Chapter Six

Discrete-time Queue Analytical Models Based on Different AQM Algorithms

6.1 Introduction

Since the congestion problem has emerged in computer networks [67], relatively few research works have been conducted on developing analytical models for the various AQM approaches [19, 83], with most studies focused on using simulation. This is due to several reasons such as the difficulty in designing analytical models [122], and the diversity of data traffic loads such as video, image and voice. The aims of building analytical models are 1) To obtain a better insight into how the modelled systems operate and 2) to enhance network performance using the models to select optimum parameter values to provide satisfactory QoS.

For modelling queueing networks, one can use either continuous-time queues or discrete-time queues [118]. In this chapter, discrete-time queues are used since the relative simplicity of the analytical techniques can be exploited in analysing complex queueing
systems [118]. Also, relatively few works have been conducted on modelling and analysing queueing networks based on discrete-time queues [59, 60, 115, 118]. The authors in [19, 83] presented analytical models based on the RED algorithm in order to erase the bias of bursty traffic in controlling the average queueing delay for the packets. Henderson and Taylor [59, 60] introduced some discrete-time queueing network analytical models that rely upon multiple arrivals and departures of packets.

A product form result was proposed by the authors in [59, 60]. This product form result uses for a certain class of queueing networks. Also in [59, 60], the equilibrium distribution was obtained by assuming batches of customers, such as packets can depart from queue nodes and route to other queue nodes to arrive in batches at a later time point.

Moreover, Woodward [118] has written a book about modelling and analysing the performance of the queueing systems in computer networks and communications. This book includes details about the discrete-time queue approaches and how they can be used to model the computer and communication systems.

In this chapter, eight different discrete-time queue analytical models based on different existing AQM approaches such as RED [46], DRED [11], GRED [50] and BLUE [39, 43], are developed. The aim is to utilise them as congestion control approaches in wired networks. Another aim is to obtain satisfactory performance measure results when congestion is present.

The proposed discrete time queue analytical models are: Two discrete-time queue models based on the RED algorithm, these models are called RED Alpha and RED Linear models. Furthermore, two discrete-time queue models are based on the GRED method; these models are called GRED Alpha and GRED Linear. Moreover, two discrete-time
queue analytical models based on the BLUE algorithm are presented (BLUE Alpha, BLUE Linear). Lastly, two discrete-time queue analytical models based on the DRED algorithm are developed and these are called DRED Alpha and DRED Linear. The details of the proposed discrete-time queue analytical models are presented in Sections (6.3-6.6).

This chapter is organised as follows: Section 6.2 covers single queue node system that is analysed using the discrete-time queue approach with respect to the algorithms of DRED, BLUE, RED and GRED. Discrete-time queue analytical models based on DRED, BLUE, RED and GRED are presented in Sections 6.3, 6.4, 6.5 and 6.6, respectively. Moreover, these sections include simulation results for the DRED, BLUE, RED and GRED algorithms and the numerical results with reference to the developed analytical models.

**6.2 The Single Queue Node System**

In this section, a single queue node system is introduced and the proposed discrete-time queue analytical models for the DRED, BLUE, RED and GRED, are based on this system. The purpose of the analysis of the single node system is to detect the congestion buffers early in order to utilise the results obtained from the models to control congestion in wired networks. The single queue node system used is shown in Figure 6.1 where the threshold locations including $th$, $min\, threshold$, $max\, threshold$ and $double\, max\, threshold$ are applied at the router buffer of Figure 6.1 in order to develop the discrete-time queue analytical models. The router buffer is analysed using the discrete-time queue approach.
The outputs from this analysis are eight discrete-time queue analytical models which are described in detail in Sections (6.3-6.6).

Moreover, every AQM method is compared with the analytical models that are based on it, i.e. the RED method is compared with the proposed RED Alpha and RED Linear with reference to the following performance measures \( mql, T, D, P_L, D_p, P_{loss} \). This is because it is desired to obtain which method or analytical model provides better performance measure results. It should be made clear at this point that we are not trying to validate the analytical models using the simulations since the analytical models essentially represent different methods. e.g. RED (simulation) uses average queue length as a congestion measure and the RED-Alpha and RED-Linear use the instantaneous queue length as the congestion measure. The comparison is to see which gives the better performance. The threshold \( th \) position at router buffers of the BLUE and its corresponded analytical models is examined, and the \( \text{min}threshold \) position at the router buffers of the RED and its corresponded analytical models is examined as well. The examination of \( th \) and \( \text{min}threshold \) helps in detecting the best \( th \) and \( \text{min}threshold \) that identifies the most satisfactory performance measure results. The results of performance measures are achieved based on setting the packet arrival probability parameter to different values. In addition, part of the performance measure results of the BLUE, BLUE analytical models, RED, and RED analytical models are carried out based on setting the \( th \) and \( \text{min}threshold \) parameters to different values. Hence, the decision which algorithm or model presents more satisfactory performance measure results is only given
with regard to the parameter values of the packet arrival probability and both \( th \) and \( \min \) threshold.

The performance measurements on the AQM method simulations are carried out only after the system reaches to a steady state. The simulations have been run ten times, each run with a different seed, for each simulation point. The performance measure results \((mql, T, D, P_{loss})\) and \((P_L, D_p)\) of the AQM methods are represented by the mean results of the ten runs and a statistical analysis was also carried out.

The AQM methods and proposed analytical models are implemented using a Java environment on a 1.66 Centrino machine with 1024 MB RAM.

\[ \alpha \]

\[ \beta \]

Packet arrival

Packets queued in the router buffer

**Figure 6.1:** The single queue node system.

In Figure 6.1, \( \alpha \) denotes the probability of packet arrival and \( \beta \) is the probability of the packet departure. In addition, this queue node has a finite system capacity \( (K) \). Packet inter-arrival times and service times are geometrically distributed with means \( \frac{1}{\alpha} \) and \( \frac{1}{\beta} \), respectively.
6.3 DRED Based Discrete-time Queue Analytical Models

This section introduces two discrete-time queue analytical models [7] based on the DRED algorithm to detect congestion in the network router buffer in the early stages. The DRED-based analytical models are called Alpha [7] and Linear. The DRED models also aim to provide satisfactory performance measure results when a high congestion exists. Another aim for the DRED-Alpha model is maintaining \( mql \) and \( D \) at values smaller than the DRED’s values in order to give less buffer overflow than DRED. This section is structured as follows: Subsections 6.3.1 and 6.3.2 give details of the Alpha and Linear analytical models, respectively. The simulation and the numerical results of DRED and the two proposed analytical models are discussed in Subsection 6.3.3. Finally, a summary is presented in Subsection 6.3.4.

6.3.1 DRED-Alpha Analytical Model

The following model is based on the DRED algorithm [11] and is called DRED-Alpha [7]. This model is analysed using the single queue node system displayed in Figure 6.2. The queue node system given in Figure 6.1 is modified by adding the threshold \( (th) \) as a position in the router buffer. The difference between Figures 6.1 and 6.2 is that the router queue in Figure 6.2 starts dropping packets at early points, and this happens when the current queue length \( (ql) \) reaches the \( th \) position, whereas the queue node in Figure 6.1
drops packets only after the queue becomes full. The analytical model and DRED are implemented at the router buffer [11]. The analytical model goal is to identify and react to the congestion at its router buffer early in order to adjust the probability of packet arrival for the network to alleviate the congestion. The proposed DRED-Alpha model also aims to provide satisfactory performance measure results when heavy congestion is present, and to maintain its $mql$ and $D$ at values lower than the DRED’s values with aim of lowering overflow buffer.

The proposed analytical model uses the discrete-time queue approach [118] to analyse the queue node system shown in Figure 6.2. Normally, the discrete-time queue approach relies on a specific time unit called a slot, where single or multiple events may occur during each slot. An example of a single event in a slot is the arrival or the departure of a packet, while packet arrival and departure in a single slot is an example of multiple events. The finite queue node capacity is $K$ packets, which includes any packet currently in service. In addition, the arrival process employed is an i.i.d Bernoulli process, $a_n \in \{0,1\}, n = 0,1,2,...,$ where $a_n$ represents the number of packet arrivals in slot $n$. It should be noted that the DRED-based analytical models use $th$ as a congestion detector [7, 11], where $th$ denotes the threshold position at the router buffer, similar to the classic DRED algorithm. In the proposed analytical model [7], $th$ is given a value equal to that of [11] and given in equation (6.1).

\[
th = \frac{0.9K}{2}
\]

(6.1)
In Figure 6.2, the probability of packet arrival ($\alpha$) starts when the $ql$ at the router buffer is less than $th$ position, and thereby no packets are dropped ($Dp = 0$). However, if the $ql$ reaches the $th$ position, the probability of packet arrival decreases from $\alpha_1$ to $\alpha_2$ aiming to reduce the congestion at the router buffer, and the $Dp$ value increases from 0 to $\left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right)$.

![Figure 6.2: The single queue node system for the DRED Alpha model.](image)

The parameters ($\alpha_1, \alpha_2$) denote the probability of packet arrival in a slot before and after the $ql$ reaches the $th$ position at the router buffer, respectively. $\beta$ represents the packet departure probability from the router buffer in a slot. The queueing discipline used in the DRED-Alpha analytical model is first come first served (FCFS), the queue node system is assumed in equilibrium, and the queue length process is a Markov chain with finite state space. The state space is $\{0, 1, 2, 3, \ldots, th - 1, th, th + 1, \ldots, K - 1, K\}$. In analysing the performance, values of $\alpha_1$ are such that $\alpha_1 > \alpha_2$ with half of $\alpha_1$ values are smaller than $\beta$ ($\beta > \alpha_1$) and the other values higher than $\beta$ ($\beta < \alpha_1$) in order to create a congestion.
situation. Furthermore, $\beta > \alpha 2$. The state transition diagram of the DRED-Alpha queue node system is illustrated in Figure 6.3.

Figure 6.3: The state transition diagram for the DRED Alpha analytical model.

