption 7.13, \(\mathcal{N}_{t_\ell}\) be a default counter process, which counts the number of default until time \(t\), that admits Definition 4.13, \(t_{\Delta \ell}\) be the payment length that follows Definition 7.2, \(\mathcal{P}_\Delta\) be the CDS periodic premium that admits Definition 7.6 and follows Assumption 7.4, \(\mathcal{B}_{t_\ell}\) be the discount factor that follows Definition 7.7, \(\mathcal{K}_i\) be the nominal amount that admits Definition 7.3 and follows Lemma 7.4, \(f_{\xi}^w\) be the spot forward rate that admits Assumption 7.9, \(\mathcal{D}_i\) be loss given default that admits Definition 7.5, \(\mathcal{N}_{t_\ell}^{(-i)^{th}}\) be Counter Process of Excluded \(i\) that follows Definition 4.14, Lemma 7.8, \(p_{\xi_i|M_t}^{M_t}\) be the probability of default conditioned upon the Systematic Market Risk Factor \(\mathcal{M}_t\) that follows Corollary 5.11, \(\mathcal{P}_L\) the premium payment leg that admits Corollary 7.1, and \(D_L\) be the \(k^{th}\) to default payment leg that admits Lemma 7.9. Then the \(\mathcal{P}_\Delta\) is given by the subsequent equality:

\[
\mathcal{P}_\Delta = \frac{\mathbb{E}^{Q^*}\left[ \int_0^T \sum_{i=1}^{n} \mathcal{B}_t \mathcal{D}_i \mathcal{Q}^* \left( \mathcal{N}_t^{(-i)^{th}} = m-1 \right \mid \mathcal{M}_t \right] dp_{\xi_i|M_t}^{M_t} \right]}{\sum_{\ell=1}^{T_{\Delta \ell}} t_{\Delta \ell} \mathcal{B}_{t_\ell} \times \left( \mathcal{R}_\Delta \sum_{k=0}^{\mathcal{R}_m-1} \mathcal{Q}^* (\mathcal{N}_{t_\ell} = k) + \sum_{k=\mathcal{R}_m}^{\mathcal{R}_M} (\mathcal{R}_M - k) \mathcal{Q}^* (\mathcal{N}_{t_\ell} = k) \right)}
\]
7.6 Pricing Of CDO’s

In this subsection, the collateralised debt obligations (CDO)’s will be explained analogously to the basket default swaps and in its context, i.e. \( m^{th} \) to default CDS and \( \left( \frac{n}{(\mathcal{R}_m,\mathcal{R}_N)} \right)^{th} \) to default CDS. The (CDO)’s mainly consists of three major parts; explicitly the issuer, the asset side, and the liability side.

The issuer is a virtual entity that is responsible for linking the asset side by the liability side by issuing notes for individual transaction type, i.e. CDOs. The CDO’s assets side could be built upon an individual or many of one or more types of reference entities, i.e. CDO tranches, CDO of CDOs, i.e. CDO-squared structures (CDO\(^2\)) and CDO\(^n\), Credit Default Swaps (CDS), Synthetic Collateralised Debt Obligations (SCDO), Collateralised Bond Obligations (CBO s), Mortgage-Backed Securities (MBS), Collateralised Loan Obligations (CLOs), etc. In opposite, the issuer issues, generally, the liability side by tranching a corresponding three positions of the capital structure of the CDO, i.e. the Equity Tranche, the Mezzanine Tranche, and finally the Senior Tranche.

The CDOs produce more adaptable and flexible product than the basket default swaps products if its structure is observed exteriorly, i.e. the liability side, and complex if its structure is viewed interiorly, i.e. the assets side that represents the reference portfolio.

In contrast, the cash flows in the CDOs are almost identical to the basket default swaps in the assets side, where default payments are due to any default event in return of periodic premium payments. On the contrary, the cash flow in liability side is quite complex as a consequence of its tranche structure that waterfalls its interests and repayments cash flow as bottom-up, tranche by tranche, while, oppositely, the loss waterfalls are structured top-down, tranche by tranche, in case of losses.
In the case of interests and repayments the senior tranche notes’ holders receives their portion firstly then the mezzanine tranche notes’ holders receives their segment secondly, and finally the equity tranche notes’ holders receives their fraction. This sequence influences the amount of payments they are receiving at each time since the losses are affecting those amounts of payments in the opposite direction. The equity tranche notes’ holders suffer the initial losses until the end of equity tranche capacity, then the mezzanine tranche notes’ holders tolerate the subsequent losses until the end of the mezzanine volume, and finally if any losses have exceeded the equity and then the mezzanine tranches, the senior tranche notes’ holders have to carry them.

The CDO’s could be seen in the same manner of the \( (\mathbb{R}_n) \) to default CDS, where the payments, in the later, depends on the number of defaulted credit entities that are in between a ranking level, and those ranking level in the CDOs corresponds to the tranches points.

Another types of CDO structure is called the Synthetic CDOs, which is symbolised as SCDO. The SCDOs referenced portfolio is purely created from a number of CDS contracts. The SCDO issuer sells those it to third parties. As a consequence of any default event in the referenced portfolio, this referenced credit entity will pass to the SCDO’s tranche holders. Analogously to the cash CDO in the previous example, instead of the direct losses in the capital and interests’ repayments, the cash flows are structured similarly to the CDS contracts. In other words, the notes holders are the protection seller and the third parties are the protection buyer. The equity tranche notes’ holders are responsible for the default legs payoffs of the CDSs until the notional principal reaches the capacity of the equity tranche, then the mezzanine tranche notes’ holders are liable for the default legs payoffs of the CDSs until the notional principal reaches the volume of the mezzanine tranche, and finally the senior tranche notes’
holders are accountable for the default legs payoffs of the residual CDS’s notional principal. In return, each tranche notes’ holders are getting periodic premium legs that reflect the amount of risk they are responsible for.

Furthermore, many alternative CDO structures are available in the market, where CDX and iTraxx indices are examples. CDX and iTraxx are a standardised CDO tranches those have launched to the market generated by an underlying portfolio.

Trading these standardised CDO tranches are known as single tranche CDO, which is signified as STCDO. A STCDO contract is an agreement that two parties agrees to enter a protection contract that one of them represents the protection seller against losses that affects that tranche and the other party corresponds to the protection buyer. Contrast to the SCDO tranches, where the referenced credit entities portfolio is tranched by selling a CDS contracts, the STCDO are not part of the SCDO, which means that its two parties are not trading the actual credit entities that build up these indices and their tranches but they are trading the movements and actions that those indices are facing. STCDO cash flows are calculated in the same manner as SCDO are carried out.

CDX.NA.IG index is an example of the CDX family indices and it presents default protection contract on 125 of equally weighted North American investment-grade rated issuers. Its equity, junior mezzanine, senior mezzanine, senior, super senior, and second super senior tranches protects the losses, respectively, between 0%-3%, 3%-7%, 7%-10%, 10%-15%, 15%-30%, and finally 30%-100%. Oppositely, iTRAXX Europe index is a member of the iTRAXX family indices and it provides default protection contract on 125 of equally weighted European investment-grade rated issuers. Its equity, junior mezzanine, senior mezzanine, senior, super senior tranches, and second super senior tranches protects the losses, respectively, between 0%-3%, 3%-6%, 6%-9%, 9%-12%, 12%-22% and finally 22%-100%.
In the following, modelling and pricing the SCDO will be studied, where it could be easily extended to price other types of CDO structure. It is important to drive the default, premium legs, and the accrued legs in sequence to equalise their expectation. In this subsection, the assumption of no accrued payments is relaxed and thus it includes four main parts after the subsequent assumptions and definitions, i.e. pricing its default leg, pricing its premium leg, pricing its accrued leg, and finally pricing its fair periodic payments.

**Assumption 7.15 (Collateralised Debt Obligation Periodic Premium $P_\Delta$)**

Let $P_\Delta$ be the CDO periodic premium that admits Definition 7.6, then $P_\Delta$ is valued to equalise the risk neutral of the tranche to zero.

The determination of modelling any type of credit risk derivatives is to find its fair price. This fair value encloses two sides: the premium and the default legs. However, the fair premium leg valuation is usually divided to be paid periodically. To standardise this concept to each and every credit risk derivatives noted in this thesis, the previous assumption expresses CDO periodic premium payments.

As stated in the introduction, the CDO is modelled through the cumulative loss distribution of the referenced portfolio. This is introduced through the following definitions.

**Definition 7.8 (CDO Cumulative Loss $C_t$)**

For a specific company $i$ at time $t$, the summation of the products of $D_t$, as the loss given default that admits Definition 7.5, and $N_t^i$, as the counter process associated with the default company $i$ that admits Definition 4.12, is called the CDO’s cumulative loss, denoted by $C_t$, i.e. $C_t = \sum_{i=1}^{n} D_t N_t^i$

The CDO’s cumulative loss $C_t$ is a pure jump process as a consequence of the counter process $N_t^i$ behaviour, where $N_t^i \in \mathbb{I}$ and is a jump process.
Definition 7.9 (CDO Cumulative Loss Excluded $i$, $C^{(-i)th}$)

For an excluded company $i$ at time $t$, the summation of the products of $\mathcal{D}_i$, as the loss given default that admits Definition 7.5, $\mathcal{N}_t^i$, as the counter process associated with the default company $i$ that admits Definition 4.12, and $\tau^{(-i)th}$ be the $i^{th}$ to default not happened at the given time $\tau_i$ that admits Assumption 4.6, is called the CDO’s cumulative loss excluded $j$, denoted by $C^{(-i)th}$, i.e. $C^{(-i)th} = \sum_{i \neq j} \mathcal{D}_i \mathcal{N}_t^i$

In view of Definition 7.8 and Definition 7.9, the subsequent lemma is an immediate consequence.

Lemma 7.10 (CDO Cumulative Loss Excluded $i$, $C^{(-i)th}$)

Let $C^{(-i)th}$ be the CDO’s cumulative loss excluded company $i$ that admits Definition 7.9, and $\mathcal{D}_i$ be the loss given default that admits Definition 7.5, then the CDO’s cumulative loss $C_t$ that admits Definition 7.8, is given by the subsequent equality:

$$C_t = C^{(-i)th} + \mathcal{D}_i$$

Proof:

It is enough to observe the equality $\sum_{i=1}^n \mathcal{D}_i \mathcal{N}_t^i = \sum_{i \neq j} \mathcal{D}_i \mathcal{N}_t^i + \mathcal{D}_j$.

In a CDO, cumulative loss of the reference credit portfolio are fragmented by some thresholds. These fragmented parts are called the tranches. This concept is expressed in the following assumption with the assumption of having three tranches, i.e. equity mezzanine, and senior, where incorporating more tranches is a straightforward process.

Assumption 7.16 (CDO Tranches)

The CDO consist of three tranches equity mezzanine, and senior. All with two berried thresholds points:

i. The attachment point $S_A$

ii. The detachment point $S_B$
where $0 \leq S_A \leq S_B \leq \sum_{i=1}^{n} K_i$ and $K_i$ is the nominal amount that follows Definition 7.3.

Taking into consideration the cumulative default loss definition and the CDO tranches assumption, the subsequent tranche cumulative default loss could be expressed as in the following definition.

**Definition 7.10 (Tranche Cumulative Loss $C^{[S_A, S_B]}_t$)**

Let $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $C^{(-i)th}_t$ be the CDO’s cumulative loss excluded company $i$ that admits Definition 7.9 and follows Lemma 7.10, and $D_i$ be the loss given default that admits Definition 7.5. Then the non-decreasing function of $C^{(-i)th}_t$’s cumulative loss excluded company $i$, denoted by $C^{[S_A, S_B]}_t$, i.e. $C^{[S_A, S_B]}_t = \omega^{[S_A, S_B]}_t = \omega^{[S_A, S_B]}_t \left( C^{(-i)th}_t + D_i \right)$

With Lemma 7.10 in mind, Definition 7.10 could be rearticulated as in the subsequent definition.

**Definition 7.11 (Excluded i Tranche’s Cumulative Loss $C^{[S_A, S_B]}_{(-i)th}$)**

Let $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, and $D_i$ be the loss given default that admits Definition 7.5, and the $C^{(-i)th}_t$ be the non-decreasing function of CDO’s cumulative loss excluded company $i$.

---

36 See Lemma 3.1 for the definition of the non-decreasing function.
37 $\omega$ is denoting a non-decreasing function.
that admits Definition 7.9 and follows Lemma 7.10. Then $C_t^{(-\ell)^{th}}$, which is berried by $S_A$ and $S_B$ is called the excluded $i$ tranche’s cumulative loss, denoted by $C_t^{[S_A,S_B]}(-\ell)^{th}$, i.e.

$$C_t^{[S_A,S_B]}(-\ell)^{th} = \omega_t^{[S_A,S_B]}(-\ell)^{th}$$

Observing the $CDO$’s analogously to the $(\frac{n}{|\mathcal{R}|})^{th}$ to default $CDS$, introduces the tranches points, where the payments, in the later, depends on the number of defaulted credit entities that are in between a ranking level, and those ranking level in the $CDO$s corresponds to the tranches points.

**Lemma 7.11 (Tranche Cumulative Default Loss $C_t^{[S_A,S_B]}$)**

Let $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $C_t^{[S_A,S_B]}$ be the tranche’s cumulative loss that admits Definition 7.10.

Then $C_t^{[S_A,S_B]}$ is given by the subsequent equality:

$$C_t^{[S_A,S_B]} = (C_t - S_A) \left( I_{[S_A,S_B]}(C_t) \right) + (S_B - S_A) \left( I_{[S_B,\Sigma_i=1^n]}(C_t) \right)$$

**Proof:**

It is enough, where the rest is a straightforward result, to observe subsequent triplet equalities of $C_t^{[S_A,S_B]}$:

$$C_t^{[S_A,S_B]} = \begin{cases} 
0, & \text{if } C_t \leq S_A \\
C_t - S_A, & \text{if } S_A \leq C_t \leq S_B \\
S_B - S_A, & \text{if } C_t \geq S_B 
\end{cases}$$

The default leg of is modelled, in the following, by two methods: the first is achieved by the mean of tranche’s cumulative loss and its first moment, where the second is accomplished when the weights of the referenced credit risk portfolio are not equal.

In order to model the default leg by mean of tranche’s cumulative loss, its first moment is introduced as in the next lemma.
Lemma 7.12 (First Moment of the Tranche’s Cumulative Loss)

Let $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $C^t_{[S_A,S_B]}$ be the tranche’s cumulative loss that admits Definition 7.10 and Lemma 7.11, and $Q_{C_t}^{B,\infty}$ be the distribution of the function $C_t$, where $C_t$ follows Definition 7.8 at the area $]B,\infty[$. Then the first moment of the $C^t_{[S_A,S_B]}$ is given by the subsequent equality:

$$\mathbb{E}^{Q^*}[C^t_{[S_A,S_B]}] = (S_B - S_A)Q_{C_t}^{B,\infty} + \int_{S_A}^{S_B} (x - S_A) dQ_{C_t}^X$$

With Lemma 7.12 in hand, the CDO’s discounted default payment could be articulated as follow.

Lemma 7.13 (CDO’s Discounted Default Payment Leg $\mathcal{D}L$)

Let $B_t$ be the discount factor that follows Definition 7.7, $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $f_t^w$ be the spot forward rate that admits Assumption 7.9, and $Q_{C_t}^{B,\infty}$ be the distribution of the function $C_t$, where $C_t$ follows Definition 7.8 at the area $]B,\infty[$. Then the discounted default payment, denoted by $\mathcal{D}L$, is given by the subsequent equality:

$$\mathcal{D}L = B_T \left( (S_B - S_A)Q_{C_t}^{B,\infty} + \int_{S_A}^{S_B} (x - S_A) dQ_{C_t}^X \right)$$

$$+ \int_0^T f_t^w B_t \left( (S_B - S_A)Q_{C_t}^{B,\infty} + \int_{S_A}^{S_B} (x - S_A) dQ_{C_t}^X \right) dt$$

Proof:

In view of Definition 7.10 and $C^t_{[S_A,S_B]}$ as a non-decreasing process, it is possible to define the $\mathcal{D}L$, sequentially, by employing Stieltjes integrals with respect to $C^t_{[S_A,S_B]}$. 

operating the Stieltjes integration by parts formula, exploiting Fubini theorem, and concluding by the result of Lemma 7.12, as the chain of equality show:

$$
\mathcal{DL} = \mathbb{E}^Q \left[ \int_0^T B_t dC_t^{[S_d, S_B]} \right] = \mathbb{E}^Q \left[ B_T C_t^{[S_d, S_B]} + \int_0^T f_T^w B_t C_t^{[S_d, S_B]} dt \right] = B_T \mathbb{E}^Q \left[ C_t^{[S_d, S_B]} \right] + \int_0^T f_T^w B_t \mathbb{E}^Q \left[ C_t^{[S_d, S_B]} \right] dt = B_T \left( (S_B - S_d) Q^{B, \infty}_{C_T}^{S_B} + \int_{S_d}^{S_B} (x - S_d) d Q^{x}_{C_T} \right) + \int_0^T f_T^w B_t \left( (S_B - S_d) Q^{B, \infty}_{C_T}^{S_B} + \int_{S_d}^{S_B} (x - S_d) d Q^{x}_{C_T} \right) dt
$$

In following a second pricing approach is proposed, which emphasizes the contribution of different names to the default leg.

**Lemma 7.14 (CDO’s Discounted Default Payment Leg \(\mathcal{DL}\))**

Let \(B_t\) be the discount factor that follows Definition 7.7, \(S_d\) and \(S_B\) be, respectively, the attachment and detachment points those follow Assumption 7.16, \(C_t^{[S_d, S_B]}\) and \(C_{(t-j)^{th}}^{[S_d, S_B]}\) respectively, be the tranche’s cumulative loss between the \(S_d\) and \(S_B\) that admits Definition 7.8 and follows Lemma 7.11, and the excluded i tranche’s cumulative loss between the \(S_d\) and \(S_B\) that admits Definition 7.10 and Definition 7.11 and follows Lemma 7.11, \(\mathcal{N}_t^i\) be a default process associated with the default credit entity i that admits Definition 4.12, and \(p^i_{t|\mathcal{M}_t}\) be the probability of default conditioned upon the Systematic Market Risk Factor \(\mathcal{M}_t\) that follows Corollary 5.11. Then the discounted default payment, denoted by \(\mathcal{DL}\), is given by the subsequent equality:

$$
\mathcal{DL} = \sum_{j=1}^n \mathbb{E}^Q \left[ \int_0^T B_t \mathbb{E}^Q \left[ C_t^{[S_d, S_B]} - C_{(t-j)^{th}}^{[S_d, S_B]} \right] | \mathcal{M}_t \right] dp^i_{t|\mathcal{M}_t}
$$
Proof:

In view of Definition 7.10, where \( \mathcal{C}_t^{\{S_A,S_B\}} = \omega^{\{S_A,S_B\}}_t \), and following Lemma 7.6 it is possible to discretise \( \int_0^T B_t d\mathcal{C}_t^{\{S_A,S_B\}} \) to \( \sum_{j=1}^{n} B_{T_j} N_i \left( C_{T_j}^{\{S_A,S_B\}} - C_{\tau^{(-j)}(j)}^{\{S_A,S_B\}} \right) \). Taking into consideration the independence between the interest rates and recovery rates of Assumption 7.1, and iterated expectation theorem on the random time \( \tau_i \) being conditionally independent on \( \mathcal{M}_t \) from Assumption 5.5, and finally by using the transfer theorem, i.e. integrating over the conditional distribution of \( \tau_j \), the chains of equalities hold:

\[
\mathcal{D}_L = \mathbb{E}^{\mathbb{Q}^*} \left[ \sum_{j=1}^{n} B_{T_j} N_i \left( C_{T_j}^{\{S_A,S_B\}} - C_{\tau^{(-j)}(j)}^{\{S_A,S_B\}} \right) \right] \\
= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}^*} \left[ B_{T_j} N_i \left( C_{T_j}^{\{S_A,S_B\}} - C_{\tau^{(-j)}(j)}^{\{S_A,S_B\}} \right) \right] \mathcal{M}_t, \tau_j = t \\
= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}^*} \left[ \int_{\tau^{(-j)}(j)}^T B_t \mathbb{E}^{\mathbb{Q}^*} \left[ C_{t}^{\{S_A,S_B\}} - C_{\tau^{(-j)}(j)}^{\{S_A,S_B\}} \right] \mathcal{M}_t \right] d\mathbb{P}_{\tau_j | \mathcal{M}_t} \\
= \sum_{j=1}^{n} \mathbb{E}^{\mathbb{Q}^*} \left[ \int_{0}^{T} B_t \mathbb{E}^{\mathbb{Q}^*} \left[ C_{t}^{\{S_A,S_B\}} - C_{\tau^{(-j)}(j)}^{\{S_A,S_B\}} \right] \mathcal{M}_t \right] d\mathbb{P}_{\tau_j | \mathcal{M}_t} \\

\]

After modelling the \( \mathcal{CDO} \)’s discounted default payment leg and with the aim to evaluate the \( \mathcal{CDO} \)’s periodic premium, the need to model the \( \mathcal{CDO} \)’s premium payment leg is important.