Figure 6.3 can be used to derive the balance equations (6.2-6.8) for the analytical model. These are:

$$p_0 = (1 - \alpha 1)p_0 + [\beta(1 - \alpha 1)]p_1 \quad \cdots \quad (6.2)$$

$$p_1 = \alpha 1p_0 + [\alpha 1\beta + (1 - \alpha 1)(1 - \beta)]p_1 + [\beta(1 - \alpha 1)]p_2 \quad \cdots \quad (6.3)$$

In general:

$$p_i = [\alpha 1(1 - \beta)]p_{i-1} + [\alpha 1\beta + (1 - \alpha 1)(1 - \beta)]p_i + [\beta(1 - \alpha 1)]p_{i+1} \quad , \quad \text{where} \quad i = 2, 3, 4, \ldots, th - 2 \quad \cdots \quad (6.4)$$

$$p_{th-1} = [\alpha 1(1 - \beta)]p_{th-2} + [\alpha 1\beta + (1 - \alpha 1)(1 - \beta)]p_{th-1} + [\beta(1 - \alpha 2)]p_{th} \quad \cdots \quad (6.5)$$

$$p_{th} = [\alpha 1(1 - \beta)]p_{th-1} + [\alpha 2\beta + (1 - \alpha 2)(1 - \beta)]p_{th} + [\beta(1 - \alpha 2)]p_{th+1} \quad \cdots \quad (6.6)$$
\[ p_i = [\alpha 2(1-\beta)]p_{i-1} + [\alpha 2\beta + (1-\alpha 2)(1-\beta)]p_i + [\beta(1-\alpha 2)]p_{i+1}, \]  
\[ \text{where } i = th + 1, th + 2, th + 3, ..., K - 1 \]

Finally,

\[ p_K = [\alpha 2(1-\beta)]p_{K-1} + [\alpha 2\beta + (1-\beta)]p_K, \text{ where } K = I + th \]  
\[ \text{(6.8)} \]

Let \( \gamma_i = \frac{\alpha i(1-\beta)}{\beta(1-\alpha i)}, i = 1, 2, \ldots \)  
\[ \text{……………………………………………………………………………………………... (6.9)} \]

In order to obtain the equilibrium probabilities \( (p_1, p_2, \ldots, p_K) \) in the proposed DRED-Alpha analytical model, equation (6.9) can be applied in the above balance equations as shown below.

In general:

\[ p_i = \frac{\alpha^i(1-\beta)^{i-1}}{\beta^i(1-\alpha)^i} p_0 \]  
\[ \text{where } i = 1, 2, 3, ..., th - 1 \]  
\[ \text{……………………………………………………………………………………………... (6.10)} \]

\[ p_{th+i} = \frac{\alpha^i \alpha^i (1-\beta)^{i-1}}{\beta^i(1-\alpha)^i(1-\alpha 2)^{i-1}} p_0 = \frac{\gamma^i \gamma^i (1-\alpha 1)}{(1-\alpha 2)(1-\beta)} p_0, \]  
\[ \text{where } i = 0, 1, 2, 3, ..., I \text{ and } I = K - th \]  
\[ \text{……………………………………………………………………………………………... (6.11)} \]

Then, the probability that the queue node system is empty (or the probability that there are no packets in the queue node system \( (p_0) \)) is computed. The \( p_0 \) result is obtained using the normalising equation (6.12).
\[ \sum_{i=0}^{\infty} p_i = 1 \] ................................................................. (6.12)

By substituting the equilibrium probabilities (equations (6.10-6.11)) in equation (6.12), the final form for the \( p_0 \) is obtained as in equation (6.13).

\[
p_0 = \left[ \frac{1 - \gamma_1(1 - \gamma_1)}{(1 - \beta)(1 - \gamma_1)} + \frac{\gamma_1(1 - \alpha_1)(1 - \gamma_2^{i+1})}{(1 - \beta)(1 - \alpha_2)(1 - \gamma_2)} \right]^{-1} \] .............................................. (6.13)

The performance measures (mean queue length \( mql \), Throughput \( T \), average queueing delay \( D \), overflow packet loss probability \( P_L \), packet dropping probability \( D_p \) and the total of overflow packet loss and dropping probabilities \( P_{loss} \)) can be calculated for the proposed analytical model to evaluate its performance. Initially, the \( mql \) can be calculated using the generating function \( P(z) \), where \( P(z) \) can be defined as in equation (6.14).

\[
P(z) = \sum_{i=0}^{\infty} z^i p_i \] ................................................................. (6.14)

Taking the first derivative of \( P(z) \) at \( z = 1 \) the \( mql \) is obtained as

\[
mql = P^{(0)}(1) = p_0 \left[ \frac{1 - \gamma_1}{(1 - \gamma_1)^2} + \frac{\gamma_1(1 - \alpha_1)(1 - \gamma_2^{i+1})}{(1 - \alpha_2)(1 - \gamma_2)^2} \right] \] ................................................................. (6.15)
Alternatively, the \( mql \) can be evaluated using the \( \sum_{i=0}^{K} i \times p_i \) equation. After the \( mql \) is calculated, the \( T \) measure can be obtained as the number of packets that pass through the system queue node per slot. Another definition of \( T \) for a single server system is the fraction of time that the router server is busy in serving packets. \( T \) is computed according to equation (6.16).

\[
T = \beta \sum_{i=1}^{K} \prod_{i} = \beta (1 - p_0) \text{ packets/slot} \quad \ldots \quad (6.16)
\]

Now, depending on the \( mql \) and the \( T \) results, and using Littles law (equation (6.17)),

\[
D = \frac{mql}{T} \text{ slots} = \frac{P^{(i)}(1)}{T} \text{ slots} = \frac{\sum_{i=0}^{K} i \times p_i}{T} \text{ slots} \quad \ldots \quad (6.17)
\]

Finally, the DRED-Alpha router buffer starts dropping packets when the \( ql \) is larger than or equal to \( th \) position and starts losing packets when its router buffer becomes full. The loss probability is defined as the probability of losing packets due to buffer overflow and denoted by \( P_L \), while the dropping probability of packets can be defined as the probability of packets being dropped probabilistically before the buffer is full, denoted by \( D_p \). Thus, the \( P_{Loss} \) can be given as the sum of probabilities of packets that have been lost and dropped at the router buffer from all the packets that have arrived; the \( P_L, D_p \) and \( P_{Loss} \) are computed using equations (6.18-6.20), respectively.
\[ P_L = (1 - \beta)P_K \] ......................................................... (6.18)

\[ D_p = \left( \frac{\alpha_1 - \alpha_2}{\alpha_2} \right) \times \sum_{i=th}^{K-1} p_i \] ......................................................... (6.19)

\[ P_{Loss} = P_L + D_p \] ......................................................... (6.20)

### 6.3.2 DRED-Linear Analytical Model

This subsection proposes a discrete-time queue analytical model based on the DRED approach called DRED-Linear. DRED-Linear is built in discrete-time based on the queue node system displayed in Figure 6.4. It is obvious from Figure 6.4 that the only difference between the model in this subsection and the model in Subsection 6.3.1 is in the values of packet arrival probability and \( Dp \) when the congestion occurs at the router buffer (\( ql \geq th \)).

![Figure 6.4: The single queue node system for the DRED Linear model.](image-url)
Hence, if the condition \( q_l \geq th \) is true, DRED-Alpha decreases its packet arrival probability from \( \alpha_1 \) to another fixed and smaller value (\( \alpha_2 \)). Whereas, DRED-Linear decreases the probability of packet arrival linearly from \( \alpha_1 \) to another probability value (\( \alpha_i \)) that depends on the state transition diagram shown in Figure 6.5, where

\[ \alpha_i = \alpha_1 - \left( 1 + i - th \right) \frac{\alpha_1}{(1 + K - th)}, \quad \text{if } i \geq th. \]

\( \alpha_i \) denotes the effective packet arrival probability in a slot (original arrival rate less rate of dropping packets) when the \( q_l \) is equal to or larger than the \( th \) position at the router buffer. In a corresponding way, the DRED-Linear model increases its \( Dp \) value linearly from 0 to \( \left( \frac{\alpha_1 - \alpha_i}{\alpha_1} \right) \) as the \( q_l \) increases from \( th \) position to \( K \). However, if the condition \( q_l \geq th \) is false, the packet arrival probability and \( Dp \) of the DRED-Linear model are as those of DRED-Alpha. The parameters (\( \alpha_1, \beta, th, K \)) of the DRED-Linear model are as those of the Alpha model, and the queueing discipline used is FCFS. It is assumed that the queue node system is in equilibrium and the queue length process is a Markov chain with finite state space. The state space is \( \{0,1,2,3,\ldots,th-1,th,th+1,\ldots,K-1,K\} \). It is also assumed that \( \alpha_1 > \alpha_i \) and \( \alpha_1 \) varies such that half of \( \alpha_1 \) values are smaller than \( \beta \) (\( \beta > \alpha_1 \)) and the rest of the values are higher than \( \beta \) (\( \beta < \alpha_1 \)). Also \( \beta > \alpha_i \). The state transition diagram for the proposed model is depicted in Figure 6.5.
The state transition diagram in Figure 6.5 can be used to compute the balance equations (6.21-6.27) for the analytical model.

\[ p_0 = (1 - \alpha_1) p_0 + [\beta(1 - \alpha_1)] p_1 \] ............................ (6.21)

\[ p_1 = \alpha_1 p_0 + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)] p_1 + [\beta(1 - \alpha_1)] p_2 \] ............................ (6.22)

In general:

\[ p_i = [\alpha_1(1 - \beta)] p_{i-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)] p_i + [\beta(1 - \alpha_1)] p_{i+1} \], where

\[ i = 2, 3, 4, ..., th - 2 \] ............................ (6.23)

\[ p_{th - 1} = [\alpha_1(1 - \beta)] p_{th - 2} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)] p_{th - 1} + [\beta(1 - \alpha_{th})] p_{th} \] ............................ (6.24)

\[ p_{th} = [\alpha_1(1 - \beta)] p_{th - 1} + [\alpha_{th} \beta + (1 - \alpha_{th})(1 - \beta)] p_{th} + [\beta(1 - \alpha_{th+1})] p_{th+1} \] ............................ (6.25)

\[ p_i = [\alpha_{i-1}(1 - \beta)] p_{i-1} + [\alpha_i \beta + (1 - \alpha_i)(1 - \beta)] p_i + [\beta(1 - \alpha_{i+1})] p_{i+1} \], ............................ (6.26)

\[ i = th + 1, th + 2, th + 3, ..., K - 1 \]
Lastly,

\[ p_K = [\alpha_{K-1}(1-\beta)]p_{K-1} + [\alpha_K \beta + (1-\beta)]p_K \text{, where } K = th + I \] ........................ (6.27)

Let \( \gamma = \frac{\alpha_{1}(1-\beta)}{\beta(1-\alpha_{1})} \) .......................................................... (6.28)

and \( \gamma_i = \frac{\alpha_i (1-\beta)}{\beta(1-\alpha_i)} \), where \( i \geq th \) ......................................................... (6.29)

Equations (6.28, 6.29) are applied within the balance equations (6.21-6.27) to produce the equilibrium probabilities (shown below) for the DRED-Linear model,

In general:

\[ p_i = \frac{\alpha_{1}^i (1-\beta)^{i-1}}{\beta' (1-\alpha')^i} p_0 = \frac{\gamma^i}{(1-\beta)} p_0, \text{ where } i = 1,2,3,...,th-1 \] ................................. (6.30)

\[ p_i = \frac{\alpha_{th}^i (1-\beta)^{i-1} \prod_{j=th}^{i-1} \alpha_j}{\beta' (1-\alpha')^{th-1} \prod_{l=th}^{i-1} (1-\alpha_l)} \frac{\gamma_{th}^i (1-\alpha')^{th-1} \prod_{j=th}^{i-1} \gamma_j}{(1-\alpha_i)(1-\beta)} p_0 \] ................................. (6.31)

where \( i = th, th+1, th+2,...,th+I \), \( I = K - th \) and \( \gamma_j = 1, \ j < th \).

Then all equilibrium probabilities are obtained except \( p_0 \), which can be evaluated by applying equations (6.30,6.31) in the normalising equation (6.12). This gives:

\[ p_0 = \left[ \frac{1 - \gamma_{th} - \beta (1-\gamma)}{(1-\beta)(1-\gamma)} + \gamma_{th} (1-\alpha_{1}) \sum_{i=th}^{th+I} (\gamma_j) \frac{1}{(1-\alpha_i)} \right]^{-1} \] ................................. (6.32)
The performance measures \((mql, T, D, P_L, D_p, P_{Loss})\) can now be calculated for the Linear analytical model. Firstly, the mql can be calculated using the \(P(z)\) given in equation (6.14). The mql result is equal to either the first derivative of the \(P(z)\) at \(z = 1\) or the \(\sum_i i \times p_i\) value. This gives:

\[
mql = P^{(1)}(1) = \sum_{i=0}^{K} i \times p_i = \frac{P_0}{(1 - \beta)} \left[ \gamma - \gamma^\theta \gamma^\theta \frac{\gamma + \theta (1 - \gamma)}{(1 - \gamma)^2} + \gamma^\theta (1 - \alpha^\gamma) \sum_{i=th}^{K} \prod_{j=th}^i \left(\gamma_j\right) \frac{i}{(1 - \alpha^\gamma)} \right]
\]

where \(\gamma_j = 1, j < th \) 

The \(P_L\) and \(P_{Loss}\) of the proposed model can be computed as in equations (6.18 and 6.20) respectively, whereas the \(D_p\) is calculated using equations (6.34).

\[
D_p = \sum_{i=th}^{K-\gamma} \left(\frac{\alpha^\gamma - \alpha_i}{\alpha^\gamma}\right) \times p_i
\]

The remaining performance measures \((T, D)\) can be evaluated using equations (6.16, 6.17), respectively.

### 6.3.3 DRED Simulation and the Results of the Analytical Models

This subsection compares the proposed DRED analytical models (Alpha, Linear) and the DRED algorithm with reference to \((mql, T, D, P_L, D_p, P_{Loss})\) to achieve the algorithm or
model offers more satisfactory performance measure results. This comparison is divided into two parts (6.3.3.1, 6.3.3.2), where Subsection 6.3.3.1 compares the proposed models and DRED with regard to the performance measures \((mql,T,D,P_{\text{Loss}})\). In Subsection 6.3.3.2, the \((P_L,D_p)\) are calculated for the proposed models and DRED.

### 6.3.3.1 Performance Evaluation of the DRED, DRED-Alpha and DRED-Linear

The DRED and proposed analytical models (Alpha, Linear) are compared with reference to the different performance measures \((mql,T,D,P_{\text{Loss}})\) to obtain which one gives better performance measure results. The parameters of DRED (probability of packet arrival \((\alpha_1)\), probability of packet departure \((\beta)\), \(K\), threshold \(C(t)\), \(\epsilon\), \(qw\), \(D_{\text{init}}\), number of slots) were set as discussed in Subsection 3.1.2, where the threshold for DRED method or \(th\) value for the DRED models has been calculated using equation (6.1) [11]. The DRED analytical model parameters \((\alpha_1,\beta,K)\) were set similar to the DRED algorithm and the \(\alpha_2\) parameter was set to 0.1.

The results of \(mql\), \(T\), \(D\) and \(P_{\text{Loss}}\) for the DRED and the proposed DRED-Alpha and DRED-Linear analytical models versus the \(\alpha_1\) are shown in Figures (6.6-6.9), respectively. The performance measure results \((mql,T,D,P_{\text{Loss}})\) of the DRED algorithm are represented by the mean results of the ten runs. In addition, Figures (6.6-6.9) show the performance measures as a function of \(\alpha_1\).
It is shown in Figures 6.6 and 6.8 that the DRED and proposed DRED-Alpha and DRED-Linear models present similar $mql$ and $D$ results when the $\alpha$ value is less than the $\beta$ value. At these values, there is either no congestion or a light congestion. When congestion is initiated by increasing the $\alpha$ to values larger than the $\beta$ value, the DRED-Alpha becomes better than the DRED-Linear and DRED with reference of $mql$ and $D$ results and also the DRED outperforms the DRED-Linear concerning to $mql$ and $D$ results.

After analysing Figure 6.7, it is apparent that the DRED and proposed DRED analytical models offer similar $T$ results whether the $\alpha$ value is smaller or larger than the $\beta$ value. In addition, Figure 6.7 shows that the $T$ results for the DRED and analytical models increase as long as the $\alpha$ value increases, but this happens only when the $\alpha$ value < $\beta$ value. On other hand, if the $\alpha$ value is larger than the $\beta$ value, the $T$ results will stabilise at the $\beta$ value (0.5). It is obvious in Figure 6.9 that the proposed DRED-Alpha and DRED-Linear analytical models provide similar results with reference to $P_{Loss}$ regardless of the $\alpha$ value. The DRED algorithm gives similar results as the DRED analytical models with regard to $P_{Loss}$ if only the $\alpha$ value was set to either 0.18 or 0.33.

Conversely, if the $\alpha$ was set to a value equal to or larger than 0.48 (near or larger than the $\beta$ value), it seems in Figure 6.9 that the DRED and the analytical models produce similar $P_{Loss}$ results. However, the DRED algorithm gives marginally higher $P_{Loss}$ results than the analytical models, but this is not clear on Figure 6.9 because the difference between DRED and its corresponding analytical models is too small. The difference can be
clear in Figure 6.10 in Subsection 6.3.3.2, where DRED algorithm, due to buffer overflow, loses slightly more packets than the proposed analytical models.

Figure 6.6: $mql$ vs. probability of packet arrival ($\alpha 1$).

Figure 6.7: $T$ vs. probability of packet arrival ($\alpha 1$).

Figure 6.8: $D$ vs. probability of packet arrival ($\alpha 1$).

Figure 6.9: $P_{Loss}$ vs. probability of packet arrival ($\alpha 1$).
6.3.3.2 The Overflow Loss and Dropping Probability Results For the DRED, DRED-Alpha and DRED-Linear

This subsection examines which algorithm among the proposed models and DRED exhibits less buffer overflow and drops fewer packets. The parameters of DRED and the analytical models are set similar to the values described in Subsection 3.1.2 and 6.3.3.1, respectively. The performance measure results of the DRED algorithm and the proposed DRED-Alpha and DRED-Linear analytical models regarding $P_L$ and $D_p$ versus $\alpha 1$ are exhibited in Figures 6.10 and 6.11, respectively. The performance measure results ($P_L, D_p$) of the DRED algorithm are represented by the mean results of the ten runs. Figures (6.10-6.11) show the $P_L$ and $D_p$ as a function of $\alpha 1$.

It is obvious from Figure 6.10 that the DRED algorithm and proposed DRED-Alpha and DRED-Linear lose similar amounts of packets due to overflow their routers buffers when
the $\alpha_1$ value is equal to 0.18 or 0.33. On the other hand, if the $\alpha_1$ value equals 0.48 or is larger than the $\beta$ value, congestion may occur. This makes the router buffer of DRED overflow more often than those of DRED-Alpha and DRED-Linear analytical models. That is, random fluctuations of queue length upwards in the simulated DRED will cause overflows. However, the analytical models are in the steady state so such random fluctuations in queue length do not occur. Thus, DRED loses more packets in total than DRED-Alpha and DRED-Linear when the $\alpha_1$ value equals 0.48 or greater than the $\beta$ value. Figure 6.11 shows that the DRED and the two DRED models drop similar amount of packets whether the $\alpha_1$ value is high or not.

### 6.3.4 Section 6.3 Summary

In Section 6.3, two discrete-time analytical models were proposed based on DRED called Alpha and Linear. In the DRED-Alpha model, if congestion occurs then the packet arrival probability value decreases to another smaller fixed one in order to alleviate the congestion. On the other hand, in the Linear model the packet arrival probability value reduces linearly as the queue length increases from $t_h$ to $K$. All the performance measure results are performed with reference to the $\alpha_1$ values. Hence a decision of which algorithm or model is offering more satisfactory performance measure results is only issued based on the $\alpha_1$ values. From the simulation and numerical results that are conducted in Subsection 6.3.3, the following can be concluded:
• The proposed DRED analytical models (Alpha, Linear) and DRED algorithm give similar \( mql \) and \( D \) results once the \( \alpha_1 \) value is smaller than the \( \beta \) value. Conversely, if the \( \alpha_1 \) was set to a value larger than the \( \beta \) (occurrence of congestion), then the DRED-Alpha outperforms DRED-Linear according to \( mql \) and \( D \) results as well as the \( mql \) and \( D \) results for DRED are better than their corresponding results of the DRED-Linear.