**Lemma 7.15 (Premium Payment Leg \( \mathcal{P}_L \))**

Let \( \tau_{\Delta} \) be the payment length that follows Definition 7.2, \( \mathcal{P}_\Delta \) be the \( \mathcal{CDO} \) periodic premium that admits Definition 7.6 and follows Assumption 7.15, \( B_{\tau_i} \) be the discount factor that follows Definition 7.7, \( S_A \) and \( S_B \) be, respectively, the attachment and detachment points those follow Assumption 7.16, \( \mathcal{C}_t^{\{S_A,S_B\}} \) be the tranche’s cumulative...
loss between the $S_A$ and $S_B$ that admits Definition 7.10 and follows Lemma 7.11, then the premium payment leg, denoted by $\mathcal{P}L$, is given by summing over all possible premium payment dates as stated in the subsequent equality:

$$\mathcal{P}L = \mathcal{P}_\Delta \sum_{\ell=1}^{L} B_{t_\ell} t_{\Delta_\ell} \mathbb{E}^Q \left[ S_B - S_A - C_t^{[S_A, S_B]} \right]$$

Proof:

In view of Lemma 7.11 the tranche’s outstanding nominal is equal to the tranches’ initial nominal excluding the tranche’s cumulative loss, i.e. $(S_B - S_A) - C_t^{[S_A, S_B]}$, taking into consideration the independence between the interest rates and recovery rates of Assumption 7.1, given that the random time $\tau_i$ being conditionally independent on $\mathcal{M}_t$, from Assumption 5.5, and finally by $t_\ell$ as the premium payment dates that follows Definition 7.1, with $t_0 = 0$, the bi-equalities hold:

$$\mathcal{P}L = \mathbb{E}^Q \left[ \mathcal{P}_\Delta \sum_{\ell=1}^{L} B_{t_\ell} t_{\Delta_\ell} \left( S_B - S_A - C_t^{[S_A, S_B]} \right) \right]$$

$$= \mathcal{P}_\Delta \sum_{\ell=1}^{L} B_{t_\ell} t_{\Delta_\ell} \mathbb{E}^Q \left[ S_B - S_A - C_t^{[S_A, S_B]} \right]$$

As mentioned in the introduction, the assumption of no accrued payments, which was articulated in Assumption 7.10, is relaxed and thus modelling the accrued payment leg is another important element to evaluate the CDO’s periodic premium.

The accrued leg could be modelled by the same two methods used to model the default leg. The first is accomplished when the weights of the referenced credit risk portfolio are not equal, where the second is achieved by the mean of tranche’s cumulative loss and its first moment.

Lemma 7.16 (Accrued Payment Leg $\mathcal{A}L$)

Let $\mathcal{P}_\Delta$ be the CDO periodic premium that admits Definition 7.6 and follows Assumption
7.15, $\mathcal{B}_t$, be the discount factor that follows Definition 7.7, $t_{k_t}$ be the payment date immediately before $\tau_t$, where $\tau_t \in [t_{k_{t-1}}, t_{k_t}]$. $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16. $C_t^{[S_A,S_B]}$ and $C_t^{[S_A,S_B]}(e^{-i j h})$, respectively, be the tranche’s cumulative loss between the $S_A$ and $S_B$ that admits Definition 7.8 and follows Lemma 7.11, and the excluded i tranche’s cumulative loss between the $S_A$ and $S_B$ that admits Definition 7.10 and Definition 7.11 and follows Lemma 7.11, $N_i$ be a default process associated with the default credit entity i that admits Definition 4.12, and $p_{\xi_i}^{\xi_i} | \mathcal{M}_t$ be the probability of default conditioned upon the Systematic Market Risk Factor $\mathcal{M}_t$ that follows Corollary 5.11. Then the accrued payment leg, denoted by $\mathcal{A}_t$, is given by the subsequent equality:

$$\mathcal{A}_t = \mathcal{P}_\Delta \sum_{i=1}^{n} \mathbb{E}^{\mathcal{Q}^*} \left[ \sum_{t=1}^{L} \mathcal{B}_t \left( t - t_{t-1} \right) \mathbb{E}^{\mathcal{Q}^*} \left[ C_t^{[S_A,S_B]}(e^{-i j h}) \right] \bigg| \mathcal{M}_t \right] d p_{\xi_i}^{\xi_i} | \mathcal{M}_t$$

**Proof:**

In view of Lemma 7.12 and Lemma 7.15 and their proof, given Definition 7.10 where $C_t^{[S_A,S_B]} = \omega C_t^{[S_A,S_B]}$, taking into consideration the independence between the interest rates and recovery rates of Assumption 7.1, given that the random time $\tau_i$ being conditionally independent on $\mathcal{M}_t$ from Assumption 5.5, and finally by integrating over the conditional distribution of $\tau_i$, the chain of equalities hold:

$$\mathcal{A}_t = \mathbb{E}^{\mathcal{Q}^*} \left[ \sum_{i=1}^{n} \mathcal{B}_t \left( t - t_{t-1} \right) \mathbb{E}^{\mathcal{Q}^*} \left[ C_t^{[S_A,S_B]}(e^{-i j h}) \right] \bigg| \mathcal{M}_t \right]$$

$$= \mathcal{P}_\Delta \mathbb{E}^{\mathcal{Q}^*} \left[ \sum_{i=1}^{n} \mathcal{B}_t \left( t - t_{t-1} \right) \mathbb{E}^{\mathcal{Q}^*} \left[ C_t^{[S_A,S_B]}(e^{-i j h}) \right] \bigg| \mathcal{M}_t \right]$$

$$= \mathcal{P}_\Delta \sum_{i=1}^{n} \mathbb{E}^{\mathcal{Q}^*} \left[ \sum_{t=1}^{L} \mathcal{B}_t \left( t - t_{t-1} \right) \mathbb{E}^{\mathcal{Q}^*} \left[ C_t^{[S_A,S_B]}(e^{-i j h}) \right] \bigg| \mathcal{M}_t \right] d p_{\xi_i}^{\xi_i} | \mathcal{M}_t$$
The above lemma could be seen analogously to Lemma 7.14, where the next could be seen analogously to Lemma 7.13. The next lemma makes use of the tranche’s first moment cumulative.

**Lemma 7.17 (Accrued Payment Leg $\mathcal{AL}$)**

Let $P_{\Delta}$ be the CDO periodic premium that admits Definition 7.6 and follows Assumption 7.15, $B_t$ be the discount factor that follows Definition 7.7, $f^w_t$ be the spot forward rate that admits Assumption 7.9, $t_{kj}$ be the payment date immediately before $\tau_j$, where $\tau_j \in [t_{k_{j-1}}, t_{kj}]$, $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $C_t^{[S_A, S_B]}$ be the tranche’s cumulative loss between the $S_A$ and $S_B$ that admits Definition 7.9 and follows Lemma 7.11, and $\mathbb{Q}^{[B, \infty]}_{C_t}$ be the distribution of the function $C_t$, where $C_t$ follows Definition 7.8 at the area $[B, \infty]$. Then the accrued payment leg, denoted by $\mathcal{AL}$, is given by the subsequent equality:

$$
\mathcal{AL} = P_{\Delta} \sum_{i=1}^{n} \left[ B_t(t_i - t_{i-1}) \left( (S_B - S_A)\mathbb{Q}^{[B, \infty]}_{C_t} + \int_{S_A}^{S_B} (x - S_A) d\mathbb{Q}^{x}_{C_t} \right) \right.

- \left. \int_{t_{i-1}}^{t_i} B_t(f^w_t(t - t_{i-1}) + 1) \left( (S_B - S_A)\mathbb{Q}^{[B, \infty]}_{C_t} + \int_{S_A}^{S_B} (x - S_A) d\mathbb{Q}^{x}_{C_t} \right) dt \right]
$$

**Proof:** (Similar to Lemma 7.13)

In sequence, the CDO’s periodic payment is also modelled by the same two methods used to model the default leg and accrued leg. The first is accomplished when the weights of the referenced credit risk portfolio are not equal, where the second is achieved by the mean of tranche’s cumulative loss and its first moment.

**Theorem 7.4 (CDO’s Periodic Premium $P_{\Delta}$)**

Let $B_t$ be the discount factor that follows Definition 7.7, $S_A$ and $S_B$ be, respectively, the attachment and detachment points those follow Assumption 7.16, $C_t^{[S_A, S_B]}$ and
\(C_t^{[S_A, S_B]}(\ell(-i)^{th})\), respectively, be the tranche’s cumulative loss between the \(S_A\) and \(S_B\) that admits Definition 7.8 and follows Lemma 7.11, and the excluded \(i\) tranche’s cumulative loss between the \(S_A\) and \(S_B\) that admits Definition 7.10 and Definition 7.11 and follows Lemma 7.11, \(N_t^i\) be a default process associated with the default credit entity \(i\) that admits Definition 4.12, \(p_t^{\xi_i|M_t}\) be the probability of default conditioned upon the Systematic Market Risk Factor \(M_t\) that follows Corollary 5.11, \(\Delta\) be the payment length that follows Definition 7.2, \(\mathcal{P}_\Delta\) be the CDO periodic premium that admits Definition 7.6 and follows Assumption 7.15, \(t_k\ell\) be the payment date immediately before \(\tau_\ell\), where \(\tau_\ell \in [t_{k-1}, t_k]\), \(S_A\) and \(S_B\) be, respectively, the attachment and detachment points those follow Assumption 7.16, \(D_L\) be the default payment leg that admits Lemma 7.14, \(\mathcal{P}_L\) be premium payment leg that admits Lemma 7.15, and \(\mathcal{A}_L\) be the accrued payment leg that admits Lemma 7.16. Then the \(\mathcal{P}_\Delta\) is given by the subsequent equality:

\[
\mathcal{P}_\Delta = \frac{\sum_{j=1}^{n_\ell} E^Q \left[ \int_0^{\tau_\ell} B_{t\ell} E^Q \left[ C_t^{[S_A, S_B]} - c_t^{[S_A, S_B]}(\ell(-j)^{th}) \right] \left| M_t \right| dp_t^{\xi_i|M_t} \right] }{\sum_{j=1}^{n_\ell} \left( \sum_{i=1}^{7} \left( B_t^{\ell \Delta} E^Q \left[ S_B - S_A - \mathcal{Q}_t^{[S_A, S_B]} \right] + E^Q \left[ \int_{t_{k-1}}^{\tau_\ell} B_t(t-t_{k-1}) E^Q \left[ c_t^{[S_A, S_B]}(\ell(-j)^{th}) \right] \left| M_t \right| dp_t^{\xi_i|M_t} \right) \right) }}
\]

The other method that the CDO’s periodic premium could be evaluated by is articulated in the subsequent theorem.

**Theorem 7.5 (CDO’s Periodic Premium \(\mathcal{P}_\Delta\))**

Let \(B_t\) be the discount factor that follows Definition 7.7, \(f_t^w\) be the spot forward rate that admits Assumption 7.9, \(S_A\) and \(S_B\) be, respectively, the attachment and detachment points those follow Assumption 7.16, \(C_t^{[S_A, S_B]}\) be the tranche’s cumulative loss between the \(S_A\) and \(S_B\) that admits Definition 7.9 and follows Lemma 7.11, \(Q_t^{[S_A, S_B]}\) be the distribution of the function \(C_t\), where \(C_t\) follows Definition 7.8 at the period \(]B, \infty[\), \(N_t^i\)
be a default process associated with the default credit entity $i$ that admits Definition 4.12, $p_{\xi_i}^{M_t}$ be the probability of default conditioned upon the Systematic Market Risk Factor $M_t$ that follows Corollary 5.11, $t_{\Delta t}$ be the payment length that follows Definition 7.2, $P_{\Delta}$ be the CDO periodic premium that admits Definition 7.6 and follows Assumption 7.15, $t_{k_t}$ be the payment date immediately before $\tau_t$, where $\tau_j \in [t_{k_{t-1}}, t_{k_t}]$.

$D_L$ be the default payment leg that admits Lemma 7.14, $P_L$ be premium payment leg that admits Lemma 7.15, and $\Delta L$ be the accrued payment leg that admits Lemma 7.17.

Then the $P_{\Delta}$ is given by the subsequent equality:

$$P_{\Delta} =$$

$$\sum_{i=1}^{n} E^{Q_t} \left[ \sum_{t_{\Delta t}} B_t \mathbb{E}^{Q_t} \left[ \frac{\xi_i^{S_A,S_B}}{(t_{\Delta t})^{M_t}} \right] \right]$$

$$\sum_{i=1}^{n} \left( B_t \mathbb{E}^{Q_t} \left[ S_B - S_A - C_{t_{\Delta t}}^{S_A,S_B} \right] + B_t \mathbb{E}^{Q_t} \left[ C_{t_{\Delta t}}^{S_A,S_B} \right] \right)$$

Where $E^{Q_t} [C_t^{S_A,S_B}] = (S_B - S_A) Q_{C_t}^t + \int S_B(x - S_A) dQ_{C_t}^x$

### 7.7 Mathematical Summary

#### The $k^{th}$ to Default CDS Periodic Premium $P_{\Delta}$

$$P_{\Delta} = \left( \frac{1-\delta}{1-G^f_B t + \int_0^T f_t^w B_t G^k_t dt} \right) \left( \frac{1}{\sum_{t_{\Delta t}} B_t \mathbb{E}^{Q_t} \left( N_t=k \right)} \right)$$

#### The Homogeneous $\left( \left( \frac{R_{m,R_M}}{R_{m,R_M}} \right) \right)^{th}$ to Default CDS Periodic Premium $P_{\Delta}$

$$P_{\Delta} = \left( D \left( 1 - G^f_B t + \int_0^T f_t^w B_t G^k_t dt \right) \right) \left( \frac{1}{\sum_{t_{\Delta t}} B_t \mathbb{E}^{Q_t} \left( N_t=k \right)} \right)$$

#### The Non-Homogeneous $\left( \left( \frac{R_{m,R_M}}{R_{m,R_M}} \right) \right)^{th-to-default}$ CDS Periodic Premium $P_{\Delta}$

$$P_{\Delta} = \left( D \left( 1 - G^f_B t + \int_0^T f_t^w B_t G^k_t dt \right) \right) \left( \frac{1}{\sum_{t_{\Delta t}} B_t \mathbb{E}^{Q_t} \left( N_t=k \right)} \right)$$

Chapter Seven: Pricing Basket Default Swap and Collateralised Debt Obligation  Page 296
CDO’s Periodic Premium $\mathcal{P}_\Delta$

$$
\mathcal{P}_\Delta = \frac{\sum_{t=1}^{n} \mathbb{E}_c^\mathbb{Q} \left[ \int_{t_0}^{T} \mathbb{E}_c^\mathbb{Q} \left[ c_t^{[\mathcal{S}_t, \mathcal{S}_h]} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} \right] \mathcal{M}_t dP_{t_i} \right] \right]}{\sum_{t=1}^{n} \left( \sum_{f=1}^{F} (B_{t_i}^{t_f} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} + \mathbb{E}_c^\mathbb{Q} \left[ c_t^{[\mathcal{S}_t, \mathcal{S}_h]} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} \right] \mathcal{M}_t dP_{t_i} \right) \right)}
$$

Or

$$
\mathcal{P}_\Delta = \frac{\sum_{t=1}^{n} \mathbb{E}_c^\mathbb{Q} \left[ \int_{t_0}^{T} \mathbb{E}_c^\mathbb{Q} \left[ c_t^{[\mathcal{S}_t, \mathcal{S}_h]} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} \right] \mathcal{M}_t dP_{t_i} \right] \right]}{\sum_{t=1}^{n} \left( \sum_{f=1}^{F} (B_{t_i}^{t_f} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} + \mathbb{E}_c^\mathbb{Q} \left[ c_t^{[\mathcal{S}_t, \mathcal{S}_h]} - c_t^{[\mathcal{S}_t, \mathcal{S}_h]} \right] \mathcal{M}_t dP_{t_i} \right) \right)}
$$

Where $\mathbb{E}_c^\mathbb{Q} \left[ c_t^{[\mathcal{S}_t, \mathcal{S}_h]} \right] = (\mathcal{S}_B - \mathcal{S}_A) \mathbb{Q}^{[B, \infty]} + \int_{\mathcal{S}_t}^{\mathcal{S}_B} (\mathcal{S}_B - \mathcal{S}_A) d\mathbb{Q}_c$.
Chapter Eight

Numerical Evaluation via DFT’s Fast (FFT) and Very Fast (VFFT) Forms

8.1 Outline

- **Introduction**
- **Discrete, Fast, and Very Fast Fourier Transform**
  - Discrete Fourier Transform
  - Fast Fourier Transform
  - Very Fast Fourier Transform
  - Comparing DFT, FFT, and VFFT
- **Characteristic Functions’ of Number of Defaults, Cumulative Loss, and Loss Given Default**
  - Number of Defaults’ Characteristic Function
  - Cumulative loss Characteristic Function
  - Loss Given Default’s Characteristic Function
- **Numerical Evaluation via Discrete, Fast, Very Fast Fourier Transform**
  - Extracting the Number of Default’s by the Inverse Fourier Transform and the DFT
  - Evaluating the Number of Defaults’ Characteristic Function
  - Extracting the Cumulative Loss’s by the Inverse Fourier Transform and the DFT
  - Evaluating the Cumulative Loss’s Characteristic Function
  - Evaluating the Loss Given Default’s Characteristic Function
- **Mathematical Summary**
8.2 Introduction

In general, the need for fast and efficient numerical methods in finance is essential due to the complexity of measuring and pricing risks in a contingent claims framework, and the difficulties of solving the dimensionality problem. Credit derivatives contain both problems, especially, the latter problem which stems from the fact that many credit products consist of a large number of underlying assets, for example, CDOs.

As mentioned in the introduction of Chapter 5, the market standard model, the Gaussian Factor Copula model, is compatible for analytical computation of loss distributions, and in sequence, the Lévy Factor Copula Model does. The foremost feature of this model is that the default events are independent, conditionally on some latent state variables. This simplifies the computation of aggregate loss distributions due to dimensionality reduction. This factor method is harmonious for huge dimensional problems.

Recently the Discrete Fourier Transform, which is abbreviated as $\mathcal{DFT}$, and more precisely its fast form; $(\mathcal{FFT})$, has captured the attention of researchers and practitioners seeking the high-speed evaluation of complex and large financial pricing problems. Increasingly researchers are employing the characteristic function, which is calculated by the $\mathcal{FFT}$, in order to price European option contracts. Since the influential paper of (Heston, 1993) on stochastic volatility, chronological papers has improved the use and practice of the $\mathcal{FFT}$ in this field with the purpose of overcoming the complexity of measuring and pricing it with a fast and stable/reliable numerical method, (see (Bates, 1998), (Madan et al., 1998), (Duffie et al., 2000), (Bakshi and Madan, 2000), (Heston and Nandi, 2000) and (Carr and Wu, 2004)). Where (Dempster and Hong, 2002) has proved, by the mean of the $\mathcal{FFT}$ that the difficulties of solving the dimensionality problem is not a problematic issue anymore.
The successful of implementing the $\mathcal{FFT}$ in Option pricing has enthused the practitioners and scholars to extend this technique to other financial instruments. (Gregory and Laurent, 2003), (Hull and White, 2004), (Laurent and Gregory, 2005) and (Mortensen, 2005) have implemented the $\mathcal{FFT}$, correspondingly, in credit derivatives aspect, but it was mentioned implicitly. Except in (Debuysscher, 2003, Debuysscher and Szegö, 2003a, Debuysscher and Szegö, 2003b, Debuysscher and Szegö, 2005) it was implemented in different way.

This chapter, mainly, have three directions; the first proposes explicitly the implementation of the $\mathcal{DFT}$ and its fast and very fast forms, where it starts by the Fourier matrix, which is the $\mathcal{DFT}$ base, and defining the $\mathcal{DFT}$ and its numerical complexity, in section 8.3.1. In sequence, two forms of the $\mathcal{DFT}$ will be considered: the first is $\mathcal{FFT}$ (Cooley and Tukey, 1965), in section 8.3.2, where the second is the proposed and recommended form, i.e. the very fast form of $\mathcal{DFT}$ called Very Fast Fourier Transform ($\mathcal{VFFT}$) (Shepherd et al., 2003), that is used in engineering fields, in section 8.3.3. This subsection will be concluded by a numerical complexity and accuracy comparison between the three forms mentioned above, in section 8.3.4.

The second is describing and proving the Linearly Correlated, Stochastically Correlated, and Randomly Loaded Factor Copula characteristic functions of the number of defaults, in section 8.4.1, the accumulated loss, in section 8.4.2, and proposing some new loss given default, in section 8.4.3. Computing these characteristic functions are essential to evaluate the basket CDS, i.e. $n^{th}$ to default $\mathcal{CDS}$ and $\left(\frac{n}{\mathcal{R}_m \mathcal{R}_d}\right)^{th}$ to default $\mathcal{CDS}$, and the $\mathcal{CDO}$. Where extracting the number of defaults distribution, the accumulated loss, and the loss given defaults from their corresponding characteristic functions by the $\mathcal{DFT}$ is the last direction that this section is containing.
Describing how the characteristic functions, mentioned previously, are numerically evaluated by the \( \mathcal{DFT} \) is expressed first. This part will be illustrated in section 8.5 on the number of defaults in the Lévy Factor copula Model and sequentially it is pulled over to numerically evaluate the extended models, specifically Lévy Binary Stochastic Correlated Factor Copula Model, Lévy Symmetric Stochastic Correlated Factor Copula Model, and Lévy Random Factor Loading Copula Model.

### 8.3 Discrete, Fast, and Very Fast Fourier Transform

The Discrete Fourier Transform (\( \mathcal{DFT} \)) is one of the most important mathematical techniques in the 20\(^{th}\) century. \( \mathcal{DFT} \) was written as a transform on the orthogonal basis functions by Gauss 1805.