• The proposed DRED analytical models (Alpha, Linear) and DRED algorithm provide similar \( T \) results regardless the \( \alpha_1 \) value.

• Both of the proposed analytical models give similar \( P_{loss} \) results whether the \( \alpha_1 \) value is high or not. It is clear that both analytical models fail to capture the increase in \( P_L \) exhibited by DRED. This is likely due to DRED being based on a simulation with its random fluctuations in queue length, whereas the analytical models are in the steady state and random fluctuations are absent.

• Both the DRED models and DRED algorithm drop similar amounts of packets at their router buffers whether the \( \alpha_1 \) value is high or not. This suggests the linear change in drop probability of DRED-Linear has little additional effect on \( D_p \) compared to the step change of \( D_p \) in DRED-Alpha.
6.4 BLUE-based Discrete-time Queue Analytical Models

In this section, two discrete-time queue analytical models [1, 6] are presented based on the BLUE algorithm [39, 43] to detect the congestion at the router buffers incipiently, and to offer satisfactory performance measure results when a high congestion occurs. The proposed BLUE-based discrete-time queue analytical models are called BLUE-Alpha [6] and the BLUE-Linear [1]. This section is organised as follows: Subsections 6.4.1 and 6.4.2 present the BLUE-Alpha and BLUE-Linear analytical models, respectively. The simulation and numerical results of the classical BLUE and the analytical models are given in Subsection 6.4.3. A summary is given in Subsection 6.4.4.

6.4.1 BLUE-Alpha Analytical Model

This analytical model is implemented at the router buffer as the traditional BLUE method [39, 43]. Similar to the DRED models, the BLUE Alpha model [6] analyses a single queue node (shown in Figure 6.2) using the discrete-time queue approach. The system capacity \( K \) and the arrival process are similar to those of the DRED models described earlier in this chapter. In addition, the BLUE Alpha relies on a single threshold \( th \) position at the router buffer, where this \( th \) is used as a congestion detector. It should be noted that the threshold \( th \) of the BLUE Alpha has a different value than that of the DRED models.
In Figure 6.2, the probability of packet arrival starts with $\alpha_1$ probability only when the $ql$ at the router buffer is equal to or smaller than the $th$ position. Whereas, if the $ql$ is larger than the $th$ position, the $Dp$ value increases from 0 to $\left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right)$ to reduce the effective arrival rate and alleviate the congestion at the router buffer. The arrival process and $a_n$ of the BLUE-Alpha as those of the DRED models. $\alpha_1$ and $\alpha_2$ stand for the packet arrival probabilities in a slot when the $ql$ at the router buffer is equal to or less than $th$ and greater than the $th$, respectively. The queueing discipline used in the proposed BLUE Alpha model is FCFS. It is assumed that the queue is in equilibrium, and the queue length process is a Markov chain with finite state space. Furthermore, the state space and $(K, \beta)$ of BLUE Alpha are similar to those of the DRED models. It is assumed that $\alpha_1 > \alpha_2$ and half of $\alpha_1$ values are smaller than $\beta$ ($\beta > \alpha_1$) and the rest of the values are larger than $\beta$ ($\beta < \alpha_1$), and $\beta > \alpha_2$. The state transition diagram for the BLUE Alpha model is shown in Figure 6.12. The difference between the state transition diagram of DRED-Alpha model (Figure 6.3) and the BLUE-Alpha state transition diagram (Figure 6.12) is when the congestion occurs at a router buffer. In the DRED models (Alpha, Linear) a congestion situation occurs similar to DRED method when the $ql$ is equal to or larger than the $th$ position at its router buffer [39, 43], whereas in the BLUE models (Alpha, Linear) a congestion is exists (similar to the BLUE method) only when $ql$ is greater than the $th$ position.
The balance equations for the BLUE Alpha model based on the state transition diagram are as follows:

\[ p_0 = (1 - \alpha_1)p_0 + [\beta(1 - \alpha_1)]p_1 \] ................................................................. (6.35)

\[ p_1 = \alpha_1p_0 + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_1 + [\beta(1 - \alpha_1)]p_2 \] ................................................................. (6.36)

In general:

\[ p_i = [\alpha_1(1 - \beta)]p_{i-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1}, \]

where \( i = 2, 3, 4, \ldots, th - 1 \) ................................................................. (6.37)

\[ p_{th} = [\alpha_1(1 - \beta)]p_{th-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_{th} + [\beta(1 - \alpha_2)]p_{th+1} \] ................................................................. (6.38)

\[ p_{th+1} = [\alpha_1(1 - \beta)]p_{th} + [\alpha_2 \beta + (1 - \alpha_2)(1 - \beta)]p_{th+1} + [\beta(1 - \alpha_2)]p_{th+2} \] ................................................................. (6.39)

\[ p_i = [\alpha_2(1 - \beta)]p_{i-1} + [\alpha_2 \beta + (1 - \alpha_2)(1 - \beta)]p_i + [\beta(1 - \alpha_2)]p_{i+1}, \]

where \( i = th + 2, th + 3, th + 4, \ldots, K - 1 \)
Finally,

$$p_K = \left[\alpha 2(1 - \beta)\right] p_{K-1} + \left[\alpha 2 \beta + (1 - \beta)\right] p_K$$, where \( K = th + I \) .......................... (6.41)

To derive the equilibrium probabilities in the BLUE Alpha model, we use equation (6.9) with the above balance equations (6.35-6.41). Thus, the equilibrium probabilities for the Alpha model are as follow:

In general:

$$p_i = \frac{\alpha^i(1 - \beta)^{i-1}}{\beta^i(1 - \alpha)^i} p_0 = \frac{\gamma^i}{(1 - \beta)} p_0$$, where \( i = 1, 2, 3, \ldots, th \) .......................... (6.42)

$$p_{th+i} = \frac{\alpha^{th+i} \alpha 2^{i-1}(1 - \beta)^{th+i-1}}{\beta^{th+i}(1 - \alpha)^{th}(1 - \alpha 2)} p_0 = \frac{\gamma^{th+i} \gamma 2^{i-1}(1 - \alpha 1)}{(1 - \alpha 2)(1 - \beta)} p_0$$, where \( i = 1, 2, 3, \ldots, I \) ........... (6.43)

The \( p_0 \) probability is obtained using the normalising equation given in equation (6.12).

Then, by applying equations (6.42, 6.43) in equation (6.12), the final form for the \( p_0 \) is obtained as follows:

$$p_0 = \left[ \frac{1 - \gamma^{th+1} - \beta(1 - \gamma 1)}{(1 - \beta)(1 - \gamma 1)} + \frac{\gamma^{th+1}(1 - \alpha 1)(1 - \gamma 2)}{(1 - \beta)(1 - \alpha 2)(1 - \gamma 2)} \right]^{-1}$$ .......................... (6.44)

To evaluate the performance measures, first the \( mql \) can be computed using either the \( \sum_{i=0}^{K} i \times p_i \) equation or the generating function \( P(z) \) (shown in equation (6.14)) as done previously in the DRED models. This results in the following:
\[ mql = P^{(i)}(1) = \frac{P_0}{(1-\beta)} \left[ \gamma (1-\gamma^1)^{th} + (1-\gamma^1)^{th} + \gamma^2 (1-\gamma^2)^{th} + \gamma^2 (1-\gamma^2)^{th} \right] \]

The rest of the performance measures \((T, D)\) are computed according to equations (6.16, 6.17). Lastly, the \(P_L\) and \(P_{Loss}\) results can be obtained using equations 6.18 and 6.20, respectively and \(D_p\) is computed as follows:

\[ D_p = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \times \sum_{i=th+1}^{K-1} p_i \]  

**6.4.2 BLUE-Linear Analytical Model**

In this subsection a discrete-time queue analytical model based on BLUE is proposed that aims to achieve the goals discussed in the Subsection 6.3.1. The new analytical model analyses the single queue node shown in Figure 6.4 using the discrete-time approach. It should be noted that the BLUE Linear model is implemented like the BLUE Alpha model discussed in Subsection 6.4.1. However, the model here differs from the BLUE Alpha model when the \(q_i\) is above the \(th\) position. Figure 6.4 shows that in the BLUE Linear model the probability of packet arrival reduces linearly from \(\alpha_1\) to \(\alpha_i\) in order to alleviate the congestion at the router buffer, where \(\alpha_i = \alpha_1 - (1 + i - th) \frac{\alpha_1}{(1 + K - th)}\), if \(i > th\). The \(\alpha_i\) value is obtained using the state transition diagram shown in Figure 6.13. Moreover, the
$D_p$ result increases linearly from 0 to $\frac{\alpha_1 - \alpha_i}{\alpha_1}$ as the $ql$ increases from $th + 1$ to $K$.

Because of the linearity in decreasing and increasing the probability of packet arrival and the $D_p$ respectively, the model is referred to as "Linear". $\alpha_i$ represents the packet arrival probability in a slot when the $ql$ is above the $th$ position at the router buffer. The remaining parameters ($\alpha_1$, $\beta$, $th$, $K$) of the BLUE Linear model are similar to those of the BLUE-Alpha model, and the queueing discipline used is FCFS. The queue node system is considered in equilibrium and the queue length process is a Markov chain with finite state space. The state space of this model is similar to that of the BLUE Alpha model. Further, it is assumed $\alpha_1 > \alpha_i$ and half of $\alpha_1$ values are smaller than $\beta$ ($\beta > \alpha_1$) and the rest of the values are greater than $\beta$ ($\beta < \alpha_1$), and $\beta > \alpha_i$. The state transition diagram for the BLUE Linear analytical model is shown in Figure 6.13.

![Figure 6.13: The state transition diagram for the BLUE Linear analytical model.](image-url)
Figure 6.13 is used to derive the balance equations for the BLUE Linear model, which are as follows:

\[ p_0 = (1-\alpha_1)p_0 + [\beta(1-\alpha_1)]p_1 \] .................................................. (6.47)

\[ p_1 = \alpha_1 p_0 + [\alpha_1\beta + (1-\alpha_1)(1-\beta)]p_1 + [\beta(1-\alpha_1)]p_2 \] .......................................... (6.48)

In general:

\[ p_i = [\alpha_1(1-\beta)]p_{i-1} + [\alpha_1\beta + (1-\alpha_1)(1-\beta)]p_i + [\beta(1-\alpha_1)]p_{i+1}, \]

where \( i = 2, 3, 4, \ldots, th - 1 \) ................................................................. (6.49)

\[ p_{th} = [\alpha_1(1-\beta)]p_{th-1} + [\alpha_1\beta + (1-\alpha_1)(1-\beta)]p_{th} + [\beta(1-\alpha_{th+1})]p_{th+1} \] ................................................ (6.50)

\[ p_{th+1} = [\alpha_1(1-\beta)]p_{th} + [\alpha_{th+1}\beta + (1-\alpha_{th+1})(1-\beta)]p_{th+1} + [\beta(1-\alpha_{th+2})]p_{th+2} \] .................................................. (6.51)

\[ p_i = [\alpha_{i-1}(1-\beta)]p_{i-1} + [\alpha_i\beta + (1-\alpha_i)(1-\beta)]p_i + [\beta(1-\alpha_{i+1})]p_{i+1}, \] .................................................. (6.52)

where \( i = th + 2, th + 3, th + 4, \ldots, K - 1 \)

Finally,

\[ p_K = [\alpha_{K-1}(1-\beta)]p_{K-1} + [\alpha_K\beta + (1-\beta)]p_K, \] where \( K = th + 1 \) ............................................. (6.53)

Equations (6.28, 6.29) are substituted in the balance equations (6.47-6.53) to derive the equilibrium probabilities in the BLUE Linear model. Equation (6.29) is applied only when \( i > th \). The equilibrium probabilities are given below.
In general:

\[ p_i = \frac{\alpha 1^i (1 - \beta)^{i-1}}{\beta^i (1 - \alpha 1)^i} p_0 = \frac{\gamma^i}{(1 - \beta)} p_0 \], where \( i = 1, 2, 3, ..., th \) ......................... (6.54)

\[ p_i = \frac{\alpha 1^{th+1} (1 - \beta)^{th+1} \prod_{j=th+1}^{i-1} \alpha_j}{\beta^i (1 - \alpha 1)^i} p_0 = \frac{\gamma^{th+1} (1 - \alpha 1) \prod_{j=th+1}^{i-1} \gamma_j}{(1 - \alpha 1) (1 - \beta)} p_0 \] ......................... (6.55)

where \( i = th + 1, th + 2, ..., th + I \) and \( \gamma_j = 1, j \leq th \).

The equilibrium probabilities equations (6.54, 6.55) of the Linear model are substituted in the normalising equation (6.12) to evaluate \( p_0 \) to give the following:

\[ p_0 = \left[ \frac{1 - \gamma^{th+1} - \beta (1 - \gamma)}{(1 - \beta) (1 - \gamma)} + \gamma^{th+1} \frac{(1 - \alpha 1) \sum_{i=th+1}^{K} \prod_{j=th+1}^{i-1} (\gamma_j) \frac{1}{(1 - \alpha_i)}}{(1 - \beta) (1 - \gamma)} \right]^{-1} \] ......................... (6.56)

The performance measures (\( T, D, P_L, P_{loss} \)) are obtained as before and the rest are given as follows:

\[ mql = P^{(1)}(1) = \frac{p_0}{(1 - \beta)} \left[ \frac{\gamma - \gamma^{th+1} [1 + th (1 - \gamma)]}{(1 - \gamma)^2} + \gamma^{th+1} (1 - \alpha 1) \sum_{i=th+1}^{K} \prod_{j=th+1}^{i-1} (\gamma_j) \frac{i}{(1 - \alpha_i)} \right] \ldots (6.57) \]

where \( \gamma_j = 1, j \leq th \).

\[ D_p = \sum_{i=th+1}^{K} \left( \frac{\alpha 1 - \alpha_i}{\alpha 1} \right) \times p_i \] ......................... (6.58)
6.4.3 BLUE Simulation and the Results of the Analytical Models

In this subsection, the proposed BLUE analytical models (Alpha, Linear) are compared with the classic BLUE algorithm regarding to the following performance measures \(mql, T, D, P_L, D_p, P_{Loss}\) to evaluate the performance. Also the th position at the router buffer that gives the most satisfactory performance measure results for the BLUE and proposed BLUE-Alpha and BLUE-Linear models, is investigated. Subsection 6.4.3.1 demonstrates the simulation and numerical results of BLUE and the proposed models with respect to different performance evaluation measures \((mql, T, D, P_{Loss})\). The calculations of the \((P_L, D_p)\) for BLUE and the two BLUE models are given in Subsection 6.4.3.2. Subsection 6.4.3.3 investigates the th position at the router buffer that provides the most satisfactory performance measure results \((mql, T, D, P_L, D_p, P_{Loss})\) for BLUE and the two models.

6.4.3.1 Performance Evaluation of the BLUE, BLUE-Alpha and BLUE-Linear

The proposed BLUE-based analytical models (Alpha, Linear) and the original BLUE method are compared with regard to \((mql, T, D, P_{Loss})\) aiming to identify the one that obtain better performance measure results. The BLUE parameter values \((\alpha, \beta, K, \text{threshold or th, } P_{inc}, P_{dec}, D_{init}, \text{number of slots})\) were given in Subsection 3.1.2. The
BLUE-Alpha and Linear models parameters ($\alpha_1, \alpha_2, \beta, K$) are set to values similar to those of the DRED models (see Subsection 6.3.3.1).

The performance measures ($mql, T, D, P_{Loss}$) of the BLUE and proposed BLUE-Alpha and Linear models versus $\alpha_1$ are shown in Figures (6.14-6.17), respectively, where the results of the BLUE are represented by the mean results of ten runs. Figures (6.14-6.17) show the performance measures as a function of $\alpha_1$.

**Figure 6.14:** $mql$ vs. probability of packet arrival ($\alpha_1$).

**Figure 6.15:** $T$ vs. probability of packet arrival ($\alpha_1$).

**Figure 6.16:** $D$ vs. probability of packet arrival ($\alpha_1$).

**Figure 6.17:** $P_{Loss}$ vs. probability of packet arrival ($\alpha_1$).
It is clear in Figures 6.14 and 6.16 that the proposed BLUE-Alpha model and BLUE method provide similar $mql$ and $D$ results whether the value of $\alpha_1$ is larger or smaller than the value of $\beta$. In other words, the proposed BLUE-Alpha and BLUE method give similar $mql$ and $D$ results whether there is congestion or not. Also, the proposed BLUE-Linear model obtains similar $mql$ and $D$ results as the BLUE-Alpha and BLUE only when the $\alpha_1$ value is smaller than the $\beta$ value. However, when the $\alpha_1$ value is larger than the $\beta$ value, both BLUE-Alpha and BLUE are better than the BLUE-Linear according to $mql$ and $D$ results. These values for $\alpha_1$ create a congestion situation.

Figure 6.15 shows that the BLUE and proposed BLUE-Alpha and Linear models offer similar $T$ results whether the $\alpha_1$ value is high or not. Moreover, the results of $T$ for the BLUE and the models are increasing as long as the $\alpha_1$ value increasing but this only happens when the $\alpha_1$ value is less than the $\beta$ value. On the other hand, when the $\alpha_1$ value increases to be larger than the $\beta$ value, the $T$ results are stabilised at the value of $\beta$. It is obvious in Figure 6.17 where the proposed BLUE models and BLUE produce similar $P_{loss}$ results when the $\alpha_1$ value $< \beta$ value or $\alpha_1$ value $= 0.63$. It seems in Figure 6.17 that the proposed BLUE models and BLUE offer similar $P_{loss}$ results when heavy congestion is present, i.e. $\alpha_1$ value $= 0.78$ or $0.83$, but actually at these values the BLUE-Alpha model is slightly better than BLUE-Linear or BLUE with reference to $P_{loss}$ results since it has buffer overflow lost slightly fewer packets than them (see Figure 6.18).
6.4.3.2 The Overflow Loss and Dropping Probability Results for the BLUE, BLUE-Alpha and BLUE-Linear

In this subsection, the BLUE method has been compared with the proposed BLUE-Alpha and BLUE-Linear analytical models with regard to \((P_L, D_p)\) to determine the one that loses and drops the fewest packets at its router buffer. The parameters of BLUE were set to values as in Subsection 3.1.2 and the models’ parameters were given values similar to those in Subsection 6.4.3.1. The results of the \(P_L\) and \(D_p\) versus \(\alpha_1\) for the BLUE and BLUE-Alpha and BLUE-Linear are illustrated in Figures 6.18 and 6.19, respectively, where the \(P_L\) and \(D_p\) of the BLUE are represented by the mean results. Figures 6.18 and 6.19 display the \(P_L\) and \(D_p\) as a function of \(\alpha_1\).

Figure 6.18: \(P_L\) vs. probability of packet arrival (\(\alpha_1\)).

Figure 6.19: \(D_p\) vs. probability of packet arrival (\(\alpha_1\)).

Figure 6.18 indicates that the BLUE method and the developed analytical models lose similar amounts of packets at their router buffers when the \(\alpha_1\) is smaller than the \(\beta\) or
equal to 0.63, where these packet loses for the BLUE and BLUE models are due to overflow their routers buffers. However, when $\alpha 1$ is set to 0.78 or 0.93 (heavy congestion), the BLUE-Linear loses more number of lost packets. It is noted in Figure 6.19 that the BLUE packets than the others and the proposed models drop similar amounts of packets whether the $\alpha 1$ value is high or not (with or without congestion).

6.4.3.3 Evaluation of $th$ for the BLUE, BLUE-Alpha and BLUE-Linear

The examination of the optimal $th$ position at the router buffers of BLUE and the proposed BLUE-Alpha and Linear models that produce the satisfactory performance measure results are presented in this subsection. The optimal $th$ position for the BLUE-based models and the BLUE algorithm is given based on the values of the $\alpha 1$. The BLUE’s parameters were set to values as in Subsection 3.1.2 except $\alpha 1$ and $th$, which were set to 0.78 and [4-14], respectively. The reason for selecting $\alpha 1 = 0.78$ is because this value can create a congestion situation. Also, the $th$ was set to [4-14] since these numbers cover the buffer length (4 and 16 denote 0.2 and 0.7 of the buffer length, respectively). In other words, the examination of $th$ is applied upon $[0.2 \times K - 0.7 \times K]$. The parameters of the BLUE analytical models were set to values similar to the BLUE’s parameters in this subsection with setting $\alpha 2$ to 0.1.