Its notability came after the publication of the Cooley-Tukey algorithm in (Cooley and Tukey, 1965). Its name (Cooley, 1987), the need of an efficient algorithm to numerically evaluate this transform (Cooley, 1987), the existence of the digital computers, the error reduction it provides (Gentleman and Sande, 1966), and more significantly decreasing the numerical complexity from \( N^2 \) to \( (N \log_2 (N)) \) have raised its importance. Cooley-Tukey algorithm is known today as the Fast Fourier Transform.

Nevertheless, many algorithms and factorisation methods were used since Gauss work in 1805, which was published in (Gauss, 1866), until the born of the \( \mathcal{FFT} \). It has many keystones starting from Gauss in 1805 studied a matrix of any composite integer (Gauss, 1866), in (Carlini, 1828) a matrix of size 12 was studied, in (Smith and Sabine, 1846) matrices of size 4, 8, 16, and 32 were investigated, in (Everett, 1860) a matrix of size 12 was examined, in (Danielson and Lanczos, 1942) a matrix of size \( 2^n \) was studied, in 1948 matrices of size any integer with relatively prime factors was studied in (Thomas, 1963), in the same direction it was studied in (Good, 1958), and finally the
revolutionary study in (Cooley and Tukey, 1965) for a matrix of any composite integer size.

However, as articulated in (Heideman and Burrus, 1984) that there is more than 2500 titles after (Cooley and Tukey, 1965) has been published. In these publications the intention was to implement the Fourier transform in a way that reduces its complexity, i.e. based of 2, 4, 8, or prime and as a convolution, in parallel, or in series, as well as any combination of them. However, the original $\mathcal{FFT}$ complexity has not been defeated by a general form (Shepherd et al., 2003). In spite of this, in (Winograd, 1977) and (Winograd, 1978) an optimised algorithm to calculate the $\mathcal{FFT}$ that could be considerably faster, i.e. up to a factor of 2 (Press et al., 2007), has been launch. Its limitation is that it computes small $\mathcal{FFT}$'s sizes only.

An acknowledged study was carried out in (Shepherd et al., 2003) and followed in (Zhou, 2006) and (Linardatos, 2008) on the Fourier matrix of any even size $N$, conditional on $N$ being not divisible by the square of an odd prime. This factorisation leads to a linear complexity of the Fourier transform, i.e. $3N$ (Shepherd et al., 2003), and called the Very Fast Fourier Transform ($\mathcal{VFFT}$).

In this subsection, the $\mathcal{DFT}$, the $\mathcal{FFT}$, and the $\mathcal{VFFT}$ will be presented with a limited scope, to be precise their definition, matrix representation and complexity. Finally comparing their accuracy, complexity, and speed among each other are articulated.

### 8.3.1 Discrete Fourier Transform

The $\mathcal{DFT}$ was written as a transform on the orthogonal basis functions by Gauss 1805.

In this subsection the original Fourier matrix and the $\mathcal{DFT}$ will be presented in order to compare it with the $\mathcal{FFT}$ and $\mathcal{VFFT}$.
Definition 8.1 (Fourier Matrix $\mathcal{F}_N$)

Let $\mathcal{F}_N$ be an $N \times N$ complex matrix, $\left(\frac{j2\pi}{N}\right)$ be the $N^{th}$ root of unity, and $j = \sqrt{-1}$. Then if its $\mathcal{F}_{(g,r)}$ entry is given by $e^{(g)(r)\left(-\frac{j2\pi}{N}\right)}$, $\mathcal{F}_N$ is called the Fourier Matrix, i.e.

$$\mathcal{F}_N = \begin{bmatrix}
e^{(0)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(0)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(0)(N-1)\left(-\frac{j2\pi}{N}\right)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
e^{(g)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(g)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(g)(N-1)\left(-\frac{j2\pi}{N}\right)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
e^{(N-1)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(N-1)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(N-1)(N-1)\left(-\frac{j2\pi}{N}\right)}
\end{bmatrix}$$

A direct injection of the Fourier matrix presents the $\mathcal{DFT}$, where it could be used to transform a real or complex sequence.

Theorem 8.1 (Discrete Fourier Transform $\mathcal{DFT}$)

Let $[\varphi]_0^{N-1}$ be an $N$ complex sequence, $\mathcal{F}_N$ be an $N \times N$ Fourier Matrix that admits Definition 8.1. Then $[f]_0^{N-1}$ is its transformed sequence by the mean of Discrete Fourier Transform, which is denoted by $\mathcal{DFT}$. It could be represented by:

- **Sequence representation:** $[f_g]_0^{N-1} = \mathcal{DFT}[\varphi_r]_0^{N-1} = \sum_{r=0}^{N-1} \varphi_r e^{gr\left(-\frac{j2\pi}{N}\right)}$

- **Matrix representation:** $f = \mathcal{F}_N \cdot \varphi$ or

$$\begin{bmatrix}f_0 \\
f_g \\
f_{N-1}\end{bmatrix} = \begin{bmatrix}e^{(0)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(0)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(0)(N-1)\left(-\frac{j2\pi}{N}\right)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
e^{(g)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(g)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(g)(N-1)\left(-\frac{j2\pi}{N}\right)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
e^{(N-1)(0)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(N-1)(r)\left(-\frac{j2\pi}{N}\right)} & \cdots & e^{(N-1)(N-1)\left(-\frac{j2\pi}{N}\right)}
\end{bmatrix} \cdot \begin{bmatrix}\varphi_0 \\
\varphi_g \\
\varphi_{N-1}\end{bmatrix}$$

The Fourier matrix contains an equal spaced phase points on the unit circle. This could be seen in the subsequent Corollary.
Corollary 8.1 (DFT representation of $\mathcal{F}_8$)

Let $\mathcal{F}_8$ be an $8 \times 8$ Fourier matrix, then it could be represented by an 8 equally spaced phase points of $e\left(-\frac{j2\pi}{N}\right)$; as could be seen in the following matrix:

$$
\mathcal{F}_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
i & -1 & -j & 1 & j & -1 & -j & -j \\
1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & -j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \\
1 & -i & -1 & i & 1 & -j & -1 & j \\
1 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -j & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & j & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}
\end{bmatrix}
$$

or in the subsequent figure:

One of the most important issues when utilising an algorithm is how long it will take in order to run this algorithm. Accordingly, calculating the DFT’s complexity is essential.
Theorem 8.2 (DFT Complexity)

Let \([\varphi_r]_0^{N-1}\) be an \(N\) complex sequence, where \([f_g]_0^{N-1}\) be its transformed sequence by the mean of DFT that admits Definition 8.1. Then \([f_g]_0^{N-1} = \mathcal{DFT}[\varphi_r]_0^{N-1}\) complexity is \(O(N^2)\), i.e. \(4N^2\) multiplications and \(N(4N - 2)\) additions.

Proof:

Since \(e^{\frac{j2\pi r}{N}}\) is complex and \(\varphi_r\) could be complex, the computing \(f_g\) could be achieved by the subsequent dual equality:

\[
[f_g]_0^{N-1} = \mathcal{DFT}[\varphi_r]_0^{N-1}
 = \left[ \sum_{r=0}^{N-1} \left[ \left( \operatorname{Re} \left( e^{\frac{j2\pi r}{N}} \right) \right) + i \operatorname{Im} \left( e^{\frac{j2\pi r}{N}} \right) \right] \right]_0^{N-1} \left[ \operatorname{Re}(\varphi_r) + i \operatorname{Im}(\varphi_r) \right]
 = \left[ \sum_{j=0}^{N-1} \left[ \operatorname{Re} \left( e^{\frac{j2\pi j}{N}} \right) \operatorname{Re}(\varphi_r) - \operatorname{Im} \left( e^{\frac{j2\pi j}{N}} \right) \operatorname{Im}(\varphi_r) \right] 
 + \left[ \operatorname{Im} \left( e^{\frac{j2\pi j}{N}} \right) \operatorname{Re}(\varphi_r) + \operatorname{Re} \left( e^{\frac{j2\pi j}{N}} \right) \operatorname{Im}(\varphi_r) \right] \right]_0^{N-1}

Then by observing \(\mathbb{M}\) as the number of multiplications and \(\mathbb{A}\) as the number of additions of the previous double equalities, it could be concluded that:

\[
\mathbb{M} = \left[ \sum_{r=0}^{N-1} [4] \right]_0^{N-1} = [4N]_0^{N-1} = 4N^2
\]

\[
\mathbb{A} = \left[ \sum_{r=0}^{N-1} [4] \right]_0^{N-1} = [4N - 2]_0^{N-1} = N(4N - 2)
\]

In view of Theorem 8.1 Theorem 8.2, it is obvious to observe how unacceptable amount of time is required to compute large DFT's.

8.3.2 Fast Fourier Transform

The FFT, i.e. the Fourier Transform of size \(2^n \times 2^n\), described in (Cooley and Tukey, 1965) is derived from the possibility of reducing the DFT processes to \(\log_2 (N)\) steps,
which means its complexity is reduced to \((N \log_2 (N))\). In other words, the Fourier matrix could be factorised into \((\log_2 (N)+1)\) factors one of them is a permutation matrix and it lies in the real numbers, where the others are sparse and lie in the complex numbers field.

**Theorem 8.3 (FFT)**

Let \(\{\varphi_r\}_{0}^{N-1}\) be an \(N\) complex sequence, \(F_N\) be a Fourier matrix, which admits Definition 8.1, of an even size \(N \times N\), conditional on \(N = 2^E\). Then \(\{\varphi_r\}_{0}^{N-1}\) could be transformed \(\{f_g\}_{0}^{N-1}\) by the mean of the Fast Fourier Transform, denoted by \(\mathcal{F}\), by decomposing \(F_N\) into two sequences recursively until it reaches the binary level.

- **Matrix representation:** \(f = F_N \cdot \varphi\), where \(F_N = \mathcal{A}\{0\} \cdots \mathcal{A}\{s\} \cdot \mathcal{P}\), \(\mathcal{A}\{\cdot\}\) is factorisation of \(F_N\), and \(\mathcal{P}\) is its permutation matrix.

- **Sequence representation:**

\[
\begin{align*}
\{f_g\}_{0}^{N-1} &= \mathcal{DFT}\{\varphi_r\}_{0}^{N-1} \\
&= \left[ \sum_{r=0}^{N-1} \varphi_r \ e^{g r (-\frac{j 2 \pi}{N})} \right]_{0}^{N-1} \\
&= \left[ \sum_{r=0}^{\frac{N}{2}} \varphi_{(2r)} \ e^{g (2r) \left(-\frac{j 2 \pi}{N}\right)} + \sum_{r=0}^{\frac{N}{2}} \varphi_{(2r+1)} \ e^{g (2r+1) \left(-\frac{j 2 \pi}{N}\right)} \right]_{0}^{N-1} \\
&= \left[ \sum_{r=0}^{\frac{N}{2}} \varphi_{(2r)} \ e^{g r \left(-\frac{j 2 \pi}{N}\right)} + e^{g \left(-\frac{j 2 \pi}{N}\right)} \sum_{r=0}^{\frac{N}{2}} \varphi_{(2r+1)} \ e^{g r \left(-\frac{j 2 \pi}{N}\right)} \right]_{0}^{N-1} \\
&= \mathcal{FFT}\{\varphi_r\}_{0}^{N-1}
\end{align*}
\]

Applying this theorem to the Fourier matrix of size 8, could give a first indication of the \(\mathcal{FFT}\) complexity; especially when comparing it to the original representation of the Fourier matrix in Corollary 8.1.
Corollary 8.2 (FFT representation of $\mathcal{F}_8$)

Let $\mathcal{F}_8$ be a Fourier matrix of size $2^3 \times 2^3$ that admits Corollary 8.1. Then by utilising the FFT factorisation of Theorem 8.3, $\mathcal{F}_8 = \mathcal{A}_0 \cdot \mathcal{A}_1 \cdot \mathcal{A}_2 \cdot \mathcal{P}$, where

$$
\mathcal{P} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\mathcal{A}_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -j & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}
\end{bmatrix}
$$

$$
\mathcal{A}_1 = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -j & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -j & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
$$

$$
\mathcal{A}_2 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
$$

The amount of calculation required to multiply the Fourier matrix, which is given in Corollary 8.1, by a sequence is significantly higher than the stepwise multiplication of sequence by the product of factorised Fourier matrix given by Corollary 8.2.

Theorem 8.4 (Complexity of FFT)

Let $[\varphi]^N_{0}^{-1}$ be an N complex sequence, where $[f]^N_{0}^{-1}$ be its transformed sequence by the mean of FFT that admits Definition 8.3. Then $[f_g]^N_{0}^{-1} = \mathcal{F}\mathcal{F}\mathcal{T}[\varphi]^N_{0}^{-1}$ complexity is $O(N \log_2 N)$. 

Chapter Eight: Numerical Evaluation Via DFT’s Fast (FFT) and Very Fast (VFFT) forms  Page 307
Proof:

In view of the sequence representation of Theorem 8.2 and Theorem 8.3, it could be observed that:

- **Step1**: requires $4N^2$ multiplications and $N(4N - 2)$ additions.

- **Step2**: requires $2 \left( \frac{N}{2} \right)^2 + N$ multiplications and $2 \frac{N}{2} \left( \frac{N}{2} - 1 \right) + N$ additions, i.e. requires $N + 2 \left( \frac{N}{2} \right)^2$ multiplications and additions.

- **Step3**: requires $N + 2 \left( \frac{N}{2} + 2 \left( \frac{N}{4} \right)^2 \right)$ multiplications and additions.

- **Step4**: requires $N + 2 \left( \frac{N}{2} + 2 \left( \frac{N}{4} + 2 \left( \frac{N}{8} \right)^2 \right) \right)$ multiplications and additions.

- \vdots

- **Step $\log_2 N$**: requires $N \log_2 N$ multiplications and additions.

8.3.3 Very Fast Fourier Transform

The $\mathcal{VFFT}$, presented in (Shepherd et al., 2003) and expansion followed in (Zhou, 2006), factorises the Fourier matrix into two parts $\mathcal{F}_N = G_N . H_N$, where $G_N$ is complex and encloses the phase point information and $H_N$ is real and includes the amplitude information. $G_N$ and $H_N$ are then factorised recursively.

**Theorem 8.5 ($\mathcal{VFFT}$)**

Let $\mathcal{F}_N$ be a Fourier matrix, which admits Definition 8.1, of an even size $N \times N$, conditional on $N$ being not divided by the square of an odd prime. Then the $\mathcal{VFFT}$ representation of $\mathcal{F}_N$ could achieved by factorising it into two parts:

- $G$ be the complex matrix that contains $\mathcal{F}_N$'s information regarding its phase points; $G$ is recursively regenerated (factorised) into three main parts: its left factor matrix, denoted by $G^L_{(m)}$, its block diagonal matrix $G^D$, and its right factor matrix, denoted by $G^R_{(m)}$. $G$ is factorised until $G^D$ are of size $2 \times 2$, i.e.
\[ G = G_1^L \cdots G_m^L G_n^D G_n^R \cdots G_1^R \]

- \( H \) be the real and sparse matrix that contains \( F_N \)'s information regarding its amplitude. \( H \) is fully diagonalised, block by block separately, i.e.

\[ H = P \cdot H_1^L \cdots H_m^L H_n^D H_n^R \cdots H_1^R \cdot P^T \]

Presenting the \( VFFT \) to the Fourier matrix of size 8 shows the advantage that it has over the \( FFT \) in Corollary 8.2, which in sequence shows it over the original Fourier Transform in Corollary 8.1. For more details on the \( VFFT \) factorisation, the reader is referred to (Shepherd et al., 2003) and (Zhou, 2006).

**Corollary 8.3 (VFFT representation of \( F_8 \))**

Let \( F_8 \) be a Fourier matrix of size \( 8 \times 8 \) that admits Corollary 8.1. Then by utilising the \( VFFT \) factorisation of Theorem 8.5, \( F_8 \) is given by three steps:

1. \( F_8 \) is factorised: \( F_8 = G_8 \cdot H_8 \)

   where

   \[
   G_8 = \begin{bmatrix}
   1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
   1 & 1+j & j & -1+j & -1 & -j & -1-j & 1-j \\
   1 & j & -1 & -j & 1 & j & -1 & -j \\
   1 & -1+j & -j & 1+j & -1 & 1-j & j & -1-j \\
   1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
   1 & -1-j & j & 1-j & 1 & 1-j & -1 & j \\
   1 & -j & 1 & -j & 1 & -j & 1 & -j \\
   1 & 1-j & -j & -1-j & -1 & 1-j & j & 1+j
   \end{bmatrix}
   \]

2. \( G_8 \) is factorised: \( G_8 = G_1^L \cdot G_2^L \cdot G_n^D \cdot G_n^R \cdot G_1^R \)

   \[
   H_8 = \begin{bmatrix}
   1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & \frac{2+\sqrt{2}}{4} & 0 & 0 & 0 & \frac{2-\sqrt{2}}{4} & 0 & 0 \\
   0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & \frac{2+\sqrt{2}}{4} & 0 & 0 & \frac{2-\sqrt{2}}{4} & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
   0 & \frac{2-\sqrt{2}}{4} & 0 & 0 & 0 & \frac{2+\sqrt{2}}{4} & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & \frac{2-\sqrt{2}}{4} & 0 & 0 & 0 & \frac{2+\sqrt{2}}{4}
   \end{bmatrix}
   \]
3. $\mathcal{H}_8$ is factorised: $\mathcal{H}_8 = P \cdot \mathcal{H}_1^L \cdot \mathcal{H}_1^D \cdot \mathcal{H}_1^R \cdot P^T$
By observing the $VFFT$ factorisation of the Fourier matrix in the Corollary 8.3, it is obvious that multiplying a sequence by the $VFFT$ requires significantly smaller number of process than the Fourier matrix given by Corollary 8.1 and its fast form by Corollary 8.2.

**Theorem 8.6 (Complexity of $VFFT$)**

Let $[\varphi]_{0}^{N-1}$ be an $N$ complex sequence, where $[f]_{0}^{N-1}$ be its transformed sequence by the mean of $VFFT$ that admits Theorem 8.5. Then $[f]_{0}^{N-1} = VFFT[\varphi]_{0}^{N-1}$ complexity is $O(3N)$.

**Proof:** (Shepherd et al., 2003) and Figure 2.6.

**8.3.4 Comparing $DFT$, $FFT$, and $VFFT$**

Accuracy, complexity, and stability are three main factors those affects any numerical evaluation. This subsection compares the Fourier matrix represented by $DFT$, $FFT$, and $VFFT$ complexity depending on, respectively, Theorem 8.2, Theorem 8.4, and Theorem 8.6, and accuracy depending on the study done in (Linardatos, 2008). The accuracy investigation was carried out by multiplying 1000 random vectors by three different sizes of the Fourier matrix, specifically 8, 16, and 32. Then this multiplication is inversed and difference between the theoretical output and the original is accumulated by repeating this method 100 times. Finally this error is averaged.
The VFFT is proven to have the minimum complexity and accumulated error, which means that it has the higher speed and accuracy. For complete comparison between them see Table 8.1.

<table>
<thead>
<tr>
<th>Size</th>
<th>VFFT</th>
<th>FFT</th>
<th>DFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>O(3N)</td>
<td>Errors</td>
<td>O(N log₂ N)</td>
</tr>
<tr>
<td>18</td>
<td>2.009</td>
<td>24</td>
<td>3.399 × 10⁻¹⁵</td>
</tr>
<tr>
<td>16</td>
<td>4.86 × 10⁻¹⁵</td>
<td>164</td>
<td>6.241 × 10⁻¹⁵</td>
</tr>
<tr>
<td>32</td>
<td>1.055</td>
<td>1.167 × 10⁻¹⁴</td>
<td>1024</td>
</tr>
</tbody>
</table>

Table 8.1: Comparing the DFT, FFT, and VFFT complexity and accuracy (Linardatos, 2008).

### 8.4 Characteristic Functions’ of Number of Defaults, Cumulative Loss, and Loss Given Default

This section describes and prove the Linearly Correlated, Stochastically Correlated, and Randomly Loaded Factor Copula characteristic functions of the number of defaults, in section 8.4.1, the accumulated loss, in section 8.4.2, and proposing some new loss given default, in section 8.4.3. Computing these characteristic functions are necessary to evaluate the basket CDS, i.e. \( n^{th} \) to default \( CDS \) and \( \left( \frac{n}{m} \right)^{th} \) to default \( CDS \), and the CDO.

#### 8.4.1 Number of Defaults’ Characteristic Function

Computing the probability of \( m \) companies, where \( m \in \mathbb{K} \), out of \( n \) being in default at time \( t \) requires computing \( m \) companies being in default at time \( t \) as an initial step; consecutively to obtain the distribution function from its characteristic function or its corresponding moment generating function. As mentioned previously, when the number
of defaults’ characteristic function is computed, the DFT is used to compute the number of defaults.

**Theorem 8.7 (Lévy Factor Copula Model: Unconditional Number of Default’s Characteristic Function \( \varphi_{N_t} \))**

\( \mathcal{N}_t \) be a default counter process that admits Definition 4.13, \( \mathcal{N}_t^i \) be a default process associated with the default credit entity \( i \) that admits Definition 4.12, \( q_{t_i}^{\xi_i|M_t} \) and \( p_{t_i}^{\xi_i|M_t} \) be, respectively, the conditional survival and default upon the Systematic Market Risk Factor \( \mathcal{M}_t \) in the “Lévy Factor Copula” model that admits Corollary 5.10. Then the unconditional number of default’s characteristic function, denoted by \( \varphi_{N_t} \), is given by:

\[
\varphi_{N_t}(u) = \int \prod_{i=1}^{n} \left( 1 - (1 - e^{ju}) p_{t_i}^{\xi_i|M_t} \right) f_{\mathcal{M}_t}(m) dm
\]

**Proof:**

In view of Theorem 5.4, Corollary 5.11 and Corollary 5.12, \( \mathcal{N}_t^i \) as Bernoulli random variable and conditionally independent, and the iterated expectation theorem, the subsequent chain of equalities hold:

\[
\varphi_{N_t}(u) = \mathbb{E}^{Q^*}[e^{juN_t}] = \mathbb{E}^{Q^*}\left[\mathbb{E}^{Q^*}[u^{N_t^i} | \mathcal{M}_t]\right] = \mathbb{E}^{Q^*}\left[\prod_{i=1}^{n} \left( q_{t_i}^{\xi_i|M_t} + p_{t_i}^{\xi_i|M_t} e^{ju} \right)\right] = \int \prod_{i=1}^{n} \left( q_{t_i}^{\xi_i|m} + p_{t_i}^{\xi_i|m} e^{ju} \right) f_{\mathcal{M}_t}(m) dm = \int \prod_{i=1}^{n} \left( (1 - p_{t_i}^{\xi_i|m}) + p_{t_i}^{\xi_i|m} e^{ju} \right) f_{\mathcal{M}_t}(m) dm = \int \prod_{i=1}^{n} \left( 1 - (1 - e^{ju}) p_{t_i}^{\xi_i|m} \right) f_{\mathcal{M}_t}(m) dm
\]

Alternative procedures could be achieved when the corresponding moment generating function is used. As many researchers, such as (Brunlid, 2006) and (Laurent and...
Gregory, 2005, etc, have stated that for small dimensional problems, specifically when the number of credit entities do not exceed 200, it is possible to utilise the formal expansion of the moment generating function, i.e. $\prod_{i=1}^{n} (q_{t_i}^{\xi_i|M_t} + p_{t_i}^{\xi_i|M_t} \times u)$. The following corollary represents the corresponding moment generating function, while the Lemma that follows is its formal expansion representation.

**Corollary 8.4 (Lévy Factor Copula: Unconditional Number of Default’s Moment Generating Function $\psi_{N_t}$)**

Let $N_t$ be a default counter process, which counts the number of default until time $t$ that admits Definition 4.13, $q_{t_i}^{\xi_i|M_t}$ and $p_{t_i}^{\xi_i|M_t}$ be, respectively, the conditional survival and default upon the Systematic Market Risk Factor $M_t$ in the “Lévy Factor Copula” model that admits Corollary 5.10, then the moment generating function of $N_t$, denoted by $\psi_{N_t}$, is given by the subsequent equality:

$$\psi_{N_t}(u) = \int \prod_{i=1}^{n} (1 - (1 - u)p_{t_i}^{\xi_i|m}) f_{M_t}(m) dm$$

**Lemma 8.1 (Lévy Factor Copula: Unconditional Number of Default’s Formal Expansion $\psi_{k}$)**

Let $N_t$ be a default counter process that admits Definition 4.13, $q_{t_i}^{\xi_i|M_t}$ and $p_{t_i}^{\xi_i|M_t}$ be, respectively, the conditional survival and default upon the Systematic Market Risk Factor $M_t$ in the “Lévy Factor Copula” model that admits Corollary 5.10, and $\psi_{N_t}$ be moment generating function of $N_t$ that follows Corollary 8.4, then the formal expansion of $\psi_{N_t}$, is given by the expression underneath the subsequent equality:

$$\psi_{N_t}(u) = \int \prod_{i=1}^{n} (q_{t_i}^{\xi_i|m} + p_{t_i}^{\xi_i|m} \times u) f_{M_t}(m) dm$$

And accordingly, the probability of $k$ credit entities in default at time $t$ could be
computed by the subsequent equality:

\[ Q^*(N_t = k) = \int \phi_k(m)f_{M_t}(m)dm \]

Proof:

Initially, \( \psi_{N_t}(u) \) could be expanded by the following equality\(^{38}\):

\[
\psi_{N_t}(u) = E[u^{N_t}] = \sum_{k=0}^{n} Q^*(N_t = k)u^k
\]

Next, by observing that if \( \psi_{N_t}^{\xi_k|M_t}(u) = \prod_{i=1}^{n} \left( q_{\xi_i}^{\xi_i|m} + p_{\xi_i}^{\xi_i|m} \times u^i \right) \), then \( \psi_{N_t}^{\xi_{k+1}|M_t}(u) \) is given by:

\[
\psi_{N_t}^{\xi_{k+1}|M_t}(u) = \psi_{N_t}^{\xi_k|M_t}(u) \times \left( q_{\xi_i}^{\xi_{k+1}|m} + p_{\xi_i}^{\xi_{k+1}|m} \times u^{k+1} \right)
\]

Consequently, if \( \phi_k(M_t) \) is assumed to be the polynomial expansions’ coefficients of \( E[u^{N_t}] \), then the default time’s probability distribution function is given by the next equality:

\[ Q^*(N_t = k) = \int \phi_k(m)f_{M_t}(m)dm \]

After introducing the characteristic function of the unconditional number of default in the context of Lévy Factor copula Model, it could be rephrased to incorporate the extended models. It is worth to note that the need for rephrasing the based model in not essential, where the conditional time of default probability could be injected directly in the previous equations. The following three theorems are presented in order to organise the representation and standardise it.

By evoking the equality of the conditional time of default probability distribution in the Lévy Binary Stochastic Correlated Factor Copula Model, Theorem 8.7 could be rephrased as shown in the following theorem.

\(^{38}\) Note that \( Q^*(N_t = 0) = \mathbb{E}_Q^*[\prod_{i=1}^{n} (q_{\xi_i}^{\xi_i|M_t})] \)
Theorem 8.8 (Lévy Binary Stochastic Correlated Factor Copula Model: Unconditional Number of Default’s Characteristic Function $\varphi_{\mathcal{N}_t}$)

Let $\mathcal{N}_t$ be a default counter process that admits Definition 4.13, $p_{\xi_i|\mathcal{M}_t}^{t_i}$ be the conditional default upon the Systematic Market Risk Factor $\mathcal{M}_t$ in the “Lévy Binary Stochastic Correlated Factor Copula” model that follow Lemma 5.8. Then the unconditional number of default’s characteristic function, denoted by $\varphi_{\mathcal{N}_t}$, is given by:

$$\varphi_{\mathcal{N}_t}(u) = \int \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi_i|\mathcal{M}_t}^{t_i, B_i=0} \right) f_{\mathcal{M}_t}(m) dm$$

$$+ \int \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi_i|\mathcal{M}_t}^{t_i, B_i=1} \right) f_{\mathcal{M}_t}(m) dm$$

In the same line, the equality of the conditional time of default probability distribution in Theorem 8.7 could be rearticulated to suite the Lévy Symmetric Stochastic Correlated Factor Copula Model. This could be seen in the following theorem.

Theorem 8.9 (Lévy Symmetric Stochastic Correlated Factor Copula Model: Unconditional Number of Default’s Characteristic Function $\varphi_{\mathcal{N}_t}$)

Let $\mathcal{N}_t$ be a default counter process that admits Definition 4.13, $p_{\xi_i|\mathcal{M}_t}^{t_i}$ be the conditional default upon the Systematic Market Risk Factor $\mathcal{M}_t$ in the “Lévy Symmetric Stochastic Correlated Factor Copula” model that follow Lemma 5.9. Then the unconditional number of default’s characteristic function, denoted by $\varphi_{\mathcal{N}_t}$, is given by:

$$\varphi_{\mathcal{N}_t}(u) = \int \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi_i|\mathcal{M}_t}^{t_i, B_s=1} \right) f_{\mathcal{M}_t}(m) dm$$

$$+ \int \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi_i|\mathcal{M}_t}^{t_i, B_s=0, B_i=0} \right) f_{\mathcal{M}_t}(m) dm$$

$$+ \int \prod_{i=1}^{n} \left( 1 - (1 - e^{iu}) p_{\xi_i|\mathcal{M}_t}^{t_i, B_s=0, B_i=1} \right) f_{\mathcal{M}_t}(m) dm$$

When the Lévy Random Factor Loading Copula Model’s conditional time of default
probability distribution replace the based model articulated in Theorem 8.7, the subsequent theorem results.

**Theorem 8.10 (Lévy Random Factor Loading Copula Model: Unconditional Number of Default’s Characteristic Function \( \varphi_{N_t} \))**

Let \( N_t \) be a default counter process that admits Definition 4.13, \( N_t^i \) be a default process associated with the default credit entity \( i \) that admits Definition 4.12, \( p_{\xi_t^i|M_t} \) be the conditional default upon the Systematic Market Risk Factor \( M_t \) in the “Lévy Random Factor Loading Copula” model that follows Lemma 5.10. Then the unconditional number of default’s characteristic function, denoted by \( \varphi_{N_t} \), is given by:

\[
\varphi_{N_t}(u) = \int \prod_{i=1}^{\kappa} \left( 1 - (1 - e^{ju}) p_{\xi_t^i|M_t} \right) f_{M_t}(m) dm 
+ \int \prod_{i=1}^{+\infty} \left( 1 - (1 - e^{ju}) p_{\xi_t^i|M_t} \right) f_{M_t}(m) dm
\]

Where \( k_i = -\ell_1 \int_{-\infty}^{\kappa} m f_{M_t}(m) dm - \ell_2 \int_{\kappa}^{+\infty} m f_{M_t}(m) dm \)

**Proof:**

Note that the representation of distributions and parameters when associated with the number 1, they represent the period \( M_t < \kappa \) and when associated with the number 2 they represent the period \( M_t \geq \kappa \). Initialising the number of defaults’ characteristic function in the first step, extracting its conditional characteristic function on both periods, extracting its unconditional characteristic function, and then by Definition 7.8 combining the following three steps the theorem proved by adding the second and third steps and by taking \( k_i \) as in the final step.

Firstly: by following the steps of Theorem 8.7, the number of defaults’ characteristic function established by the subsequent chain of equality:
\[ \varphi_{N_t}(u) = E^{Q^*}\left[e^{iuN_t}\right] \\
= E^{Q^*}\left[Q^*\left[u^{N_t} \mid \mathcal{M}_t \right]\right] \\
= E^{Q^*}\left[Q^*\left[u^{N_t} \mid \mathcal{M}_t, \mathcal{M}_t < \kappa \right]\right] \\
+ E^{Q^*}\left[Q^*\left[u^{N_t} \mid \mathcal{M}_t, \mathcal{M}_t \geq \kappa \right]\right] \\
\]

Secondly: when \( \mathcal{M}_t < \kappa \), the transfer theorem is utilised, and the iterated expectation theorem is applied, \( \varphi_{N_t}(u) \) is given by the subsequent chain of equalities:

\[ \varphi_{N_t}^{\mathcal{M}_t < \kappa}(u) = E^{Q^*}\left[Q^*\left[u^{N_t} \mid \mathcal{M}_t, \mathcal{M}_t < \kappa \right]\right] \\
= E^{Q^*}\left[Q(\mathcal{M}_t < \kappa)Q^*\left[u^{N_t} \mid \mathcal{M}_t \right]\right] \\
= \int_{-\infty}^{\kappa} \prod_{i=1}^{n} \left(1 - (1 - e^{iu})p_{t_i}^{\xi_i|m_1}\right) f_{\mathcal{M}_t}(m) dm \\
\]

Thirdly: when \( \mathcal{M}_t \geq \kappa \), the transfer theorem is utilised, and the iterated expectation theorem is applied, \( \varphi_{N_t}(u) \) is given by the subsequent sequential equalities:

\[ \varphi_{N_t}^{\mathcal{M}_t \geq \kappa}(u) = E^{Q^*}\left[Q^*\left[u^{N_t} \mid \mathcal{M}_t, \mathcal{M}_t \geq \kappa \right]\right] \\
= E^{Q^*}\left[Q(\mathcal{M}_t \geq \kappa)Q^*\left[u^{N_t} \mid \mathcal{M}_t \right]\right] \\
= \int_{\kappa}^{+\infty} \prod_{i=1}^{n} \left(1 - (1 - e^{iu})p_{t_i}^{\xi_i|m_2}\right) f_{\mathcal{M}_t}(m) dm \\
\]

Finally: \( \kappa_i \) is a straightforward result of Corollary 5.16 and Assumption 5.11; as:

\[ \kappa_i = -E\left[\rho_i(\mathcal{M}_t, \mathcal{M}_t)\right] \\
= -E\left[Q(\mathcal{M}_t < \kappa)\ell_1\mathcal{M}_t + Q(\mathcal{M}_t \geq \kappa)\ell_2\mathcal{M}_t\right] \\
= -E\left[Q(\mathcal{M}_t < \kappa)\ell_1\mathcal{M}_t\right] - E\left[Q(\mathcal{M}_t \geq \kappa)\ell_2\mathcal{M}_t\right] \\
= -\ell_1 \int_{-\infty}^{\kappa} m f_{\mathcal{M}_t}(m) dm - \ell_2 \int_{\kappa}^{+\infty} m f_{\mathcal{M}_t}(m) dm \\
\]

The last three theorems could be rewritten by the moment generating function and accordingly the formal expansion could be utilised, but as stated previously that the formal expansion method is useful for small dimensional problems, specifically when the number of credit entities do not exceed 200.
8.4.2 Cumulative loss Characteristic Function

The Lévy Factor Copula model eases computing the accumulated loss’s characteristic function at a particular time. This characteristic could be recovered by the DFT in order of getting the accumulated loss. The corresponding moment generating function could be in the same manner that it was extracted in the number of defaults case. Accordingly, the formal expansion is direct implication of the moment generating function. The subsequent theorem constructs the unconditional accumulated loss’s characteristic function from the probability of default time conditioned upon the systematic market risk factor.

**Theorem 8.11 (Lévy Factor Copula Model: The Unconditional Cumulative Loss $\varphi_{C_t}$)**

Let $C_t$ be the CDO’s cumulative loss that admits Definition 7.8, $D_i$ be the loss given default that admits Definition 7.5, $N_t$ be a default counter process, which counts the number of default until time $t$ that admits Definition 4.13, and finally let $q_{t_i}^{M_t}$ and $p_{t_i}^{M_t}$ be, respectively, the conditional survival and default upon the Systematic Market Risk Factor $M_t$ in the “Lévy Factor Copula” model that follow Corollary 5.10. Then the unconditional cumulative loss’s characteristic function, denoted by $\varphi_{C_t}$, is given by the subsequent equality:

$$\varphi_{C_t}(u) = \int \prod_{i=1}^{n} \left( 1 - (1 - \varphi_{D_i}(u)) p_{t_i}^{M_t} \right) f_{M_t}(m) dm$$

**Proof:**

In view of Theorem 5.4, Corollary 5.11 and Corollary 5.12, $N_t^i$ be a default process associated with the default credit entity $i$ that admits Definition 4.12, and the iterated expectation theorem, the subsequent chain of equalities hold:
\[
\varphi_{c_t}(u) = \mathbb{E}^{\mathbb{Q}}[e^{ju c_t}]
\]
\[
= \mathbb{E}^{\mathbb{Q}}\left[ \mathbb{E}^{\mathbb{Q}'}[e^{ju c_t} | \mathcal{M}_t] \right]
\]
\[
= \mathbb{E}^{\mathbb{Q}'}\left[ \prod_{i=1}^{n} \mathbb{E}^{\mathbb{Q}'}[e^{j u d_{i}^{\text{m}} | \mathcal{M}_t}] \right]
\]
\[
= \mathbb{E}^{\mathbb{Q}'}\left[ \prod_{i=1}^{n} \left( q_{t_i}^{\xi_{i}^{\text{m}}} + p_{t_i}^{\xi_{i}^{\text{m}}} \varphi_{d_i}(u) \right) \right]
\]
\[
= \int \prod_{i=1}^{n} \left( q_{t_i}^{\xi_{i}^{\text{m}}} + p_{t_i}^{\xi_{i}^{\text{m}}} \varphi_{d_i}(u) \right) f_{\mathcal{M}_t}(m) dm
\]
\[
= \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_{i}^{\text{m}}} \right) f_{\mathcal{M}_t}(m) dm
\]

Where \( \varphi_{d_i} \) is the loss given default’s characteristic and it could follow any distribution function as it could be constant or deterministic\(^{39} \).

The loss given default’s characteristic function, in the previous theorem, could be easily extended to incorporate stochastic or any other structure.

As carried out in the previous section, the accumulated loss’s characteristic function of Lévy Factor copula Model could be rearticulated to incorporate with the extended models. This rephrasing, again, is not essential, where the conditional time of default probability could be inserted in the previous equations. The subsequent three theorems are offered consecutively to systematise and standardise the representation.

By evoking the equality of the conditional accumulated loss’s distribution in the Lévy Binary Stochastic Correlated Factor Copula Model, Theorem 8.11 could be rephrased as shown in the following theorem.

**Theorem 8.12 (Lévy Binary Stochastic Correlated Factor Copula Model: The Unconditional Cumulative Loss \( \varphi_{c_t} \))**

Let \( c_t \) be the CDO’s cumulative loss that admits Definition 7.8, \( d_i \) be the loss given

\(^{39} \) For more details see the next subsection.
default that admits Definition 7.5, $\mathcal{N}_t$ be a default counter process that admits Definition 4.13, $\mathcal{N}_t^i$ be a default process associated with the default credit entity $i$ that admits Definition 4.12, and finally let $q_{t_i}^{\xi_{i}^{\mathcal{M}_t}}$ and $p_{t_i}^{\xi_{i}^{\mathcal{M}_t}}$ be, respectively, the conditional survival and default upon the Systematic Market Risk Factor $\mathcal{M}_t$ in the “Lévy Binary Stochastic Correlated Factor Copula” model that admits Lemma 5.8. Then the unconditional cumulative loss’s characteristic function, denoted by $\varphi_{\xi_t}$, is given by the subsequent equality:

$$\varphi_{\xi_t}(u) = \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{\xi_t}(u) \right) p_{t_i}^{\xi_{i}^{\mathcal{M}_t}B_i=0} \right) f_{\mathcal{M}_t}(m) dm$$

$$+ \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{\xi_t}(u) \right) p_{t_i}^{\xi_{i}^{\mathcal{M}_t}B_i=1} \right) f_{\mathcal{M}_t}(m) dm$$

**Proof:** (see Theorem 8.11)

In the same line, the equality of the conditional accumulated loss’s distribution in Theorem 8.11 could be rearticulated to suite the Lévy Symmetric Stochastic Correlated Factor Copula Model. This could be seen in the following theorem.

**Theorem 8.13 (Lévy Symmetric Stochastic Correlated Factor Copula Model: The Unconditional Cumulative Loss $\varphi_{\xi_t}$)**

Let $\xi_t$ be the CDO’s cumulative loss that admits Definition 7.8, $\mathcal{D}_t$ be the loss given default that admits Definition 7.5, $\mathcal{N}_t$ be a default counter process that admits Definition 4.13, $\mathcal{N}_t^i$ be a default process associated with the default credit entity $i$ that admits Definition 4.12, and finally let $q_{t_i}^{\xi_{i}^{\mathcal{M}_t}}$ and $p_{t_i}^{\xi_{i}^{\mathcal{M}_t}}$ be, respectively, the conditional survival and default upon the Systematic Market Risk Factor $\mathcal{M}_t$ in the “Lévy Symmetric Stochastic Correlated Factor Copula” model that admits Lemma 5.9. Then the unconditional cumulative loss’s characteristic function, denoted by $\varphi_{\xi_t}$, is given by the subsequent equality:
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

Chapter Eight: Numerical Evaluation Via DFT’s Fast (FFT) and Very Fast (VFFT) forms

\[
\varphi_{c_t}(u) = \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_i | M_t, B_t = 1} \right) f_{M_t}(m) \, dm \\
+ \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_i | M_t, B_t = 0, B_i = 0} \right) f_{M_t}(m) \, dm \\
+ \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_i | M_t, B_t = 0, B_i = 1} \right) f_{M_t}(m) \, dm
\]

**Proof:** (see Theorem 8.11)

When the Lévy Random Factor Loading Copula Model’s conditional accumulated loss’s distribution replaces the based model articulated in Theorem 8.11, the subsequent theorem results.

**Theorem 8.14 (Lévy Random Factor Loading Copula Model: The Unconditional Cumulative Loss \( \varphi_{c_t} \))**

Let \( C_t \) be the CDO’s cumulative loss that admits Definition 7.8, \( D_t \) be the loss given default that admits Definition 7.5, \( N_t \) be a default counter process that admits Definition 4.13, \( N_t^i \) be a default process associated with the default credit entity \( i \) that admits Definition 4.12, and finally let \( q_{t_i}^{\xi | M_t} \) and \( p_{t_i}^{\xi | M_t} \) be, respectively, the conditional survival and default upon the Systematic Market Risk Factor \( M_t \) in the “Lévy Random Factor Loading Copula” model that admits Lemma 5.10. Then the unconditional cumulative loss’s characteristic function, denoted by \( \varphi_{c_t} \), is given by the subsequent equality:

\[
\varphi_{c_t}(u) = \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_i | M_t, M_t^{i \uparrow}} \right) f_{M_t}(m) \, dm \\
+ \int \prod_{i=1}^{n} \left( 1 - \left( 1 - \varphi_{d_i}(u) \right) p_{t_i}^{\xi_i | M_t, M_t^{i \downarrow}} \right) f_{M_t}(m) \, dm
\]

**Proof:** (see Theorem 8.11)

Again, the above four theorems could be rewritten by the moment generating function
and accordingly the formal expansion could be utilised. However, the formal expansion method is useful for small dimensional problems, i.e. do not exceed 200.