The performance measure results ($mql$, $T$, $D$, $P_{Loss}$, $P_L$, $D_p$) versus $th$ of the BLUE and BLUE-Alpha and Linear models are depicted in Figures (6.20-6.25), respectively.
Figure 6.20: $mql$ vs. $th$.

Figure 6.21: $T$ vs. $th$.

Figure 6.22: $D$ vs. probability vs. $th$.

Figure 6.23: $P_{loss}$ vs. $th$.

Figure 6.24: $P_L$ vs. $th$.

Figure 6.25: $D_p$ vs. $th$. 
From Figures 6.20 and 6.22, it is noted that the most satisfactory results for the BLUE and proposed models concerning \( mql \) and \( D \) results are obtained when the \( th \) is set to the smallest tested value and this value is equal to \( 0.2 \times K = 4 \). In other words, as long as the \( th \) value increases, both \( mql \) and \( D \) results increase and vice versa. Generally, in order to achieve better \( mql \) and \( D \) results for the BLUE and BLUE models, the \( th \) position at their routers buffers must be set to a value as small as possible.

Figures 6.21, 6.23, 6.24 and 6.25 indicate that the optimal \( th \) position at the router buffers of BLUE and BLUE-Alpha and Linear which gives the most satisfactory performance measure results regarding \( T \), \( P_{Loss} \), \( P_L \) and \( D_p \) is equal to any tested value. Therefore the performance measures \((T, P_{Loss}, P_L, D_p)\) results are not affected with the \( th \) value.

6.4.4 Section 6.4 Summary

Two discrete-time queue analytical models based on the BLUE algorithm have been developed with the objective of identifying congestion in its preliminary stages. These analytical models are called Alpha and Linear. If the Alpha router buffer is congested, its packet arrival probability value gets reduced to another constant and smaller value to alleviate the congestion. Furthermore, in the Linear model when congestion arises, this is managed through reducing the packet arrival probability value linearly. The results of different performance measures of BLUE and its corresponding analytical models were
obtained by setting the $\alpha_1$ parameter to different values and others by giving the $th$ parameter different values. The simulation and numerical results obtained in Subsection 6.4.3 can be summarised as follows:

- Both BLUE and the proposed BLUE-Alpha model offer similar $mql$ and $D$ results regardless the $\alpha_1$. Furthermore, similar $mql$ and $D$ results for the BLUE-Alpha and BLUE are obtained by the BLUE-Linear model when the $\alpha_1$ is set to a value lower than the value of $\beta$. This suggests that BLUE-Alpha is a good analytical model for BLUE. The BLUE-Alpha and BLUE outperform the BLUE-Linear in terms of $mql$ and $D$ results when the $\alpha_1$ was set to a value larger than the $\beta$.

- Whatever the $\alpha_1$ value, the BLUE and BLUE models give similar $T$ results.

- The results of $P_{Loss}$ for BLUE and developed BLUE-Alpha and BLUE-Linear models are similar when the $\alpha_1$ value is smaller than the $\beta$ value or equal to 0.63. When $\alpha_1$ value = 0.78 or 0.83, the proposed BLUE-Alpha is marginally different from BLUE and BLUE-Linear according to $P_{Loss}$ since it loses slightly fewer packets than BLUE-Linear or BLUE, where packet loss is due to router buffer overflows.

- If the $\alpha_1$ value is set to a value lower than the $\beta$ value or equal to 0.63, then the proposed models and BLUE lose similar amounts of packets from overflowing their router buffers. Conversely, if the $\alpha_1$ is given a value equal to 0.78 or 0.93, then BLUE loses more packets ($P_L$) than BLUE-Alpha and fewer packets than BLUE-Linear.
• The BLUE models and BLUE method drop similar numbers of packets regardless of the value of $\alpha_1$.
• The most satisfactory $mql$ and $D$ results can be achieved when the $th$ position at BLUE and BLUE analytical models is set to a value as small as possible.
• The most satisfactory $T, P_{loss}, P_d, D_p$ results for the BLUE and BLUE models are obtained when the $th$ value was given to any tested value [4-14].
• Overall, the results suggest a linear version of BLUE offers no advantages and is inferior in some aspects of its performance.

6.5 Discrete-time Queue Analytical Models based on the RED Algorithm

Two discrete-time queue analytical models based on the RED algorithm [46] are proposed. The two analytical models are called RED-Alpha and RED-Linear and it should be noted that these both rely on queue length rather than average queue length (as in RED) as a congestion measure. The RED models also aim to stabilise their $mql$ and $D$ at values smaller than RED to reduce buffer overflow compared to RED. The RED-Alpha and Linear models use an instantaneous queue length rather than the average queue length to overcome one of RED’s shortcomings. This shortcoming is explained in the following example: At a short interval, the arrival rate momentarily increases; this may cause the RED router buffer to fill up and overflow. The average queue length may be less than the min threshold and
takes a time to build up. Thus, no packets can be dropped although the RED router buffer is overflowing. Whereas using the instantaneous queue length makes it pass the min\textit{threshold} and cause probabilistic dropping to occur before the buffer has an opportunity to overflow.

This section is organised as follows: Subsections 6.5.1 and 6.5.2 present the proposed RED-based Alpha and the Linear analytical models, respectively. The simulation and the numerical results of the RED and the proposed models are given in Subsection 6.5.3. Lastly, Subsection 6.5.4 summarises this section.

\textbf{6.5.1 RED-Alpha Analytical Model}

This subsection presents an Alpha analytical model that is based on the RED algorithm and aims to do the following: 1) Detects congestion at an early stage and 2) Stabilises \textit{mql} and \textit{D} at smaller values than RED to give less buffer overflow than RED.

The two proposed RED models (Alpha, Linear) use two thresholds (\textit{min threshold}, \textit{max threshold}) at the router buffers as in RED. The \textit{min threshold} is employed to detect the initial stages of the congestion, whereas the \textit{max threshold} is used to determine heavy congestion. The single queue node capacity (\textit{K}) and the type of the arrival process are similar to those in the DRED and BLUE models. A single queue node which analyses to build a RED-Alpha model is based is shown in Figure 6.26.
Figure 6.26: The single queue node system for RED Alpha model.

Figure 6.26 shows the packet arrival probability initially at $\alpha_1$ if the $ql$ is smaller than the min threshold, and therefore no packets are dropped ($Dp = 0$). However, when the $ql$ is between the min threshold and the max threshold, the probability of packet arrival decreases from $\alpha_1$ to $\alpha_2$ to control the congestion at the router buffer, and the $Dp$ value increases from 0 to $\frac{\alpha_1 - \alpha_2}{\alpha_1}$. Moreover, if the $ql$ is above than the max threshold position, the packet arrival probability reduces from $\alpha_2$ to $\alpha_3$ to alleviate the congestion, and the $Dp$ value increases from $\frac{\alpha_1 - \alpha_2}{\alpha_1}$ to $\frac{\alpha_1 - \alpha_3}{\alpha_1}$.

Similar to the BLUE and DRED Alpha models, the packet arrival probability is reduced to a lower value when the router buffer becomes congested. Therefore, this model is called the RED Alpha model. $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the probabilities of packet arrival in a slot when the $ql$ at the router buffer is a) less than the min threshold position, b) between the min threshold and the max threshold positions, and c) equal to or greater than the max threshold position, respectively. The $\beta$ parameter and the queueing discipline for the Alpha model are similar to those of the DRED and BLUE models. The queue is considered
to be in equilibrium and the queue length process is a Markov chain with finite state space \( \{0, 1, 2, 3, ..., \text{min } \text{threshold}, ..., \text{max } \text{threshold}, ..., K - 1, K \} \). It is also assumed that \( \alpha_1 > \alpha_2, \alpha_2 > \alpha_3 \) and half of \( \alpha_1 \) values are smaller than \( \beta \) (\( \beta > \alpha_1 \)) and the rest of the values are larger than \( \beta \) (\( \beta < \alpha_1 \)). In addition, \( \beta > \alpha_2 \) and \( \beta > \alpha_3 \). The state transition diagram for the Alpha model is shown in Figure 6.27.

The balance equations \( (6.59-6.68) \) for the Alpha-RED are derived from the state transition diagram shown in Figure 6.27.

\[
p_0 = (1 - \alpha_1)p_0 + [\beta(1 - \alpha_1)]p_1 \tag{6.59}
\]
\[
p_i = \alpha_1 p_0 + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1} \tag{6.60}
\]

In general:

\[
p_i = [\alpha_1(1 - \beta)]p_{i-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1},
\]

where \( i = 2, 3, 4, ..., \text{min } \text{threshold} - 2 \) \tag{6.61}
\[ p_{\min \text{threshold} - 1} = [\alpha_1(1 - \beta)]p_{\min \text{threshold} - 2} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_{\min \text{threshold} - 1} + [\beta(1 - \alpha_2)]p_{\min \text{threshold}} \] .............................. (6.62)

\[ p_{\min \text{threshold}} = [\alpha_1(1 - \beta)]p_{\min \text{threshold} - 1} + [\alpha_2 \beta + (1 - \alpha_2)(1 - \beta)]p_{\min \text{threshold}} + [\beta(1 - \alpha_2)]p_{\min \text{threshold} + 1} \] .............................. (6.63)

\[ p_i = [\alpha_2(1 - \beta)]p_{i-1} + [\alpha_2 \beta + (1 - \alpha_2)(1 - \beta)]p_i + [\beta(1 - \alpha_2)]p_{i+1}, \]

where \( i = \min \text{threshold} + 1, \min \text{threshold} + 2, \min \text{threshold} + 3, \ldots, \max \text{threshold} - 2 \) . (6.64)

\[ p_{\max \text{threshold} - 1} = [\alpha_2(1 - \beta)]p_{\max \text{threshold} - 2} + [\alpha_2 \beta + (1 - \alpha_2)(1 - \beta)]p_{\max \text{threshold} - 1} + [\beta(1 - \alpha_3)]p_{\max \text{threshold}} \] .............................. (6.65)

\[ p_{\max \text{threshold}} = [\alpha_2(1 - \beta)]p_{\max \text{threshold} - 1} + [\alpha_3 \beta + (1 - \alpha_3)(1 - \beta)]p_{\max \text{threshold}} + [\beta(1 - \alpha_3)]p_{\max \text{threshold} + 1} \] .............................. (6.66)

\[ p_i = [\alpha_3(1 - \beta)]p_{i-1} + [\alpha_3 \beta + (1 - \alpha_3)(1 - \beta)]p_i + [\beta(1 - \alpha_3)]p_{i+1}, \]

where \( i = \max \text{threshold} + 1, \max \text{threshold} + 2, \max \text{threshold} + 3, \ldots, K - 1 \) . (6.67)