8.4.3 Loss Given Default’s Characteristic Function

In consideration of Definition 7.5, Assumption 7.1, Assumption 7.2, and Assumption 7.3, the Loss Given Default \( \mathcal{D}_i = \mathcal{K}_i(1 - \delta_i) \) is fully determined though its recovery rate. The recovery rate could be modelled deterministically or independently from default rates and follow any distribution conditional on returning significant Loss Given Default (\( \mathcal{D} \)) probability, i.e. decreasing the mean or increasing the standard deviation of \( \mathcal{D} \) could produces an insignificant probability when \( \mathcal{D} \not\in [0\%, 100\%] \) (Debuysscher and Szegö, 2003b).

The Credit risk’s reduced form model was initiated in (Jarrow and Turnbull, 1995) by considering an independent recovery rates from their corresponding default times, where the in later it is deliberated as unpredictable stopping times. In (Jarrow and Turnbull, 1995), (Credit-Suisse-Financial-Products 1997), and (Canabarro et al., 2003) the recovery rate was modelled as a constant, where in Crosby and Bohn [2002], (Gupton et al., 1997) it was modelled as a stochastic recovery rate in Moody’s KMV and in (Crosbie and Bohn, 2002) it was modelled as a stochastic recovery rate in CreditMatrix. A benchmark Credit risk evaluation framework, in the same direction, has been lunched in Basel II (see (Basel Committee on Banking Supervision 2001b, Basel Committee on Banking Supervision 2001c, Basel Committee on Banking Supervision 2001a)) and discussed in (Crouhy et al., 2000a), (Gordy, 2000), (Gordy, 2002), and (Chabaane et al., 2003).

This subsection starts by the most common loss given default (\( \mathcal{D} \)) model, i.e. homogenous and equal loss given default. Subsequently Lévy Skewed Alpha-Stable, Gaussian, Standard Lévy, Generalized Hyperbolic, Variance Gamma, and Normal
Inverse Gaussian loss given default’s characteristic functions will be modelled, where all these models are, except the homogeneous and the Gaussian, proposed models.

**Lemma 8.2 (Homogenous and Equal Loss Given Default’s Characteristic Function)**

Let $\mathcal{D}$ be the homogeneous loss given default that admits Lemma 7.3, with an equal recovery rate, i.e. $\delta_1 = \delta$ that follows Assumption 7.8 and an equal nominal, i.e. $\mathcal{K}_i = \mathcal{K} = 1$ that admits Assumption 7.9. Then the loss given default’s characteristic function, denoted by $\varphi_{\mathcal{D}_i}$, is given by the subsequent equality:

$$
\varphi_{\mathcal{D}_i}(u) = e^{ju(1-\delta)}
$$

**Proof:**

By returning the characteristic function from its expectation the dual equality hold:

$$
\varphi_{\mathcal{D}_i}(u) = \mathbb{E}^Q\left[ e^{ju\mathcal{D}_i} \right] = e^{ju\mathcal{K}_i(1-\delta)}
$$

and when all credit entities are equally weighted its individual characteristic function is equal to:

$$
\varphi_{\mathcal{D}}(u) = e^{ju(1-\delta)}
$$

Statistically supported by the Central Limit Theorem (CLT), i.e. the sum of a large number of independent and identical distributed variables those admits a finite-variance distribution will tend to be normally distributed, the financial assets are modelled by the Gaussian distribution; as stated by the pioneer work in (Bachelier, 1900). Nonetheless, based on empirical verifications in (Mandelbrot, 1963) and (Fama, 1965), financial asset returns usually have heavier tails than what Gaussian distribution can provide. Accordingly, the Lévy Skewed Alpha Stable distributions, which were introduced in (Lévy, 1925), were proposed as an alternative framework in (Mandelbrot, 1963) and (Fama, 1965). Those distributions are supported by at least two strong factors. The first
is that these distributions are supported by the Generalised Central Limit Theorem (GCLT), i.e. the only possible limit distributions for accurately normalised and centred sums of independent and identically distributed random variables are the Lévy stable laws (Laha and Rohatgi, 1979). The second is that they are leptokurtic.

The next lemmas express one of the proposed loss given default characteristic function, i.e. the Lévy Skewed Alpha Stable loss given default characteristic function. This distribution will be represented same as in (Nolan, 2009).

**Lemma 8.3 (Lévy Skewed Alpha-Stable Loss Given Default’s Characteristic Function)**

Let $D_i$ be the loss given default that admits Definition 7.5 with $K_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $D_i$ admits the Lévy Skewed Alpha-Stable distribution, i.e. Definition 6.1. Then the loss given default’s characteristic function, denoted by $\phi_{D_i}^{\mathcal{L}(\alpha,\beta,\gamma;\delta)}$, is given by the subsequent equality$^{40}$:

$$
\phi_{D_i}^{\mathcal{L}(\alpha,\beta,\gamma;\delta)}(u) = \begin{cases} 
  e^{-\gamma |uD_i|^\alpha \left(1-j\beta \text{sign}(uD_i) \left(\tan\left(\frac{\pi \alpha}{2}\right)\right) + j\delta uD_i\right)}, & \alpha \neq 1 \\
  e^{-\gamma |uD_i|^\alpha \left(1+j\beta \text{sign}(uD_i) \left(\tan\left(\frac{\pi \alpha}{2}\right)\right) + j\delta uD_i\right)}, & \alpha = 1
\end{cases}
$$

It is proved as a limiting case of the Lévy skewed alpha stable by property D6.1.2, by substituting $a = \frac{\sigma}{\sqrt{2}}, b = \mu, \alpha = 2$ and $\beta = 0$ (Nolan, 2009). In most of the literature, the Gaussian distribution is called the “Normal distribution” and also abbreviated as $\mathcal{N}(\mu, \sigma^2)$. In the subsequent lemma the Gaussian loss given default’s characteristic function is presented in order to standardise the representation.

$^{40}$ When $\delta$ is associated with $i$, i.e $\delta_i$, it represent the recovery rate. Otherwise it represent some distribution parameter except for Lemma 8.2.
Lemma 8.4 (Gaussian Loss Given Default’s Characteristic Function)

Let $\mathcal{D}_i$ be the loss given default that admits Definition 7.5 with $\mathcal{K}_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $\mathcal{D}_i$ admits the Gaussian distribution, i.e. Definition 6.2. Then the loss given default’s characteristic function, denoted by $\varphi_{\mathcal{D}_i}(\mu, \sigma)$, is given by the subsequent equality:

$$\varphi_{\mathcal{D}_i}(\mu, \sigma)(u) = e^{\left(\mu u \mathcal{D}_i + \frac{(\sigma u \mathcal{D}_i)^2}{2}\right)}$$

The second loss given default proposition is presented in the subsequent lemma with the Standard Lévy loss given default’s characteristic function. The Standard Lévy could be seen as Lévy Skewed Alpha-Stable distribution, with $\gamma = 1$ and $\delta = 0$.

Lemma 8.5 (Standard Lévy Loss Given Default’s Characteristic Function)

Let $\mathcal{D}_i$ be the loss given default that admits Definition 7.5 with $\mathcal{K}_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $\mathcal{D}_i$ admits the Standard Lévy distribution, i.e. Definition 7.5. Then the loss given default characteristic function, denoted by $\varphi_{\mathcal{D}_i}(\mu, \sigma)$, is given by the subsequent equality:

$$\varphi_{\mathcal{D}_i}(\mu, \sigma)(u) = \begin{cases} e^{\left(-1^a |u\mathcal{D}_i|^a \left(1 - j\beta \left(\text{sign}(u\mathcal{D}_i)\left(\tan\left(\frac{\pi a}{2}\right)\right)\right)\right)\right)}, & \alpha \neq 1 \\ e^{\left(-|u\mathcal{D}_i| \left(1 + j\beta \frac{2}{\pi} \left(\text{sign}(u\mathcal{D}_i)\left(\tan\left(\frac{\pi \alpha}{2}\right)\right)\right)\right)\right)}, & \alpha = 1 \end{cases}$$

As introduced, the financial assets are modelled as stochastic processes; influenced by distributional assumptions on the dependence structure and its increments. The empirical studies showed that these assets have semi-heavy tails, i.e. their kurtoses are greater than the normal one (Mandelbrot, 1963). This has led to the Lévy Skewed Alpha Stable distributions to be introduced. However, the limitation of the Lévy Skewed Alpha Stable distribution, i.e. it does not always have mean, does not have variance or...
higher moments except for the Gaussian case Definition 6.2, lead to an alternative distributions known as the Generalised Hyperbolic distributions.

The Generalised Hyperbolic distribution was introduced in (Barndorff-Nielsen, 1977a). This distribution is proved to be infinitely divisible and thus admits the Lévy process (Barndorff-Nielsen, 1977b), have semi-heavy tails that could almost fit the assets returns (Prause, 1999), and its density is identified explicitly.

Therefore, Generalized Hyperbolic distribution could introduce more flexibility when extreme cases are required to model the loss given default. The following lemma proposes the Generalized Hyperbolic loss given default’s characteristic function.

**Lemma 8.6 (Generalized Hyperbolic Loss Given Default’s Characteristic Function)**

Let $\mathcal{D}_i$ be the loss given default that admits Definition 7.5 with $\mathcal{K}_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $\mathcal{D}_i$ admits the Generalized Hyperbolic distribution, i.e. Definition 6.6. Then the loss given default characteristic function, denoted by $\varphi_{\mathcal{D}_i}^{\mathcal{GH}(\lambda, \alpha, \beta, \delta, \mu)}$, is given by the subsequent equality:

$$
\varphi_{\mathcal{D}_i}^{\mathcal{GH}(\lambda, \alpha, \beta, \delta, \mu)}(u) = e^{juD_i \mu} \left( \frac{\omega}{\alpha^2 - (\beta + juD_i)^2} \right) \frac{\frac{1}{2} K_{\lambda}(\delta \sqrt{\alpha^2 - (\beta + juD_i)^2})}{K_{\lambda}(\delta \sqrt{\omega})}
$$

The Generalized Hyperbolic also has many special and limiting cases. Variance Gamma distribution introduced in (Madan and Seneta, 1990) is one of the most important Generalized Hyperbolic distribution limiting case; since it is close under convulsion. The subsequent lemma proposes the Variance Gamma loss given default’s characteristic function.
Lemma 8.7 (Variance Gamma Loss Given Default’s Characteristic Function)

Let $D_i$ be the loss given default that admits Definition 7.5 with $K_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $D_i$ admits the Variance Gamma distribution, i.e. Definition 6.12. Then the loss given default characteristic function, denoted by $\phi_{\mathcal{V}G(\lambda, \alpha, \beta, \mu)}^D$, is given by the subsequent equality:

$$
\phi_{\mathcal{V}G(\lambda, \alpha, \beta, \mu)}^D(u) = e^{i\mu u D_i} \left( \frac{\omega}{\alpha^2 - (\beta + ju D_i)^2} \right)^\lambda
$$

The last proposed model is another limiting case of the Generalized Hyperbolic distribution, which is the Normal Inverse Gaussian Distribution introduced in (Barndorff-Nielsen, 1997). It is also close under convolution.

Lemma 8.8 (Normal Inverse Gaussian Loss Given Default’s Characteristic Function)

Let $D_i$ be the loss given default that admits Definition 7.5 with $K_i$ as the nominal amount that admits Definition 7.3 and follows Assumption 7.2, and $(1 - \delta_i)$ as the unrecovered rate that admits Definition 7.9. Let $D_i$ admits the Normal Inverse Gaussian distribution, i.e. Definition 6.15. Then the loss given default characteristic function, denoted by $\phi_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)}^D$, is given by the subsequent equality:

$$
\phi_{\mathcal{NIG}(\alpha, \beta, \delta, \mu)}^D(u) = e^{i\mu u D_i} \frac{e^{\delta \sqrt{\omega}}}{e^{\delta \sqrt{\alpha^2 - (\beta + ju D_i)^2}}}
$$

8.5 Numerical Evaluation via Discrete, Fast, Very Fast Fourier Transform

Recalling Definition 5.7, the characteristic function could be seen as a Fourier-Stieltjes Transform of the distribution function. In other words, the distribution’s characteristic function and its density function represent a Fourier transform pair. To precise the
scope of the Fourier Transform, the usual terminologies used in Fourier Transform literature, i.e. time and frequency domain, will be avoided; as a substitution the probability domain and transform domain will substitute them.

The density function is distributed over the real number domain \((-\infty, \infty)\). This function is a non-periodic function and its mass, mostly, is concentrated within a diminutive range of the infinite domain. To compute its transform, i.e. the characteristic function transform, numerically on a computer, discretizing the function is an essential step. Consequently, the numerical integration of the characteristic function step is followed. This step leads to an approximation of the true analytically-defined Fourier transform of this function, i.e. \(\mathcal{F}\).

This subsection will start by extracting the unconditional number of defaults’, the cumulative loss’, and the loss given default distribution functions by inversing their characteristic functions, truncating them, discret ing them, and finally applying the \(\mathcal{F}\). Additionally, the \(\mathcal{F}\) and \(\mathcal{F}\) replaces the \(\mathcal{F}\) conditionally on choosing the correct matrix accuracy.

8.5.1 Extracting the Number of Default’s by the Inverse Fourier Transform and the \(\mathcal{DFT}\)

The density function is truncated and then approximated in order to employ the \(\mathcal{DFT}\)’s algorithms. The truncation process will reduce the range and thus will decrease the number of computation steps needed.

**Definition 8.2 (Truncation)**

*Let \(f_X(x)\) be an extracted density function of a random variable \(X\) by the Inverse Fourier Transform from its corresponding characteristic function, \(\varphi_X(u)\) that follows*
Property D.5.7.6. Let \( v_{\min} \) be \( f_X(x) \) (reasonable\footnote{It is assumed to have a zero probability outside this interval, where theoretically it is possible to have some realisation outside these boundaries but their probability is so small, for example: lower than 1E-20.}) highest possible minimum value, i.e.

\[
v_{\min} = \sup\{v_{\min}: x \geq v_{\min}, f(x) \geq f(v_{\min}) \equiv 0, x, v_{\min} \in \mathbb{R}\}
\]

and \( v_{\max} \) be \( f_X(x) \) (reasonable) lowest possible maximum value, i.e.

\[
v_{\max} = \inf\{v_{\max}: x \leq v_{\max}, f_X(x) \geq f_X(v_{\max}) \equiv 0, x, v_{\max} \in \mathbb{R}\}
\]

where \( \text{abs}(v_{\min}) = \text{abs}(v_{\max}) \). Then \( f_X(x) \) could be truncated by \( L \), where \( L = v_{\max} - v_{\min} \).

In view of the truncation definition, the density function extracted from its corresponding characteristic function could truncate as in the following lemma.

**Lemma 8.9 (Truncated Density Function)**

Let \( f_X(x) \) be an extracted density function of a random variable \( X \) by the Inverse Fourier Transform from its corresponding characteristic function, \( \varphi_X(u) \) that follows Property D.5.7.6. Let \( L \) be its truncation limit. Then the truncated \( f_X(x) \) could be given by the subsequent equality:

\[
f_X(x) = \frac{1}{2\pi} \int_{-L_u/2}^{L_u/2} e^{-jux} \varphi_{\Delta_x}(u) du + \varepsilon_{\text{trunc}}^u
\]

where \( \varepsilon_{\text{trunc}}^u \) is the absolute difference between the truncated function's limits and the actual solution.

**Definition 8.3 (Density Step Function \( \Delta \))**

Let \( f_X(x) \) be a density function of a random variable \( X \) and \( \mathbb{D} \) be its domain, then its density step function, denoted by \( \Delta_x \), is equal to the greatest common divisor function, or its fraction, of its domain \( \mathbb{D} \), i.e. \( \Delta_x = \frac{1}{d} \text{GCD}(\mathbb{D}) \).

The choose of the density step is a critical process; especially when the FFT and the
\(\mathcal{VFFT}\) replaces the original \(\mathcal{DFT}\).

**Definition 8.4 (Discrete Density Resolution \(r_{\text{Disc}}\))**

Let \(f_X(x)\) be a density function of a random variable \(X\), \(\Delta_x\) be its density step function that admits Definition 8.3, and \(L\) be its truncation limit that admits Definition 8.2. Then the Discrete Density Resolution, denoted by \(r_{\text{Disc}}\), is given by the subsequent equality:

\[
r_{\text{Disc}} = \left\lfloor \frac{L}{\Delta_x} \right\rfloor + 1
\]

where \(r_{\text{Disc}} \in \mathbb{N}^{42}\).

Alternatively, the discrete density resolution and the density step function could be looked analogously, where one could define the other. As noted previously, this could be a better way when replacing the original \(\mathcal{DFT}\) by the \(\mathcal{FFT}\) and the \(\mathcal{VFFT}\).

**Lemma 8.10 (Density’s Step Function \(\Delta\))**

Let \(f_X(x)\) be a density function of a random variable \(X\), \(r_{\text{Disc}}\) be its discrete density resolution that admits Definition 8.4, and \(L\) be its truncation limit that admits Definition 8.2. Then its density step function, denoted by \(\Delta_x\), is given by the subsequent equality:

\[
\Delta_x = \frac{L}{r_{\text{Disc}}} - 1
\]

Another concept needed when numerically evaluating a function is to choose its step function elements resolution. This concept could be defined as in the subsequent definition.

**Definition 8.5 (Discrete Density Step Function Elements)**

Let \(f_X(x)\) be a density function of a random variable \(X\), \(r_{\text{Disc}}\) be its discrete density resolution that admits Definition 8.4, and \(\Delta_x\) be its density step function that admits Lemma 8.10. Then its \(r\) discrete density sequence elements of the step function, denoted by \(x_r\), is given by the subsequent equality:

\[\text{--}\]

\(^{42}\mathbb{N}\) is the set of non-negative integers.
\[ [x_r]^r_{Disc} = a + r\Delta \]

There are numerous methods to approximate the integrations. Simpson’s rule is one of them, where it could be replaced by any suitable one.

**Definition 8.6 (Simpson’s Rule)**

Let \( f \) be a real integrable function, \([a, b]\) be its integration bounded interval, i.e. \( I = \int_a^b f(x)dx \), \( r_{Disc} \) be an even number that represents the number of discretisation steps, \( \Delta_x \) density step function that follows Lemma 8.10, \( x_r \) be \( r \) discrete density sequence elements of the step function that admits Definition 8.5, and \( \omega \) be the integration quadratic curve approximation weights at each step. Then I could be numerically evaluated by the subsequent equality:

\[
I = \int_a^b f(x)dx = \Delta_x \sum_{r=0}^{r_{Disc}-1} \omega_n f(x_r) + \varepsilon^x_{discr}
\]

Where \( \varepsilon^x_{discr} \) is the absolute difference between the discrete and continuous solution and

\[
\omega_n = \begin{cases} 
1/3, & \text{n is zero} \\
2/3, & \text{n is even.} \\
4/3, & \text{n is odd}
\end{cases}
\]

The extracted truncated density function by the Inverse Fourier Transform from its corresponding characteristic function could be approximated by utilising the Simpson rule articulated in Definition 8.6.

**Lemma 8.11 (Discrete Density Function: Simpson’s Rule)**

Let \( f_X(x) \) be an extracted truncated density function of a random variable \( X \) by the Inverse Fourier Transform from its corresponding characteristic function, \( \varphi_X(u) \) that admits Lemma 8.9, \( r_{Disc} \) be its discrete density resolution, \( L \) be its truncation limit that admits Definition 8.2, \( \Delta_u \) be its density step function that follows Lemma 8.10, \( x_r \) be the value step function that admits Definition 8.5, \( \omega \) be the integration quadratic curve approximation weights at each step. Then I could be numerically evaluated by the subsequent equality:

\[
I = \int_a^b f(x)dx = \Delta_x \sum_{r=0}^{r_{Disc}-1} \omega_n f(x_r) + \varepsilon^x_{discr}
\]

Where \( \varepsilon^x_{discr} \) is the absolute difference between the discrete and continuous solution and

\[
\omega_n = \begin{cases} 
1/3, & \text{n is zero} \\
2/3, & \text{n is even.} \\
4/3, & \text{n is odd}
\end{cases}
\]
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

approximation’s weight’s at each step, and \( \varepsilon_{(tr+di)}^u \) be the summation of the truncation and discretisation errors. Then the \( f_X(x) \) could be numerically evaluated by Simpson’s rule, which is defined in Definition 8.6, as stated in the subsequent equality:

\[
f_X(x) = \frac{\Delta_u}{2\pi} \sum_{r=0}^{r_{Disc}^{-1}} \omega_r \varphi_{N_t}(u_r) e^{(-ju_r t)} + \varepsilon_{(tr+di)}^u \]

**Proof:**

With \( \Delta_u = \frac{L}{r_{Disc}^{-1}} \), \( u_r = \frac{-L}{2} + r\Delta_u \).

and \( \omega_n = \begin{cases} 
\frac{1}{3}, & \text{n is zero} \\
\frac{2}{3}, & \text{n is even} \\
\frac{4}{3}, & \text{n is odd} 
\end{cases} \)

It is obvious that the truncated integration of \( f_X(x) \) could be approximated as:

\[
f_X(x) = \frac{1}{2\pi} \int_{-\frac{L_u}{2}}^{\frac{L_u}{2}} e^{(-jut)} \varphi_{N_t}(u) du + \varepsilon_{trunc}^u \\
= \frac{\Delta_u}{2\pi} \sum_{r=0}^{r_{Disc}^{-1}} \omega_r \varphi_{N_t}(u_r) e^{(-ju_r t)} + \varepsilon_{trunc}^u + \varepsilon_{discr}^u \\
= \frac{\Delta_u}{2\pi} \sum_{r=0}^{r_{Disc}^{-1}} \omega_r \varphi_{N_t}(u_r) e^{(-ju_r t)} + \varepsilon_{(tr+di)}^u 
\]

With the definitions, lemmas, and theorems stated above, employing the DFT at this point requires only some manipulating of the representation and a good choose of parameters.

**Theorem 8.15 (Extracting the Lévy Factor Copula’s Unconditional Number of Default’s Distribution Function by the Inverse Fourier Transform and the DFT)**

Let \( N_t \) be a default counter process that admits Definition 4.13 and \( \varphi_{N_t} \) be its characteristic function that follows Theorem 8.7, \( \tau_{DFT} \) be the DFT resolution that follows Definition 8.4, \( \Delta_u \) and \( \Delta_t \) be, respectively, the number of defaults density step

\[^{43}\text{This representation is, also, identical to Theorem 8.7-8.10.}\]
function and its characteristic step function those follow Lemma 8.10, \( u_r \) and \( t_g \) be, respectively, the \( r \) discrete density and the \( g \) characteristic sequence elements of the step function those admits Definition 8.5, \( \omega_r \) be the integration quadratic curve approximation’s weight’s at each \( r \) step, \( \epsilon^n_{(tr+di)} \) be the summation of the truncation and discretisation error, and \( \text{DFT}_g[\varphi] \) be the \( \sum_{r=0}^{N-1} \varphi_r e^{i \omega r \frac{\pi}{N}} \) that admits Theorem 8.1. Then the unconditional number of default’s function could be numerically evaluated by the subsequent equality:

\[
N_{t,g} = \frac{\Delta u}{2\pi} e^{\left(\frac{j\pi g}{2}\right)} \text{DFT}\left(\omega_r \varphi_{N_{t,g}}(u_r) e^{(-j\Delta u \Delta o)}\right) + \epsilon^n_{(tr+di)}
\]

**Proof:**

The unconditional number of default’s distribution function, \( N_t \), could be extract in two main steps:

Firstly: by property D.5.7.6., i.e. inverse Fourier Transform, \( N_t \) will be extract from its \( \varphi_{N_t} \). In order to utilise the \( \text{DFT} \), \( N_t \) is truncated in the second equality as in Lemma 8.9 with \( L_u \) as its truncation limit and discreted in the third equality as in Lemma 8.11 by Simpson’s rule. These three sub-steps are represented mathematically by the sequence of equalities:

\[
N_t = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(-jut)} \varphi_{N_t}(u) du \\
= \frac{1}{2\pi} \int_{-L_u/2}^{L_u/2} e^{(-jut)} \varphi_{N_t}(u) du + \epsilon^n_{\text{trunc}} \\
= \frac{\Delta u}{2\pi} \sum_{r=0}^{\tau_{\text{DFT}}-1} \omega_r \varphi_{N_t}(u_r) e^{(-j\omega_r t)} + \epsilon^n_{(tr+di)}
\]

With \( \Delta_u = \frac{L}{r_{\text{DFT}}-1} \), \( [u_r]_{0}^{r_{\text{DFT}}} = \frac{-L}{2} + r\Delta_u \), and \( \omega_n = \begin{cases} 
\frac{1}{3}, & n \text{ is zero} \\
\frac{2}{3}, & n \text{ is even} \\
\frac{4}{3}, & n \text{ is odd} 
\end{cases} \)
Secondly: by replacing the number of defaults density step function and its characteristic step function by some equivalent step function elements, i.e. \( \Delta_g = \frac{L}{\tau_{DFT} - 1} \)

\[
[t_g]^{r_{DFT}}_0 = t_0 + \frac{2\pi g}{\Delta_g \tau_{DFT}} \quad \text{and} \quad [u_r]^{r_{DFT}}_0 = \left[ -\frac{L}{2} + r \Delta_u \right]^{r_{DFT}}_0,
\]

and by Theorem 8.1, the fourth equality could be replaced by the \( \mathcal{F} \). These two sub-steps are represented mathematically by the subsequent chain of equalities:

\[
N_{t,g} = \frac{\Delta u}{2\pi} \sum_{r=0}^{N-1} \omega_r \phi_{N_{t,g}}(u_r) e^{-j u_r t_g} + \varepsilon^u_{(t+r+di)}
\]

\[
= \frac{\Delta u}{2\pi} \sum_{r=0}^{N-1} \omega_r \phi_{N_{t,g}}(u_r) e^{\left(-j \left(\frac{L u + r \Delta u}{2}\right) \right)} e^{\left(j \left(\frac{L u}{2} - \frac{2\pi g}{N \Delta u}\right)\right)} e^{-j r \Delta u \left(\frac{2\pi g}{N \Delta u}\right)} + \varepsilon^u_{(t+r+di)}
\]

\[
= \frac{\Delta u}{2\pi} \sum_{r=0}^{N-1} \omega_r \phi_{N_{t,g}}(u_r) e^{\left(j t_0 \left(\frac{L u}{2}\right) \right)} e^{\left(j \left(\frac{L u}{2} - \frac{2\pi g}{N \Delta u}\right)\right)} e^{-j r \Delta u \left(\frac{2\pi g}{N \Delta u}\right)} + \varepsilon^u_{(t+r+di)}
\]

\[
= \frac{\Delta u}{2\pi} \left(\frac{t_g}{2}\right) \mathcal{F} \left(\frac{L u}{2}\right) \omega_r \phi_{N_{t,g}}(u_r) e^{-j r \Delta u \left(\frac{2\pi g}{N \Delta u}\right)} + \varepsilon^u_{(t+r+di)}
\]

With the density function of the \( \text{Lévy Factor Copula’s unconditional number of default’s been extracted by the inverse Fourier transform and the } \mathcal{F} \text{, replacing the } \mathcal{F} \text{ by the } \mathcal{F} \text{ to increase the speed requires only replacing the } \mathcal{F} \text{ resolution and the related variables those are stated in Lemma 8.10 with ones those are suitable to employ the } \mathcal{F}. \text{ The subsequent lemma state the accepted } \mathcal{F} \text{ resolution.}

**Lemma 8.12 (FFT Resolution } \tau_{FFT})**

Let \( \tau_{FFT} \) be the \( \mathcal{F} \) resolution that follows Definition 8.4 and \( F_N \) be the Fourier matrix that admits Definition 8.1. Then to evaluate the \( F_N \) by the \( \mathcal{F} \) that admits Theorem 8.3, \( \tau_{FFT} \) has to be equal to the subsequent equality:

\[
\tau_{FFT} = 2^{\log_2(\tau_{DFT}^{-1}) + k}
\]

where \( k = 0, 1, \cdots \) and used to increase the \( \mathcal{F} \) resolution.
Proof:

It enough to revoke that the size of the FFT has to be $2^E \times 2^E$.

Taking into account Theorem 8.15 and Lemma 8.12, extracting the density function of the Lévy Factor Copula’s unconditional number of default’s by the inverse Fourier transform and FFT requires only adjusting density’s step function articulated in Lemma 8.10. This important result is summarised in the following corollary.

**Corollary 8.5 (Extracting the Lévy Factor Copula’s Unconditional Number of Default’s Distribution Function by the Inverse Fourier Transform and the FFT)**

Let $N_t$ be a default counter process that admits Definition 4.13 and $\varphi_{N_t}$ be its characteristic function that follows Theorem 8.7. $\mathcal{F}_{FFT}$ be the FFT resolution that admits Lemma 8.12, $\Delta_u$ and $\Delta_t$ be, respectively, the number of defaults density step function and its characteristic step function those follow Lemma 8.10, $u_r$ and $t_g$ be, respectively, the discrete density and the characteristic sequence elements of the step function that admits Definition 8.5, $\omega_r$ be the integration quadratic curve approximation’s weight’s at each $r$ step, $\varepsilon^u_{(tr+di)}$ be the summation of the truncation and discretisation error, and $\mathcal{F}_{FFT}[\varphi]$ be the $\sum_{r=0}^{N-1} \varphi_r e^{i2\pi\frac{rj\pi}{N}}$ that admits Theorem 8.3. Then the unconditional number of default’s function could be numerically evaluated by the subsequent equality:

$$N_{t_g} = \frac{\Delta_u}{2\pi} e^{\left(\frac{i\pi}{2}\frac{t_g}{\Delta_t}\right)} \mathcal{F}_{FFT}\left(\omega_r \varphi_{N_{t_g}}(u_r) e^{-tr\Delta_u t_0}\right) + \varepsilon^u_{(tr+di)}$$

Once again, substituting the DFT by the VFFT to increase the speed of recovering the density function needs only substituting the DFT resolution and the related variables those are expressed in Lemma 8.10 with ones those are suitable to employ the VFFT. The subsequent lemma articulates the accepted VFFT resolution.
Lemma 8.13 (\textit{\textbf{VFFT Resolution } }\mathbf{r}_{\textit{VFFT}}) 

Let \( \mathbf{r}_{\textit{DFT}} \) be the DFT resolution that follows Definition 8.4 and \( \mathcal{F}_N \) be the Fourier matrix that admits Definition 8.1. Then to evaluate \( \mathcal{F}_N \) by the \textit{VFFT} that admits Theorem 8.5, \( \mathbf{r}_{\textit{VFFT}} \) has to be greater than or equal to \( \mathbf{r}_{\textit{DFT}} \), where \( \mathbf{r}_{\textit{VFFT}} \) has to be any even number not divided by the square of an odd prime.

Taking into consideration Theorem 8.15 and Lemma 8.13, extracting the density function of the Lévy Factor Copula’s unconditional number of default’s by the inverse Fourier transform and \textit{VFFT} needs only modifying the density’s step function articulated in Lemma 8.10. This significant consequence is summarised in the next corollary.

Corollary 8.6 (Extracting the Lévy Factor Copula’s Unconditional Number of Default’s Distribution Function by the Inverse Fourier Transform and the \textit{VFFT})

Let \( N_t \) be a default counter process that admits Definition 4.13 and \( \varphi_{N_t} \) be its characteristic function that follows Theorem 8.7, \( \mathbf{r}_{\textit{VFFT}} \) be the \textit{VFFT} resolution that admits Lemma 8.13, \( \Delta_u \) and \( \Delta_t \) be, respectively, the number of defaults density step function and its characteristic step function those follow Lemma 8.10, \( u_r \) and \( t_g \) be, respectively, the discrete density and the characteristic sequence elements of the step function those admits Definition 8.5, \( \omega_r \) be the integration quadratic curve approximation’s weight’s at each \( r \) step, \( \sum_{(tr+di)} u \) be the summation of the truncation and discretisation error, and \( \textit{VFFT}_g[\varphi] \) be the \( \sum_{r=0}^{N-1} \varphi_r e^{i\varphi_r} (-\frac{j2\pi}{N}) \) that admits Theorem 8.5. Then the unconditional number of default’s function could be numerically evaluated by the subsequent equality:

\[
N_{tg} = \frac{\Delta_u}{2\pi} e^{\left(j\varphi_r \left(\frac{L_u}{2}\right)\right)} \textit{VFFT}_g \left( \omega_r \varphi_{N_{tg}}(u_r) e^{(-jr\Delta_u t_0)} \right) + \varepsilon^{u}_{(tr+di)}
\]
8.5.2 Evaluating the Number of Defaults’ Characteristic Function

To complete the numerical representation of the Lévy Factor Copula’s unconditional number of default’s distribution function by the Inverse Fourier Transform and the DFT, it is necessary to represent the characteristic function and its conditional default probability in that context.

In order to evaluate the intensity in the same context, it is assumed that its intensity is deterministic and discrete, where it could be easily expanded to incorporate more sophisticated structure.

Lemma 8.14 (Evaluating: the Deterministic and Continuous $\mathbb{F}$-Intensity)

Let $\gamma$ be the $\mathbb{F}$-intensity of $\tau$ with $\Gamma$ as the hazard rate those are defined, respectively, in Definition 4.16 and Definition 4.15. Then their corresponding default rate at time $t_\delta$ could be given by the sub sequent equality:

$$F_{t_\delta} = 1 - e^{(-\gamma_i t_\delta)}$$

Proof:

In consideration of Definition 4.15 and Definition 4.16 it is accepted to write the subsequent dual equalities:

$$F_t = 1 - e^{-\Gamma t} = 1 - e^{-\int_0^t \gamma_u du}$$

By associating the $t$’s to an individual credit entity $i$

$$F_{t_i} = 1 - e^{-\int_0^{t_i} \gamma_u du}$$

and finally, when discret ing the time as sequence of step function elements in Theorem 8.15, the default rate is equal to the subsequent equality:

$$F_{t_\delta} = 1 - e^{(-\gamma_i t_\delta)}$$

After representing the intensity function as a discrete and deterministic function, the characteristic function and its conditional default probability of the Lévy Factor
Copula’s unconditional number of default’s distribution function are numerically represented as in the following theorem.

**Theorem 8.16 (Lévy Factor Copula Model: Evaluating the Number of Default’s Characteristic Function and its Conditional Default Probability)**

Let $N_{tg}$ be the extracted Lévy Factor Copula’s unconditional number of default’s distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.15 and $\varphi_{N_{tg}}$ its corresponding characteristic function that admits Theorem 8.7, $p_{t_{tg}}^{\xi_{(tr+di)}}$ be the conditional default upon the Systematic Market Risk Factor $M_{i}$ in the “Lévy Factor Copula” model that admits Corollary 5.10, $\Delta_{M}$ be $M$’s density step function that follow Lemma 8.10, $m_{d}$ be the $d$ discrete density sequence elements of the step function that admits Definition 8.5, $\omega_{d}$ be the integration quadratic curve approximation’s weight’s at each $d$ step, $\varepsilon_{\xi_{(tr+di)}}$ be the summation of the truncation and discretisation error, and $F_{t_{tg}}$ be the default rate at time $t_{g}$ that admits Lemma 8.14.

Then the number of default’s characteristic function and its conditional default probability function at time $t_{g}$, denoted by $\varphi_{N_{tg}}$, could be numerically evaluated by the subsequent equality:

$$\varphi_{N_{tg}}(u_{r}) = \Delta_{M} \sum_{d=0}^{n} \prod_{i=1}^{\nu_{M}} \left(1 - (1 - e^{iu_{r}})p_{t_{tg}}^{\xi_{(tr+di)}}\right) \omega_{d}f_{M_{i}}(m_{d}) + \varepsilon_{\xi_{(tr+di)}}$$

where

$$p_{t_{tg}}^{\xi_{(tr+di)}} = F_{j_{tg}}^{-1}\left(\frac{1 - e^{-\gamma_{r}t_{tg}}}{\sqrt{1 - \rho_{i}^{2}}}\right)$$

**Proof:**

This theorem is proved in two steps; the first is to prove the evaluation step of the characteristic function, where the second is to evaluate the conditional probability.
function.

Firstly: \( \varphi_{N_t g} \) is truncated in the second equality as in Lemma 8.9 with \( L_M \) as its truncation limit and discreted in the third equality as in Lemma 8.11 by Simpson’s rule

\[
\Delta_M = \frac{L}{r^{r-1}}, \quad [m_d]_0^{r_M} = \left[ -\frac{L_M}{2} + r\Delta_M \right]^{r_M}_0, \text{ and } \omega_n = \begin{cases} 1/3, & n \text{ is zero} \\ 2/3, & n \text{ is even} \\ 4/3, & n \text{ is odd} \end{cases}
\]

These three sub-steps are represented mathematically by the series of equalities:

\[
\varphi_{N_t g}(u_r) = \int \prod_{i=1}^{+\infty} \left( 1 - (1 - e^{i u_r}) p_{t_{tg}}^{\xi_{im}} \right) f_{M_{tg}}(m) dm \\
= \int \prod_{i=1}^{-L_M/2} \left( 1 - (1 - e^{i u_r}) p_{t_{tg}}^{\xi_{im}} \right) f_{M_{tg}}(m) dm + \varepsilon_{\text{trunc}}^M \\
= \Delta_M \sum_{d=0}^n \prod_{i=1}^{r_M} \left( 1 - (1 - e^{i u_r}) p_{t_{tg}}^{\xi_{imd}} \right) \omega_d f_{M_{tg}}(m_d) + \varepsilon_{\text{trunc}}^M + \varepsilon_{\text{discr}}^M \\
= \Delta_M \sum_{d=0}^n \prod_{i=1}^{r_M} \left( 1 - (1 - e^{i u_r}) p_{t_{tg}}^{\xi_{imd}} \right) \omega_d f_{M_{tg}}(m_d) + \varepsilon_{(tr+di)}^M
\]

Secondly: by replacing \( F_{t_{tg}}^{\xi_{im}} \) that admits Lemma 8.14 by its equality, the conditional probability function could be numerically evaluated as in the following twin equality:

\[
p_{t_{tg}}^{\xi_{imd}} = F_{t_{tg}}^{\xi_{im}} \left( \frac{F_{X_{t_{tg}}}^{-1} \left( F_{t_{tg}}^{\xi_{im}} (t_{tg}^i) \right) - \rho_i m_d}{\sqrt{1 - \rho_i^2}} \right) \\
= F_{t_{tg}}^{\xi_{im}} \left( \frac{F_{X_{t_{tg}}}^{-1} \left( 1 - e^{-\gamma r t_{tg}^i} \right) - \rho_i m_d}{\sqrt{1 - \rho_i^2}} \right)
\]

By recalling the equality of the characteristic function and its conditional default probability in the Lévy Binary Stochastic Correlated Factor Copula Model, Theorem 8.16 could be rephrased as shown in the following theorem.
Theorem 8.17 (Binary Stochastic Correlated Factor Copula Model: Evaluating the Number of Default’s Characteristic Function and its Conditional Default Probability)

Let $N_{tg}$ be the extracted Lévy Factor Copula’s unconditional number of default’s distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.15 and $\varphi_{N_t}$ its corresponding characteristic function that admits Theorem 8.2, $p_{t_i}^{\xi_{|M_t}}$ be the conditional default upon the Systematic Market Risk Factor $M_t$ in the “Lévy Binary Stochastic Correlated Factor Copula” model that follow Lemma 5.8, $\Delta_M$ be $M$’s density step function that follow Lemma 8.10, $m_d$ be the d discrete density sequence elements of the step function that admits Definition 8.5, $\omega_d$ be the integration quadratic curve approximation’s weight’s at each d step, $\varepsilon_{(tr+di)}^M$ be the summation of the truncation and discretisation errors, and $F_{t_i}^{\xi}$ be the default rate at time $t_g$ that admits Lemma 8.14. Then Then the number of default’s characteristic function and its conditional default probability function at time $t_g$, denoted by $\varphi_{N_{tg}}$, could be numerically evaluated by the subsequent equality:

$$
\varphi_{N_{tg}}(u_r) = \Delta_M \sum_{d=0}^{\tau_M} \prod_{i=1}^{n} \left(1 - (1 - e^{iu_r})p_{t_i}^{\xi_{|m_d,B_i=0}}\right) \omega_d f_{M_{tg}}(m_d) + \varepsilon_{(tr+di)}^M \\
+ \Delta_M \sum_{d=0}^{\tau_M} \prod_{i=1}^{n} \left(1 - (1 - e^{iu_r})p_{t_i}^{\xi_{|m_d,B_i=1}}\right) \omega_d f_{M_{tg}}(m_d) + \varepsilon_{(tr+di)}^M
$$

Where

- $p_{t_i}^{\xi_{|m_d,B_i=0}} = (1 - q)F_{t_i}^{\xi}(\frac{F_{X_{t_i}^{\xi}}^{-1}(1-e^{(-\nu_1^2t_i^g)}) - \rho_1m_d}{\sqrt{1 - \rho_1^2}})$

- $p_{t_i}^{\xi_{|m_d,B_i=1}} = qF_{t_i}^{\xi}(\frac{F_{X_{t_i}^{\xi}}^{-1}(1-e^{(-\nu_1^2t_i^g)}) - \rho_2m_d}{\sqrt{1 - \rho_2^2}})$
Proof: (see Theorem 8.16)

In the same line, the equality of the characteristic function and its conditional default probability in Theorem 8.16 could be rearticulated to suite the Lévy Symmetric Stochastic Correlated Factor Copula Model. This could be seen in the following theorem.