Finally,
\[ p_K = [\alpha_3(1 - \beta)]p_{K-1} + [\alpha_3 \beta + (1 - \alpha_3)(1 - \beta)]p_K \] .............................. (6.68)

where \( K = \max \text{threshold} + J \) and \( \max \text{threshold} = \min \text{threshold} + I \) . Thus, \( K \) can also be expressed as follows:
\[ K = \min \text{threshold} + I + J \] .............................. (6.69)

Let \( \gamma_i = \frac{\alpha_i(1 - \beta)}{\beta(1 - \alpha_i)}, i = 1, 2, 3 \) .............................. (6.70)
After solving the balance equations (6.59-6.68), equation (6.70) is substituted to produce the equilibrium probabilities of the RED-Alpha (equations (6.71-6.73)).

In general:

\[ p_i = \frac{\alpha_1 (1 - \beta)^{i-1}}{\beta^i (1 - \alpha)^i} p_0 - \frac{\gamma_1^i}{(1 - \beta)} p_0, \text{ where } i = 1, 2, 3, ..., \min \text{threshold} - 1 \ldots (6.71) \]

\[ P_{\min \text{threshold}+i} = \frac{\alpha_{1 \min \text{threshold}} (1 - \beta_{\min \text{threshold}+i})}{(1 - \alpha_{\min \text{threshold}+i})^{i+1}} p_0 - \frac{\gamma_{1 \min \text{threshold}}^i (1 - \alpha_1)}{(1 - \alpha_{\min \text{threshold}+i})^{i+1}} p_0. \ldots (6.72) \]

where \( i = 0, 1, 2, 3, ..., I - 1 \)

\[ P_{\max \text{threshold}+i} = \frac{\alpha_{1 \max \text{threshold}} (1 - \beta_{\max \text{threshold}+i})}{(1 - \alpha_{\max \text{threshold}+i})^{i+1}} p_0 - \frac{\gamma_{1 \max \text{threshold}}^i (1 - \alpha_1)}{(1 - \alpha_{\max \text{threshold}+i})^{i+1}} p_0, \text{ where } i = 0, 1, 2, ..., J \ldots (6.73) \]

After obtaining the equilibrium probabilities of the Alpha model, \( p_0 \) can be obtained using equation (6.12). This gives the following:

\[ p_0 = \left[ \frac{1 - \gamma_{1 \min \text{threshold}}}{(1 - \beta)(1 - \gamma_1)} - \beta (1 - \gamma_1) + \frac{\gamma_{1 \min \text{threshold}}}{(1 - \beta)} \frac{(1 - \gamma_2^i)}{(1 - \alpha_2)(1 - \gamma_2)} + \frac{\gamma_2^i (1 - \gamma_3^{i+1})}{(1 - \alpha_3)(1 - \gamma_3)} \right]^{-1} \ldots (6.74) \]

The performance measures of the RED-Alpha are evaluated as follows: The \( mql \) is obtained from \( P(z) \) as before. This is shown in equation (6.75). The rest of the performance measures (\( T, D \)) are obtained in a similar way to the DRED and BLUE
models (equations (6.16-6.17)). Moreover, the results of $P_L$ and $P_{Loss}$ for the RED models are calculated as before, whereas the $D_p$ is calculated as in equation (6.76).

$$mql = \left[ \frac{P_0}{(1-\beta)} \right] \left[ \gamma_1 - \gamma_1^{\text{threshold}} \left[ \gamma_1 + \text{min threshold} (1-\gamma_1 \right) + \gamma_1^{\text{threshold}} (1-\alpha_1 \right) \right]$$

$$= \left[ \frac{\min \text{threshold}(1-\gamma_2(1-\gamma_2^2)+\gamma_2-\gamma_2^2 \left[ \gamma_2 + I(1-\gamma_2) \right]}{(1-\alpha_2)(1-\gamma_2)^2} \right]$$

$$\left[ + \gamma_2^2 \left[ \text{max threshold}(1-\gamma_3(1-\gamma_3 J_i) + \gamma_3-\gamma_3^J J_i J_{i+1} + J(1-\gamma_3) \right]}{(1-\alpha_3)(1-\gamma_3)^2} \right]$$

$$\vdots$$

$$(6.75)$$

$$D_p = \left[ \frac{\alpha_1 - \alpha_2}{\alpha_1} \right]^{\text{max threshold} - 1} \sum_{i=\text{min threshold}} \left[ \frac{\alpha_1 - \alpha_3}{\alpha_1} \right]^{\text{max threshold} - 1} \sum_{i=\text{max threshold}} \left[ P_i \right] \vdots$$

$$(6.76)$$

### 6.5.2 RED-Linear Analytical Model

This subsection proposes a discrete-time queue analytical model based on the RED approach called RED-Linear. This model aims to achieve the same goals as the RED Alpha model, and it analyses the single queue node system shown in Figure 6.28.
With reference to Figure 6.28, when the $ql$ is smaller than the $min\ threshold$ position, the packet arrival probability begins at $\alpha_1$ probability, and therefore no packets are dropped. However, when the $ql$ is between the $min\ threshold$ and the $max\ threshold$, the probability of the packet arrival reduces linearly from $\alpha_1$ to $\alpha_i$ to manage the congestion, where 

$$\alpha_i = \alpha_1 - (1 + i - min\ threshold) \frac{(\alpha_1 - \alpha_2)}{(1 + max\ threshold - min\ threshold)} , \quad if \quad min\ threshold \leq i < max\ threshold .$$

The $\alpha_i$ can be obtained from the state transition diagram (Figure 6.29), the $Dp$ value increases linearly from 0 to $\frac{\alpha_1 - \alpha_i}{\alpha_1}$ as long as the $ql$ increases from $min\ threshold$ to $max\ threshold - 1$. Now, if the $ql$ is equal to or larger than the $max\ threshold$, the packet arrival probability changes to $\alpha_2$ probability in order to reduce the heavy congestion. In addition, the router buffer drops the arriving packets at value $\left(\frac{\alpha_1 - \alpha_2}{\alpha_1}\right)$. 
Since the router buffer increases its $D_p$ value linearly, and the packet arrival probability decreases linearly if the $\min \text{threshold} \leq ql < \max \text{threshold}$, this analytical model is called RED Linear. It is considered that $\alpha_i$ and $\alpha 2$ are the packet arrival probabilities in a slot when the $ql$ is between the positions of $\min \text{threshold}$ and $\max \text{threshold}$ and larger than or equal to the $\max \text{threshold}$ position, respectively. The parameter values of the RED-Linear (arrival process, $a_n$, queueing discipline, the state space, $\alpha 1$, $\beta$, $\min \text{threshold}$, $\max \text{threshold}$, $K$) are similar to those of the RED-Alpha (See Subsection 6.5.1). It is assumed the queue is in equilibrium and the queue length process is a Markov chain with finite state space. Further, it is considered that $\alpha 1 > \alpha_i$, $\alpha_i > \alpha 2$ and half of $\alpha 1$ values are smaller than $\beta$ ($\beta > \alpha 1$) and the rest of the values are larger than $\beta$ ($\beta < \alpha 1$). Furthermore, $\beta > \alpha_i$ and $\beta > \alpha 2$.

The state transition diagram for the RED Linear model is shown in Figure 6.29.

The balance equations (6.77-6.86) for the RED-Linear can be obtained from Figure 6.29.

$$p_0 = (1 - \alpha 1)p_0 + \beta (1 - \alpha 1)p_1 \quad \text{.......................................................... (6.77)}$$
\[ p_1 = \alpha_1 p_0 + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_1 + [\beta(1 - \alpha_1)]p_2 \quad \text{……………………………………… (6.78)} \]

In general:

\[ p_i = [\alpha_1(1 - \beta)]p_{i-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1}, \]

where \( i = 2,3,4,\ldots, \min\text{threshold} - 2 \) ………………………………………… (6.79)

\[ p_{\min\text{threshold} - 1} = [\alpha_1(1 - \beta)]p_{\min\text{threshold} - 2} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_{\min\text{threshold} - 1} + [\beta(1 - \alpha_1)]p_{\min\text{threshold}} \quad \text{……………… (6.80)} \]

\[ p_{\min\text{threshold}} = [\alpha_1(1 - \beta)]p_{\min\text{threshold} - 1} + [\alpha_\min\text{threshold} \beta + (1 - \alpha_\min\text{threshold})(1 - \beta)]p_{\min\text{threshold}} + [\beta(1 - \alpha_\min\text{threshold} + 1)]p_{\min\text{threshold} + 1} \quad \text{……………… (6.81)} \]

\[ p_i = [\alpha_{i-1}(1 - \beta)]p_{i-1} + [\alpha_i \beta + (1 - \alpha_i)(1 - \beta)]p_i + [\beta(1 - \alpha_{i+1})]p_{i+1}, \]

where \( i = \min\text{threshold} + 1, \min\text{threshold} + 2, \min\text{threshold} + 3,\ldots, \max\text{threshold} - 2 \ldots (6.82) \]

\[ p_{\max\text{threshold} - 1} = [\alpha_{\max\text{threshold} - 2}(1 - \beta)]p_{\max\text{threshold} - 2} + [\alpha_{\max\text{threshold} - 1} \beta + (1 - \alpha_{\max\text{threshold} - 1})(1 - \beta)]p_{\max\text{threshold} - 1} + [\beta(1 - \alpha_2)]p_{\max\text{threshold}} \quad \text{……………… (6.83)} \]

\[ p_{\max\text{threshold}} = [\alpha_{\max\text{threshold} - 1}(1 - \beta)]p_{\max\text{threshold} - 1} + [\alpha_{\max\text{threshold} - 1} \beta + (1 - \alpha_{\max\text{threshold} - 1})(1 - \beta)]p_{\max\text{threshold}} + [\beta(1 - \alpha_2)]p_{\max\text{threshold} + 1} \quad \text{……………… (6.84)} \]

\[ p_i = [\alpha_{i-1}(1 - \beta)]p_{i-1} + [\alpha_i \beta + (1 - \alpha_i)(1 - \beta)]p_i + [\beta(1 - \alpha_{i+1})]p_{i+1}, \]

where \( i = \max\text{threshold} + 1, \max\text{threshold} + 2, \max\text{threshold} + 3,\ldots, K - 1 \) …………… (6.85)

Lastly,

\[ p_K = [\alpha_{2}(1 - \beta)]p_{K-1} + [\alpha_{2} \beta + (1 - \alpha_{2})(1 - \beta)]p_K \quad \text{………………………………………… (6.86)} \]
where \( K \) is given as in equation (6.69).