**Theorem 8.18 (Lévy Symmetric Stochastic Correlated Factor Copula Model: Evaluating the Number of Default’s Characteristic Function and its Conditional Default Probability)**

Let $\mathcal{N}_t$ be the extracted Lévy Factor Copula’s unconditional number of default’s distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.15 and $\varphi_{N_t}$ its corresponding characteristic function that admits Theorem 8.3, $\varphi_{\mathcal{M}_t}$ be the conditional default upon the Systematic Market Risk Factor $\mathcal{M}_t$ in the “Lévy Symmetric Stochastic Correlated Factor Copula” model that follow Lemma 5.9, $\Delta_{\mathcal{M}}$ be $\mathcal{M}$’s density step function that follow Lemma 8.10, $m_d$ be the d discrete density sequence elements of the step function that admits Definition 8.5, $\omega_d$ be the integration quadratic curve approximation’s weight’s at each d step, $\varepsilon_{(tr+di)}^{\mathcal{M}}$ be the summation of the truncation and discretisation error, and $F_{t,g}$ be the default rate at time $t_g$ that admits Lemma 8.14. Then the number of default’s characteristic function and its conditional default probability function at time $t_g$, denoted by $\varphi_{\mathcal{N}_t,g}$, could be numerically evaluated by the subsequent equality:
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Chapter Eight: Numerical Evaluation Via DFT’s Fast (FFT) and Very Fast (VFFT) forms

\[ \varphi_{\mathcal{N}_t}(u_r) = \Delta_M \sum_{d=0}^{r_M} \prod_{i=1}^{n} \left( 1 - (1 - e^{j\mu_r}) p_{t_i}^{\xi|m_d B_s=1} \right) \omega_d f_{\mathcal{M}_t}(m_d) + \varepsilon_{(tr+dt)}^M + \Delta_M \sum_{d=0}^{r_M} \prod_{i=1}^{n} \left( 1 - (1 - e^{j\mu_r}) p_{t_i}^{\xi|m_d B_s=0, B_i=0} \right) \omega_d f_{\mathcal{M}_t}(m_d) + \varepsilon_{(tr+dt)}^M + \Delta_M \sum_{d=0}^{r_M} \prod_{i=1}^{n} \left( 1 - (1 - e^{j\mu_r}) p_{t_i}^{\xi|m_d B_s=0, B_i=1} \right) \omega_d f_{\mathcal{M}_t}(m_d) + \varepsilon_{(tr+dt)}^M \]

Where

- \( p_{t_i}^{\xi|m_d B_s=1} = \hat{q} \beta_{\mathcal{M}_t} - \frac{1}{\varphi_{\mathcal{X}_t}^{-1}(1-e^{(-\gamma_{\mathcal{I}_t} t_{\mathcal{I}_t}^g)})} \)
- \( p_{t_i}^{\xi|m_d B_s=0, B_i=0} = (1-q)(1-q) \beta_{\mathcal{M}_t} \frac{\sqrt{1-\rho^2}}{\varphi_{\mathcal{X}_t}^{-1}(1-e^{(-\gamma_{\mathcal{I}_t} t_{\mathcal{I}_t}^g)})} \rho m_d \)
- \( p_{t_i}^{\xi|m_d B_s=0, B_i=1} = (1-q)q \beta_{\mathcal{M}_t} \frac{\sqrt{1-\rho^2}}{\varphi_{\mathcal{X}_t}^{-1}(1-e^{(-\gamma_{\mathcal{I}_t} t_{\mathcal{I}_t}^g)})} \)

**Proof:** (see Theorem 8.16)

When the Lévy Random Factor Loading Copula Model’s characteristic function and its conditional default probability distribution replaces the based model articulated in Theorem 8.16, the subsequent theorem results.

**Theorem 8.19 (Lévy Random Factor Loading Copula Model: Evaluating the Number of Default’s Characteristic Function and its Conditional Default Probability)**

Let \( \mathcal{N}_t \) be the extracted Lévy Factor Copula’s unconditional number of default’s distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.15 and \( \varphi_{\mathcal{N}_t} \) its corresponding characteristic function that admits Theorem 8.4, \( p_{t_i}^{\xi|m_d B_s} \) be the conditional default upon the Systematic Market Risk Factor \( \mathcal{M}_t \) in the “Lévy Random Factor Loading Copula” model that follows Lemma 5.10, \( \Delta_M \) be \( \mathcal{M}_t \)’s density step function that follow Lemma 8.10, \( m_d \) be the \( d \) discrete density sequence elements of the step function that admits Definition 8.5, \( \omega_d \) be the integration quadratic...
curve approximation’s weight’s at each d step, \( \varepsilon^M_{(tr+di)} \) be the summation of the truncation and discretisation error, and \( F_{t_g} \) be the default rate at time \( t_g \) that admits Lemma 8.14. Then the number of default’s characteristic function and its conditional default probability function at time \( t_g \), denoted by \( \varphi_{N(t_g)} \), could be numerically evaluated by the subsequent equality:

\[
\varphi_{N(t_g)}(u_r) = \Delta_M \sum_{d=0}^{s} \prod_{i=1}^{n} \left(1 - (1 - e^{iu_r}) \eta_{ti}^{1|m_1^d} \right) \omega_{dM_t}(m_1^d) + \varepsilon^M_{(tr+di)} \\
+ \Delta_M \sum_{d=s}^{n} \prod_{i=1}^{n} \left(1 - (1 - e^{iu_r}) \eta_{ti}^{2|m_2^d} \right) \omega_{dM_t}(m_2^d) + \varepsilon^M_{(tr+di)}
\]

Where

- \( \eta_{ti}^{1|m_1^d} = F_{\eta ti}^{1|m_1^d} \left( \frac{\int_{-\lambda t_i^1}^{\lambda t_i^1} e^{-\frac{\lambda t_i^1}{1-\lambda^2}} \lambda^2 - \lambda t_i^1 \lambda^2 d\lambda}{\sqrt{1-\lambda^2}} \right) \)
- \( \eta_{ti}^{2|m_2^d} = F_{\eta ti}^{2|m_2^d} \left( \frac{\int_{-\lambda t_i^2}^{\lambda t_i^2} e^{-\frac{\lambda t_i^2}{1-\lambda^2}} \lambda^2 - \lambda t_i^2 \lambda^2 d\lambda}{\sqrt{1-\lambda^2}} \right) \)
- \( \kappa_i = -\ell_1 \Delta_M \sum_{d=0}^{D} m_1^d \omega_{dM_t}(m_1^d) - \ell_2 \Delta_M \sum_{d=0}^{D} m_2^d \omega_{dM_t}(m_2^d) + \varepsilon^M_{(tr+di)} \)

**Proof:**

Beside the proof of Theorem 8.16 the only constrain that density’s step function’s sequence elements have to satisfy is that the \( \kappa = \frac{-L}{2} + s \Delta_M \), where \( s \in [0, \sigma_M] \).

### 8.5.3 Extracting the Cumulative Loss by the Inverse Fourier Transform and the DFT

In order to achieve a complete and understandable numerical evaluation for the Lévy Factor Copula and its skewed versions, the loss given default is assumed to be homogenous and equal in this section. However, the numerical evaluation of other loss given defaults’ characteristic functions, which were explained in Subsection 8.4.3, will be presented mathematically in Subsection 8.5.5.
Assumption 8.1 (Homogenous and Equal Recovery Rate)

The loss given default’s characteristic function, denoted by \( \varphi_D \), is assumed to contain a homogeneous and equal recovery rate and nominal and follows Lemma 8.2.

As stated previously, the density function is distributed over the real number, non-periodic function, and its mass, mostly, is concentrated within a diminutive range of the infinite domain. To extract the density function from its characteristic function the same steps articulated in Subsection 8.5.1 will be followed. Firstly by applying the inverse Fourier transform on its characteristic function, secondly by truncating and discretizing its integration domain, and thirdly by choosing suitable parameters those support replacing them with the \( \mathcal{DFT} \) algorithm. The following theorem is an immediate result of following these steps.

Theorem 8.20 (Extracting the Lévy Factor Copula’s Unconditional Cumulative loss Distribution Function by the Inverse Fourier Transform and the \( \mathcal{DFT} \))

Let \( \mathcal{C}_t \) be the CDO’s cumulative loss that admits Definition 7.8 and \( \varphi_{\mathcal{C}_t} \) be its characteristic function that follows Theorem 8.11\(^{44} \), \( \varphi_{\mathcal{C}_t} \) be the \( \mathcal{DFT} \) resolution that follows Definition 8.4, \( \Delta_u \) and \( \Delta_t \) be, respectively, the number of defaults density step function and its characteristic step function those follow Lemma 8.10, \( u_r \) and \( t_g \) be, respectively, the \( r \) discrete density and the \( g \) characteristic sequence elements of the step function those admits Definition 8.5, \( \omega_r \) be the integration quadratic curve approximation’s weight’s at each \( r \) step, \( \epsilon_{(tr+di)} \) be the summation of the truncation and discretisation error, and \( \mathcal{DFT}_g[\varphi] \) be the \( \sum_{r=0}^{N-1} \varphi_r e^{i2\pi r(N^{-1})} \) that admits Theorem 8.1. Then the unconditional cumulative loss’s characteristic function could be numerically evaluated by the subsequent equality:

\(^{44}\)This representation is, also, identical for Theorem 8.7-8.10.
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Chapter Eight: Numerical Evaluation Via DFT’s Fast (FFT) and Very Fast (VFFT) forms

\[ C_{t_g} = \frac{\Delta_u}{2\pi} e^{\left(j \frac{t_g \Delta_u}{2}\Delta_t\right)} D_{\mathcal{F}} \left( \omega \varphi C_{t_g} (u_r) e^{-j r \Delta u t_o} \right) + \varepsilon^{u}_{(t_r + d_t)} \]

**Proof:** (see Theorem 8.16)

With the density function of the Lévy Factor Copula’s unconditional cumulative loss been extracted by the inverse Fourier transform and after employing the \( D_{\mathcal{F}} \) in the context, replacing the \( D_{\mathcal{F}} \) by the \( \mathcal{F} \) to increase the speed of recovering the density function requires only replacing the \( D_{\mathcal{F}} \) resolution and the related variables with ones those are suitable to employ the \( \mathcal{F} \). Taking into account Theorem 8.20 and Lemma 8.12, it requires only adjusting density’s step function articulated in Lemma 8.10. This important result is summarised in the following corollary.

**Corollary 8.7 (Extracting the Lévy Factor Copula’s Unconditional Cumulative loss Distribution Function by the Inverse Fourier Transform and the \( \mathcal{F} \))**

Let \( C_t \) be the CDO’s cumulative loss that admits Definition 6.8 and \( \varphi_{C_t} \) be its characteristic function that follows Theorem 8.11, \( \mathcal{F} \) be the \( \mathcal{F} \) resolution that admits Lemma 8.12, \( \Delta_u \) and \( \Delta_t \) be, respectively, the number of defaults density step function and its characteristic step function those follow Lemma 8.10, \( u_r \) and \( t_g \) be, respectively ,the \( r \) discrete density and the \( g \) characteristic sequence elements of the step function those admits Definition 8.5, \( \omega_r \) be the integration quadratic curve approximation’s weight’s at each \( r \) step, \( \varepsilon^{u}_{(t_r + d_t)} \) be the summation of the truncation and discretisation error, and \( \mathcal{F} = \mathcal{F}_{\mathcal{G}} \) be the \( \mathcal{F} \) that admits Theorem 8.3. Then the unconditional cumulative loss’s characteristic function could be numerically evaluated by the subsequent equality:

\[ C_{t_g} = \frac{\Delta_u}{2\pi} e^{\left(j \frac{t_g \Delta_u}{2}\Delta_t\right)} \mathcal{F} \left( \omega \varphi C_{t_g} (u_r) e^{-j r \Delta u t_o} \right) + \varepsilon^{u}_{(t_r + d_t)} \]

**Proof:** (see Theorem 8.16)
Once again, substituting the DFT by the VFFT to increase the speed of recovering the density function needs only substituting the DFT resolution and the related variables with ones that are suitable to employ the VFFT. Taking into consideration Theorem 8.20 and Lemma 8.13, extracting the density function of the Lévy Factor Copula’s unconditional cumulative loss by the inverse Fourier transform and VFFT needs only modifying the density’s step function articulated in Lemma 8.10. This significant consequence is summarised in the next corollary.

**Corollary 8.8 (Extracting the Lévy Factor Copula’s Unconditional Cumulative loss Distribution Function by the Inverse Fourier Transform and the VFFT)**

Let $C_t$ be the CDO’s cumulative loss that admits Definition 7.8 and $\varphi_{C_t}$ be its characteristic function that follows Theorem 8.11, $r_{VFFT}$ be the VFFT resolution that Lemma 8.14, $\Delta_u$ and $\Delta_\tau$ be, respectively, the number of defaults density step function and its characteristic step function those follow Lemma 8.10, $u_\tau$ and $t_g$ be, respectively, the discrete density and the $g$ characteristic sequence elements of the step function those admits Definition 8.5, $\omega_\tau$ be the integration quadratic curve approximation’s weight’s at each $r$ step, $\varepsilon^u_{(tr+dt)}$ be the summation of the truncation and discretisation error, and $VFFT_g[\varphi]$ be the $\sum_{g=0}^{N-1} \varphi_g e^{j2\pi g(-j2\pi N)}$ that admits Theorem 8.5. Then the unconditional cumulative loss’s characteristic function could be numerically evaluated by the subsequent equality:

$$C_{tg} = \frac{\Delta_u}{2\pi} e^{j\frac{tg}{2}} VFFT_g \left( \omega_\tau \varphi_{C_{tg}} (u_\tau) e^{-j\Delta_u t_0} \right) + \varepsilon^u_{(tr+dt)}$$

**Proof:** (see Theorem 8.16)

**8.5.4 Evaluating the Cumulative Loss’s Characteristic Function**

In order to complete the numerical representation of the Lévy Factor Copula’s unconditional cumulative loss distribution function by the Inverse Fourier Transform
and the $\mathcal{DFT}$, it is essential to articulate its characteristic function and their conditional cumulative loss in that context.

With the aim of evaluating the characteristic function and its conditional cumulative loss of the Lévy Factor Copula’s unconditional cumulative loss distribution function, it is essential to evaluate the loss given default characteristic function as well.

Since numerous models were presented in Subsection 8.4.3, only the default assumption will be discussed in this subsection, i.e. the homogeneous loss given default characteristic function with equal recovery rate and nominal, where the rest are represented in Subsection 8.5.5.

**Lemma 8.14 (Evaluating: Homogenous Loss Given Default’s Characteristic Function with Equal Recovery Rate and Nominal)**

Let $\mathcal{D}$ be the homogeneous loss given default with homogeneous and equal recovery rate and nominal those admits Lemma 8.2 and follow Assumption 8.1, and $u_r$ be the $r$ discrete characteristic sequence elements of the step function that admits Definition 8.5. Then the loss given default characteristic function, denoted by $\varphi_{\mathcal{D}_r}$, could be evaluated numerically by the subsequent equality:

$$\varphi_{\mathcal{D}_r}(u_r) = e^{ju_r(1-\delta)}$$

After representing the loss given default characteristic function as a homogeneous one with equal recovery rate and nominal, the characteristic function and its conditional cumulative loss of the Lévy Factor Copula’s unconditional cumulative loss distribution function are numerically represented as in the following theorem.

**Theorem 8.21 (Lévy Factor Copula Model: Evaluating the Cumulative Loss’s Characteristic Function and its Conditional Default Probability)**

Let $\mathcal{C}_t$ be the extracted Lévy Factor Copula’s unconditional cumulative loss distribution function by the Inverse Fourier Transform and the $\mathcal{DFT}$ that admits
Theorem 8.14 and \( \varphi_{C_t} \) its corresponding characteristic function that admits Theorem 8.11, \( p_{t_i}^{\xi_iM_t} \) be the conditional default upon the Systematic Market Risk Factor \( M_t \) in the “Lévy Factor Copula” model that admits Corollary 5.10, \( \Delta_{M} \) be \( M \)’s density step function that follow Lemma 8.10, \( m_d \) be the \( d \) discrete density sequence elements of the step function that admits Definition 8.5, \( \omega_d \) be the integration quadratic curve approximation’s weight’s at each \( d \) step, \( \varepsilon_{(tr+d)}^{M} \) be the summation of the truncation and discretisation error, \( F_{t_g}^{x} \) be the default rate at time \( t_g \) that admits Lemma 8.14, and \( \varphi_{D} \) be the homogenous loss given default’s characteristic function with equal recovery rate and nominal that admits Lemma 8.15. Then the cumulative loss’s characteristic function and its conditional default probability function at time \( t_g \), denoted by \( \varphi_{C_{tg}} \), could be numerically evaluated by the subsequent equality:

\[
\varphi_{C_{tg}}(u_t) = \Delta_{M} \sum_{d=0}^{r_{M}} \prod_{i=1}^{n} \left( 1 - \left( 1 - e^{iu_r(1-\delta)} \right) p_{t_g}^{\xi_i d} \right) \omega_d f_{M_{tg}}(m_d) + \varepsilon_{(tr+d)}^{M}
\]

where

\[
p_{t_g}^{\xi_i d} = F_{t_g}^{x} \left( \frac{F_{t_g}^{x-1} \left( 1 - e^{-\gamma_{r}t_g^{d}} \right) - \rho_i m_d}{\sqrt{1 - \rho_i^2}} \right)
\]

**Proof:** (see Theorem 8.16)

By evoking the equality of the conditional time of cumulative loss distribution in the Lévy Binary Stochastic Correlated Factor Copula Model, Theorem 8.21 could be rephrased as shown in the following theorem.

**Theorem 8.22 (Binary Stochastic Correlated Factor Copula Model: Evaluating the Cumulative Loss’s Characteristic Function and its Conditional Default Probability)**

Let \( C_{tg} \) be the extracted Lévy Factor Copula’s unconditional cumulative loss
distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.14 and \( \varphi_{c,t} \) its corresponding characteristic function that admits Theorem 8.6, \( p_{t_i}^{\xi_i|_{M_i}} \) be the conditional default upon the Systematic Market Risk Factor \( M_i \) in the “Lévy Binary Stochastic Correlated Factor Copula” model that follow Lemma 5.8, \( \Delta_M \) be \( M \)’s density step function that follow Lemma 8.10, \( m_d \) be the \( d \) discrete density sequence elements of the step function that admits Definition 8.5, \( \omega_d \) be the integration quadratic curve approximation’s weight’s at each \( d \) step, \( \varepsilon_M^{(l+di)} \) be the summation of the truncation and discretisation error, \( F_{t_j} \) be the default rate at time \( t_g \) that admits Lemma 8.14 and \( \varphi_{D,1} \) be the homogenous loss given default’s characteristic function with equal recovery rate and nominal that admits Lemma 8.15. Then the cumulative loss’s characteristic function and its conditional default probability function at time \( t_g \), denoted by \( \varphi_{c,t_g} \), could be numerically evaluated by the subsequent equality:

\[
\varphi_{c,t_g}(u_r) = \Delta_M \sum_{d=0}^{n} \prod_{i=1}^{r_M} \left( 1 - (1 - e^{i u_r(1-\delta)}) p_{t_i}^{\xi_i|m_d,B_i=0} \right) \omega_d f_{M_{t_g}}(m_d) + \varepsilon_M^{(l+di)} \\
+ \Delta_M \sum_{d=0}^{n} \prod_{i=1}^{r_M} \left( 1 - (1 - e^{i u_r(1-\delta)}) p_{t_i}^{\xi_i|m_d,B_i=1} \right) \omega_d f_{M_{t_g}}(m_d) + \varepsilon_M^{(l+di)}
\]

Where

- \( p_{t_i}^{\xi_i|m_d,B_i=0} = (1 - q) F_{j_{t_g}} \left( \frac{F_X^{-1}_{t_g} \left( 1 - e^{(-\gamma t_i^t g)} \right) - \rho_1 m_d}{\sqrt{1 - \rho_1^2}} \right) \)

- \( p_{t_i}^{\xi_i|m_d,B_i=1} = q F_{j_{t_g}} \left( \frac{F_X^{-1}_{t_g} \left( 1 - e^{(-\gamma t_i^t g)} \right) - \rho_2 m_d}{\sqrt{1 - \rho_2^2}} \right) \)

**Proof:** (see Theorem 8.16)
In the same line, the equality of the conditional cumulative loss distribution in Theorem 8.21 could be rearticulated to suite the Lévy Symmetric Stochastic Correlated Factor Copula Model. This could be seen in the following theorem.

**Theorem 8.23 (Lévy Symmetric Stochastic Correlated Factor Copula Model: Evaluating the Cumulative Loss’s Characteristic Function and its Conditional Default Probability)**

Let $C_{tg}$ be the extracted Lévy Factor Copula’s unconditional cumulative loss distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.14 and $\varphi_{C_t}$ its corresponding characteristic function that admits Theorem 8.7, $p_{t_i}^{M_t}$ be the conditional default upon the Systematic Market Risk Factor $M_t$ in the “Lévy Symmetric Stochastic Correlated Factor Copula” model that follow Lemma 5.9, $\Delta_M$ be $M$’s density step function that follow Lemma 8.10, $m_d$ be the d discrete density sequence elements of the step function that admits Definition 8.5, $\omega_d$ be the integration quadratic curve approximation’s weight’s at each d step, $\epsilon_{(tr+di)}^M$ be the summation of the truncation and discretisation error, $F_{t_i}$ be the default rate at time $t_g$ that admits Lemma 8.14 and $\varphi_{D_i}$ be the homogenous loss given default’s characteristic function with equal recovery rate and nominal that admits Lemma 8.15. Then the cumulative loss’s characteristic function and its conditional default probability function at time $t_g$, denoted by $\varphi_{C_{tg}}$, could be numerically evaluated by the subsequent equality:

$$
\varphi_{C_{tg}}(u_r) = \Delta_M \sum_{d=0}^{\tau_M} \left[ \prod_{i=1}^{n} \left( 1 - e^{iu(r(1-\delta))} \right) p_{t_i}^{M_t} \right] \omega_d f_{M_t} (m_d) + \epsilon_{(tr+di)}^M \\
+ \Delta_M \sum_{d=0}^{\tau_M} \left[ \prod_{i=1}^{n} \left( 1 - e^{iu(r(1-\delta))} \right) p_{t_i}^{M_t} \right] \omega_d f_{M_{tg}} (m_d) + \epsilon_{(tr+di)}^M \\
+ \Delta_M \sum_{d=0}^{\tau_M} \left[ \prod_{i=1}^{n} \left( 1 - e^{iu(r(1-\delta))} \right) p_{t_i}^{M_t} \right] \omega_d f_{M_{tg}} (m_d) + \epsilon_{(tr+di)}^M
$$
Where

1. \( p_{t_i}^{\xi|m_d,B_i=1} = \hat{q} F_{\mathcal{M}_t \tilde{g}} \left( F_{X_{t_i}^{\tilde{g}}}^{-1} \left( 1 - e^{-\gamma_t \tau_t^{\frac{1}{2}}} \right) \right) \)

2. \( p_{t_i}^{\xi|m_d,B_i=0,B_i=0} = (1 - \hat{q}) (1 - q) F_{\mathcal{J}_{t_i} \tilde{g}} \left( \frac{F_{X_{t_i}^{\tilde{g}}}^{-1} \left( 1 - e^{-\gamma_t \tau_t^{\frac{1}{2}}} \right) - \rho m_d}{\sqrt{1 - \rho^2}} \right) \)

3. \( p_{t_i}^{\xi|m_d,B_i=0,B_i=1} = (1 - \hat{q}) q F_{\mathcal{J}_{t_i} \tilde{g}} \left( F_{X_{t_i}^{\tilde{g}}}^{-1} \left( 1 - e^{-\gamma_t \tau_t^{\frac{1}{2}}} \right) \right) \)

**Proof:** (see Theorem 8.16)

When the Lévy Random Factor Loading Copula Model’s conditional cumulative loss distribution replaces the based model articulated in Theorem 8.21, the subsequent theorem results.

**Theorem 8.24 (Lévy Random Factor Loading Copula Model: Evaluating the Cumulative Loss’s Characteristic Function and its Conditional Default Probability)**

Let \( C_{t_i \tilde{g}} \) be the extracted Lévy Factor Copula’s unconditional cumulative loss distribution function by the Inverse Fourier Transform and the DFT that admits Theorem 8.14 and \( \varphi_{c_t} \) its corresponding characteristic function that admits Theorem 8.8. \( p_{t_i}^{\xi|m_{d_t}} \) be the conditional default upon the Systematic Market Risk Factor \( \mathcal{M}_t \) in the “Lévy Random Factor Loading Copula” model that follows Lemma 5.10, \( \Delta_{\mathcal{M}} \) be \( \mathcal{M} \)’s density step function that follow Lemma 8.10, \( m_d \) be the \( d \) discrete density sequence elements of the step function that admits Definition 8.5, \( \omega_d \) be the integration quadratic curve approximation’s weight’s at each \( d \) step, \( \epsilon_{(t \tau + dt)}^{\mathcal{M}_t} \) be the summation of the truncation and discretisation error, \( F_{t_i \tilde{g}} \) be the default rate at time \( t \tilde{g} \) that admits Lemma 8.14 and \( \varphi_{d_{t_i}} \) be the homogenous loss given default’s characteristic function with equal recovery rate and nominal that admits Lemma 8.15. Then the cumulative
loss’s characteristic function and its conditional default probability function at time $t_g$, denoted by $\varphi_{t_g}$, could be numerically evaluated by the subsequent equality:

$$\varphi_{N_{t_g}}(u_r) = \Delta_M \sum_{d=0}^{s} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu_r(1-\delta)}) p_{i\mid d}^{m_1} \right) \omega_d f_{M_{t_g}}(m_1^d) + \varepsilon_{(tr+di)}^M$$  

$$+ \Delta_M \sum_{d=s}^{n} \prod_{i=1}^{n} \left( 1 - (1 - e^{iu_r(1-\delta)}) p_{i\mid d}^{m_2} \right) \omega_d f_{M_{t_g}}(m_2^d) + \varepsilon_{(tr+di)}^M$$

Where

- $p_{i\mid d}^{m_1} = F_{\frac{i}{\gamma} g} \left( \frac{F_{\frac{i}{\gamma} g}^{-1} \left( 1 - e^{-\gamma^{i \mid r \mid g}} \right) - \kappa_1 - \ell_1 m_2^d}{1 - \ell_1^2} \right)$
- $p_{i\mid d}^{m_2} = F_{\frac{i}{\gamma} g} \left( \frac{F_{\frac{i}{\gamma} g}^{-1} \left( 1 - e^{-\gamma^{i \mid r \mid g}} \right) - \kappa_1 - \ell_2 m_2^d}{1 - \ell_2^2} \right)$
- $\kappa_i = -\ell_1 \Delta_M \sum_{d=0}^{\delta} m_1^d \omega_d f_{M_{t_g}}(m_1^d) - \ell_2 \Delta_M \sum_{d=0}^{\delta} m_2^d \omega_d f_{M_{t_g}}(m_2^d) + \varepsilon_{(tr+di)}^M$

**Proof:**

Beside the proof of Theorem 8.16 the only constrain that density’s step function’s sequence elements have to satisfy is that the $\kappa = \frac{\Delta}{2} + s \Delta_M$, where $s \in [0, \sigma_M]$.

### 8.5.5 Evaluating the Loss Given Default’s Characteristic Function

Numerically evaluating the loss given defaults’ characteristic functions explained in section 8.4.3 are presented mathematically in this subsection. These characteristic functions could replace the homogeneous one.

In order to numerically evaluate the Lévy Skewed Alpha-Stable loss given default’s characteristic function presented in Lemma 8.3, the subsequent lemma is presented.

**Lemma 8.16 (Evaluating: Lévy Skewed Alpha-Stable Loss Given Default’s Characteristic Function)**

Let $\varphi_{t_g}^{D_1}$ be the Lévy Skewed Alpha-Stable loss given default’s characteristic function that admits Lemma 8.3 and $u_r$ be the $r$ discrete characteristic sequence
elements of the step function that admits Definition 8.5. Then \( \varphi^{D_i}_{L(\alpha, \beta; y, \delta; 1)} \) could be evaluated numerically by the subsequent equality:

\[
\varphi^{D_i}_{L(\alpha, \beta; y, \delta; 1)}(u_r) = \begin{cases} 
  e^{(-y^2|u_r D_i|^\alpha(1-j\beta(\text{sign}(u_r D_i))(\tan(\pi u_r D_i))) + j\delta u_r D_i))}, & \alpha \neq 1 \\
  e^{(-\gamma|u_r D_i|(1+j\beta^2 \pi (\text{sign}(u_r D_i))(\ln(\pi D_i))) + j\delta u_r D_i))}, & \alpha = 1
\end{cases}
\]

With the aim of numerically evaluating the Gaussian loss given default’s characteristic function presented in Lemma 8.4, the following lemma is articulated.

**Lemma 8.17 (Evaluating: Gaussian Loss Given Default’s Characteristic Function)**

Let \( \varphi^{D_i}_{g(\mu, \sigma)} \) be the Gaussian loss given default’s characteristic function that admits Lemma 8.4 and \( u_r \) be the \( r \) discrete characteristic sequence elements of the step function that admits Definition 8.5. Then \( \varphi^{D_i}_{g(\mu, \sigma)} \) could be evaluated numerically by the subsequent equality:

\[
\varphi^{D_i}_{g(\mu, \sigma)}(u_r) = e^{(\mu u_r + (\sigma u_r D_i)^2)}
\]

To numerically evaluate the Standard Lévy loss given default’s characteristic function articulated in Lemma 8.4, the subsequent lemma is presented.

**Lemma 8.18 (Evaluating: Standard Lévy Loss Given Default’s Characteristic Function)**

Let \( \varphi^{D_i}_{S L(\alpha, \beta)} \) be the Standard Lévy loss given default’s characteristic function that admits Lemma 8.5 and \( u_r \) be the \( r \) discrete characteristic sequence elements of the step function that admits Definition 8.5. Then \( \varphi^{D_i}_{S L(\alpha, \beta)} \) could be evaluated numerically by the subsequent equality:

\[
\varphi^{D_i}_{S L(\alpha, \beta)}(u_r) = \begin{cases} 
  e^{(-u_r D_i |\alpha(1-j\beta(\text{sign}(u_r D_i))(\tan(\pi u_r D_i))) + j\delta u_r D_i))}, & \alpha \neq 1 \\
  e^{(-u_r D_i |(1+j\beta^2 \pi (\text{sign}(u_r D_i))(\ln(\pi D_i))) + j\delta u_r D_i))}, & \alpha = 1
\end{cases}
\]

In order to numerically evaluate the Generalized Hyperbolic loss given default’s
characteristic function presented in Lemma 8.4, the subsequent lemma is noted.

**Lemma 8.19 (Evaluating: Generalized Hyperbolic Loss Given Default’s Characteristic Function)**

Let \( \varphi_{GH}^{D_1}(\lambda, \alpha, \beta, \delta, \mu) \) be the Generalized Hyperbolic loss given default’s characteristic function that admits Lemma 8.6 and \( u_r \) be the \( r \) discrete characteristic sequence elements of the step function that admits Definition 8.5. Then \( \varphi_{GH}^{D_1}(\lambda, \alpha, \beta, \delta, \mu) \) could be evaluated numerically by the subsequent equality:

\[
\varphi_{GH}^{D_1}(\lambda, \alpha, \beta, \delta, \mu)(u_r) = e^{j u_r D_1} \left( \frac{\omega}{\alpha^2 - (\beta + j u_r D_1)^2} \right)^{\frac{1}{\lambda}} K_{\lambda} \left( \delta \sqrt{\alpha^2 - (\beta + j u_r D_1)^2} \right)
\]

With the purpose of numerically evaluating the Variance Gamma loss given default’s characteristic function articulated in Lemma 8.4, the subsequent lemma is expressed.

**Lemma 8.20 (Evaluating: Variance Gamma Loss Given Default’s Characteristic Function)**

Let \( \varphi_{VG}^{D_1}(\lambda, \alpha, \beta, \mu) \) be the Variance Gamma loss given default’s characteristic function that admits Lemma 8.7 and \( u_r \) be the \( r \) discrete characteristic sequence elements of the step function that admits Definition 8.5. Then \( \varphi_{VG}^{D_1}(\lambda, \alpha, \beta, \mu) \) could be evaluated numerically by the subsequent equality:

\[
\varphi_{VG}^{D_1}(\lambda, \alpha, \beta, \mu)(u_r) = e^{j u_r D_1} \left( \frac{\omega}{\alpha^2 - (\beta + j u_r D_1)^2} \right)^{\lambda} K_{\lambda} \left( \delta \sqrt{\alpha^2 - (\beta + j u_r D_1)^2} \right)
\]

With the aim of numerically evaluating the Normal Inverse Gaussian loss given default’s characteristic function stated in Lemma 8.4, the subsequent lemma is noted.

**Lemma 8.21 (Evaluating: Normal Inverse Gaussian Loss Given Default’s Characteristic Function)**

Let \( \varphi_{NIG}^{D_1}(\lambda, \alpha, \beta, \delta, \mu) \) be the Normal Inverse Gaussian loss given default’s characteristic function that admits Lemma 8.8 and \( u_r \) be the \( r \) discrete characteristic sequence elements of the step function that admits Definition 8.5. Then \( \varphi_{NIG}^{D_1}(\lambda, \alpha, \beta, \delta, \mu) \) could be evaluated numerically by the subsequent equality:

\[
\varphi_{NIG}^{D_1}(\lambda, \alpha, \beta, \delta, \mu)(u_r) = e^{j u_r D_1} \left( \frac{\omega}{\alpha^2 - (\beta + j u_r D_1)^2} \right)^{\lambda} K_{\lambda} \left( \delta \sqrt{\alpha^2 - (\beta + j u_r D_1)^2} \right)
\]
elements of the step function that admits Definition 8.5. Then \( \varphi_{N,\tilde{g}(\alpha, \beta, \delta, \mu)}^{\mathcal{D}_1} \) could be evaluated numerically by the subsequent equality:

\[
\varphi_{N,\tilde{g}(\alpha, \beta, \delta, \mu)}^{\mathcal{D}_1}(u_r) = e^{j\mu u_r \mathcal{D}_1} \frac{e^{\delta \sqrt{\omega}}}{e^{\delta \sqrt{\alpha^2} - (\beta + j\mu \mathcal{D}_1)^2}}
\]

**8.6 Mathematical Summary**

**Discrete Fourier Transform**

- **Sequence representation:**
  \[
  [f_g]_0^{N-1} = \mathcal{DFT}[\varphi_r]_0^{N-1} = \sum_{r=0}^{N-1} \varphi_r e^{j2\pi r \left(-\frac{j2\pi}{N}\right)}
  \]

- **Matrix representation:** \( f = \mathcal{F}_N \cdot \varphi \)

- \( \mathcal{O}(N^2) \), i.e. \( 4N^2 \) multiplications and \( N(4N - 2) \) additions.

**Fast Fourier Transform**

- **Matrix representation:** \( f = \mathcal{F}_N \cdot \varphi \), where \( \mathcal{F}_N = \mathcal{A}\{0\}, \ldots, \mathcal{A}\{s\}, \mathcal{P}, \mathcal{A}\{\} \) is factorisation of \( \mathcal{F}_N \), and \( \mathcal{P} \) is its permutation matrix.

- **Sequence representation:**
  \[
  [f_g]_0^{N-1} = \mathcal{DFFT}[\varphi_r]_0^{N-1}
  \]
  \[
  = \sum_{r=0}^{N-1} \varphi_r e^{j2\pi r \left(-\frac{j2\pi}{N}\right)}
  \]
  \[
  = \sum_{r=0}^{N-1} \varphi_{(2r)} e^{j2\pi \left(-\frac{j2\pi}{2}\right)} + \sum_{r=0}^{N-1} \varphi_{(2r+1)} e^{j2\pi \left(-\frac{j2\pi}{2}\right)}
  \]
  \[
  = \sum_{r=0}^{N-1} \varphi_{(2r)} e^{j2\pi \left(-\frac{j2\pi}{2}\right)} + \sum_{r=0}^{N-1} \varphi_{(2r+1)} e^{j2\pi \left(-\frac{j2\pi}{2}\right)}
  \]
  \[
  = \mathcal{FFFT}[\varphi_r]_0^{N-1}
  \]

- **complexity is** \( \mathcal{O}(N \log_2 N) \).

**Very Fast Fourier Transform**

- \( \mathcal{F}_N = \mathcal{G}_N \cdot \mathcal{H}_N \), where \( \mathcal{G}_N = G_{1}^{\mathcal{L}} \cdots G_{m}^{\mathcal{L}} \cdot G_{n}^{\mathcal{P}} \cdot G_{1}^{\mathcal{S}} \cdots G_{1}^{\mathcal{R}} \) is factorised complex and encloses the phase point information and
\[ \mathcal{H}_N = \mathcal{P} . \mathcal{H}^L_1 . \ldots \mathcal{H}^L_s . \mathcal{H}^D . \mathcal{H}^L_1 . \ldots \mathcal{H}^R_1 . \mathcal{P}^T \] is factorised real and includes the amplitude information.

- \textit{Complexity is } \mathcal{O}(3N).

\textbf{Lévy Factor Copula (Number of Defaults’ Characteristic Functions):}

- \[ \varphi_{N_t}(u) = \int \prod_{i=1}^{n} \left( 1 - e^{i u \xi_i} \right) p_{\xi_i}^u \left( m \right) dm \]

\textbf{Lévy Factor Copula Model (Cumulative loss Characteristic Function):}

- \[ \varphi_{c_t}(u) = \int \prod_{i=1}^{n} \left( 1 - \varphi_{D_i}(u) \right) p_{\xi_i}^u \left( m \right) dm \]

\textbf{Loss Given Default’s Characteristic Function}

- \textit{Homogenous: } \[ \varphi_{D_i}(u) = e^{i u (1-\delta)} \]

\textbf{Extracting the Unconditional Number of Default’s by the Inverse Fourier Transform and the DFT}

- \textbf{Truncation: } \[ f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(-jut)} \varphi_{N_t}(u) du + \epsilon_{\text{trunc}}^{u} \]

- \textbf{Density Step Function: } \[ \Delta_x = \frac{1}{d} \tilde{GCD}(\mathbb{D}) \]

- \textbf{Discrete Density Resolution } \[ \tau_{\text{Disc}} = \left\lfloor \frac{L}{\Delta_x} \right\rfloor + 1 \]

- \textbf{Density’s Step Function: } \[ \Delta_x = \frac{L}{\tau_{\text{Disc}} - 1} \]

- \textbf{Discrete Density Step Function Elements: } \[ [x_r]_{r_{\text{Disc}}} = a + r\Delta \]

- \textbf{Simpson’s Rule: } \[ \int_a^b f(x) dx = \Delta_x \sum_{r=0}^{r_{\text{Disc}}-1} \omega_n f(x_r) + \epsilon_{\text{discr}}^x, \text{ and } \omega_n = \]

\[ \begin{cases} 1/3, & n \text{ is zero} \\ 2/3, & n \text{ is even} \\ 4/3, & n \text{ is odd} \end{cases} \]

\textbf{Number of Defaults Characteristic Function:}

- \textbf{Evaluating Lévy Factor Copula’s:}

\[ N_{t_g} = \frac{\Delta u}{2\pi} e^{\left(i g \left(\frac{lu}{\tau}\right)\right)} \tilde{DFT}_g \left( \omega_r \varphi_{N_{t_g}}(u_r) e^{(-jr \Delta u t_0)} \right) + \epsilon_{(tr+di)}^u \]
Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT

- Homogenous Loss Given Default’s Characteristic Function

\[ \varphi_{D_t}(u_r) = e^{i u_r \left(\frac{1 - \delta}{\pi}\right)} \]

- Lévy Factor Copula:

\[ \varphi_{C_t}(u_r) = \Delta^d \sum_{d=0}^{\infty} \prod_{i=1}^{n} \left(1 - e^{i u_r (1 - \delta)} p^{\xi | m_d}_{t g} \right) \varphi_{C_t g} (u_r) e^{(-j \Delta u t_0)} + \varphi_{C_t g} (u_r) e^{(-j \Delta u t_0)} + \varepsilon_{(tr+di)} \]

\[ p^{\xi | m_d}_{t g} = F_{D_t g} \left( \frac{\sum_{d=0}^{\infty} \prod_{i=1}^{n} \left(1 - e^{i u_r (1 - \delta)} p^{\xi | m_d}_{t g} \right) \varphi_{C_t g} (u_r) e^{(-j \Delta u t_0)} + \varphi_{C_t g} (u_r) e^{(-j \Delta u t_0)} + \varepsilon_{(tr+di)} }{\sqrt{1 - \rho_i^2}} \right) \]
Chapter Nine

Conclusion and Recommendations for Further Work

9.1 Conclusion

Contrary to the normality assumption of the credit derivatives’ market’s standard model introduced in (Li, 2000) this thesis has proposed a generalised and extended forms to overcome its limitations through the Lévy process. This process provides an appropriate framework that satisfactorily overcomes the Gaussian process drawbacks. The Lévy Factor Copula Model expands and homogenises the atomic approach of exploring the Gaussian Factor Copula Model’s problems. It introduces an endless number of alternative distributions. Subsequently, this thesis has proposed the “Stochastic Correlated Lévy Factor Copula Model” and “Lévy Random Factor Loading Copula Model” in order to enhance the skewness of its correlation.

One of many advantages of the Lévy process is that it could include a variety of processes structural assumptions from pure jumps to continuous stochastic. These distributions those admit this process could represent asymmetry and fat tails as they could represent symmetry and normal tails. As a consequence they could capture both high and low events probabilities. In this thesis after introducing the Lévy Factor Copula and its skewed models, a number of limiting and mixture cases of the Lévy Skew Alpha-Stable Distribution and Generalized Hyperbolic Distribution are newly proposed in this thesis, see Table 6.1.

These models are presented by the corresponding characteristic functions of the number of defaults, the accumulated loss, and loss given default, where some new loss given
defaults are newly proposed. Numerically, these characteristic functions could be evaluated by semi-explicit evaluation techniques, i.e. via the FFT’s Fast form (FFT) and the proposed Very Fast form (VFFT). The FFT was proposed implicitly by other researchers, where her it was illustrated explicitly. This technique through its fast and very fast form reduce the computational complexity from $O(N^2)$ to, respectively, $O(N \log_2 N)$ and $O(N)$.

As stated previously, the FFT has revolutionised the numerical evaluation techniques in the financial instruments. It overcomes the complexity of measuring and pricing these instruments with a fast and stable/reliable numerical method and the difficulties of solving the dimensionality problem is not a problematic issue anymore. More proficiently, the VFFT could overcome these problems with a higher speed and accuracy, which could lighten a new direction of real time evaluation and pricing.

9.2 Recommendations for Further Work

With the research presented in this thesis there are many aspects those could be investigated. To list some of many recommendations for further works, see the subsequent points:

- Examining the replacement of the linear correlation in the context of Lévy Factor Copula Models by Kendall’s tau and the Spearman’s rho as these could fill the gap of the copula invariant property under strictly monotone functions.

- Investigating the scope of applications of each proposed distribution. This could be achieved by investigating the characteristic of the credit reference entities’ default times’, Systematic Market Risk Factors’, and Idiosyncratic Risk Factors’ mean, variance, skewness, kurtosis, and even higher moments and link them to the appropriate distributions those could imply them.
- Extending the Lévy Factor Copula Models by Implementing a stochastic recovery rates.

- Expanding the one Lévy Factor Copula Models introduced here to incorporate multifactor Lévy Factor Copula Models; in order to model more complex driven credit risk products.

- Implementing the Lévy Factor Copula Model and the numerical evaluation via the $DFT$’s Fast ($FFT$) and the Very Fast ($VFFT$) forms in other credit risk derivatives.

- Implementing the $VFFT$ as a faster and more reliable alternative method to evaluate the financial instruments those are already evaluated by the $FFT$. 
10.0 References


ANDERSEN, L., SIDENIUS, J. & BASU, S. 2003. All your Hedges in One Basket, RISK.

BACHELIER, L. 1900. Théorie de la spéculacion. Annales Scientifiques de l’École Normale Supérieure, 3, 21-86


Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT


Mathematical Finance, 10, 179-195.


LÉVY, P. 1925. Calcul des Probabilites, Gauthier Villars.


Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT


Pricing Basket CDS&CDO by Lévy Factor Copula and its skewed versions and evaluated by V-FFT


VASICEK, O. 1997. The Loan Loss Distribution. KMV.


XU, G. 2006. Extending Gaussian copula with jumps to match correlation smile Wachovia Securities.