Assume \( \gamma_i = \frac{\alpha_i(1 - \beta)}{\beta(1 - \alpha_i)} \), where \( \text{min threshold} \leq i < \text{max threshold} \) ......................... (6.87)

After solving the balance equations, equations (6.9 and 6.87) can be used in the above balance equations to generate the equilibrium probabilities for the RED-Linear model as shown below.

In general:

\[
p_i = \frac{\alpha^i(1 - \beta)^{i-1}}{\beta^i(1 - \alpha^i)} \quad \text{and} \quad p_0 = \frac{\gamma^i}{(1 - \beta)} p_0, \text{ where } i = 1,2,\ldots,\text{min threshold } - 1 \quad \text{............... (6.88)}
\]

\[
p_i = \frac{\alpha^i(1 - \beta)^{i-1}}{\beta^i(1 - \alpha^i)} \prod_{j=\text{min threshold}}^{i} \alpha_j \quad \text{and} \quad p_0 = \frac{\gamma^i}{(1 - \alpha_i)(1 - \beta)} \prod_{j=\text{min threshold}}^{i} \gamma_j p_0 \quad \text{............. (6.89)}
\]

where \( i = \text{min threshold}, \text{min threshold } + 1, \text{min threshold } + 2,\ldots, \text{max threshold } - 1 \)

\[
P_{\text{max threshold } + i} = \frac{\alpha^i(1 - \beta)^{i-1}}{\beta^i(1 - \alpha^i)} \prod_{j=\text{min threshold}}^{\text{max threshold } - i} \alpha_j \frac{\gamma^i(1 - \alpha^i)^{i-1}}{(1 - \alpha^i)(1 - \beta)} \prod_{j=\text{min threshold}}^{\text{max threshold } - i} \gamma_j p_0 \quad \text{where } i = 0,1,2,\ldots,J \quad \text{............... (6.90)}
\]

Now, utilising the equilibrium probabilities in equation (6.12), the \( p_0 \) can be obtained (equation (6.91)).
\[
P_0 = \left[\frac{1 - \gamma^{\min_{\text{threshold}}}}{(1 - \beta)(1 - \gamma)} + \frac{\gamma^{\min_{\text{threshold}}}}{(1 - \beta)^{\max_{\text{threshold}}}} \sum_{i = \min_{\text{threshold}}}^{i = \max_{\text{threshold}} - 1} \prod_{j = \min_{\text{threshold}}}^{j = i + 1} \left(\frac{1}{1 - \alpha_i}\right)\right]^{-1} \tag{6.91}
\]

The evaluation of the performance measures \( (mql, T, D, P_L, P_{\text{Loss}}) \) for the RED-Linear are obtained as before, with the \( mql \) as given in equation (6.92). Finally, the \( D_p \) of the RED-Linear is computed as in equation (6.93).

\[
mql = P^{(1)}(1) = \frac{p_0}{(1 - \beta)} \left[ \frac{\gamma^{\min_{\text{threshold}}}}{(1 - \gamma)^2} \sum_{i = \min_{\text{threshold}}}^{i = \max_{\text{threshold}} - 1} \prod_{j = \min_{\text{threshold}}}^{j = i + 1} \left(\frac{1}{1 - \alpha_i}\right) + \frac{\gamma^{\min_{\text{threshold}}}}{(1 - \beta)^{\max_{\text{threshold}}}} \left[1 + J(1 - \gamma^2)\right] \prod_{l = \min_{\text{threshold}}}^{l = \max_{\text{threshold}} - 1} \left(\frac{1}{1 - \alpha_i}\right)\right] \tag{6.92}
\]

\[
D_p = \sum_{i = \min_{\text{threshold}}}^{i = \max_{\text{threshold}} - 1} \left(\frac{\alpha_1 - \alpha_i}{\alpha_1}\right) \prod_{i = \min_{\text{threshold}}}^{i = \max_{\text{threshold}} - 1} \left(\frac{\alpha_1 - \alpha_i}{\alpha_1}\right) \sum_{i = \min_{\text{threshold}}}^{i = \max_{\text{threshold}} - 1} p_i \tag{6.93}
\]

### 6.5.3 Results for the RED Simulation and the Analytical Models

In this subsection, the RED algorithm is compared with the proposed RED-based analytical models (Alpha, Linear) to identify which algorithm or model offers more satisfactory performance measure results, and to evaluate the performance of all the models considered.
This experimental subsection is split into three subsections. In Subsections 6.5.3.1 and 6.5.3.2, the proposed analytical models and RED are compared with respect to different performance measures \((mql, T, D, P_{\text{Loss}})\) and \((P_L, D_p)\), respectively. The investigation of the min threshold optimal position at the routers buffers of the RED and RED models that gives the best performance measure results is given in Subsection 6.5.3.3. It should be noted that the purpose of the RED simulation is not to validate the analytical models since they use different strategies, but simply as a basis for performance comparison.

6.5.3.1 Performance Evaluation of the RED, RED-Alpha and RED-Linear

This subsection compares RED and the developed RED-based models with regards to \((mql, T, D, P_{\text{Loss}})\) in order to acquire which algorithm or model offers more satisfactory performance measure results. The RED parameters \((\alpha_1, \beta, K, D_{\text{max}}, q_w, \text{min threshold}, \text{max threshold})\) have been set as in Subsection 3.1.2, whereas the parameters of RED analytical models \((\alpha_1, D_{\text{max}}, q_w, \text{min threshold}, \text{max threshold})\) are set as RED’s parameters in this subsection. Also \(\alpha_2\) and \(\alpha_3\) are set to 0.1 and 0.05, respectively.

The performance measure results \((mql, T, D, P_{\text{Loss}})\) versus the \(\alpha_1\) are exhibited in Figures (6.30-6.33), respectively, where the RED’s performance measure results are represented by the mean results of the ten runs. Also, the performance measures as a function of \(\alpha_1\) are shown in Figures (6.30-6.33).
After analysing Figures 6.30 and 6.32, it is shown that the RED and proposed RED-Alpha and Linear models provide similar \( mql \) and \( D \) results when the \( \alpha_1 \) value is either 0.18 or 0.33. On the other hand, when the \( \alpha_1 \) value becomes larger than the \( \beta \) value or near it, i.e. 0.48, congestion arises at the router buffers of the RED and RED models. As a
result, the \textit{mql} and \textit{D} results of the RED-based models become better than those of RED method since they maintain their \textit{mql} and \textit{D} lower than RED.

Figure 6.31 indicates the \textit{T} results for the RED and RED-Alpha and Linear models. This figure depicts that the RED models and RED provide similar \textit{T} results when the $\alpha_1$ value equals to 0.18 or 0.33 because there is either a light congestion or no congestion at their router buffers. In cases where the $\alpha_1$ value increases to be near the $\beta$ (0.48, 0.63), a congestion incident is occurred and RED offers \textit{T} results marginally higher than the RED models. Therefore, the \textit{T} results of RED method are marginally better than those for the RED-based models when the $\alpha_1$ is 0.48 or 0.63, which is expected since the \textit{mql} is higher.

In addition, the RED-Linear slightly outperforms the RED-Alpha with reference to \textit{T} results at these $\alpha_1$ values. Finally, when a high congestion exists, RED and the proposed models generate similar \textit{T} results.

Moreover, Figure 6.31 exhibits that the \textit{T} results of the RED and RED models are increased as long as the $\alpha_1$ value increases with and the $\alpha_1$ value < $\beta$ value. On the contrary, the \textit{T} results are stabilised at the $\beta$ value for the RED and RED models when $\alpha_1$ value > $\beta$ value.

It is clear in Figure 6.33 that the proposed RED models and RED offer similar $P_{\text{Loss}}$ results when the $\alpha_1$ value is equal to 0.18. When the $\alpha_1$ value is set to the following values: 0.33, 0.48 and 0.63, congestion may occur, thus the $P_{\text{Loss}}$ results of the RED method become marginally better than those of RED-Alpha and RED-Linear since the total of dropped and overflow (lost) packets of RED are smaller than those of the proposed models. Lastly, if a high congestion happens because of the high $\alpha_1$ values, i.e. 0.78 or
Then the RED and RED models achieve similar total number of dropped and overflow packets.

6.5.3.2 The Overflow Loss and Dropping Probability Results for the RED, RED-Alpha and RED-Linear

In this subsection, a comparison of the RED and RED-Alpha and RED-Linear models is introduced to identify the one that loses and drops fewer packets at its router buffer. The RED’s parameters \((a_1, \beta, K, D_{max}, q_w, \text{min threshold}, \text{max threshold})\) have been set to values as in Subsection 3.1.2, whereas the parameters of RED models \((a_1, a_2, a_3, D_{max}, q_w, \text{min threshold}, \text{max threshold})\) are set to the same values in Subsection 6.5.3.1.

The \(P_L\) and \(D_p\) results for the RED and developed models versus \(a_1\) are displayed in Figures (6.34-6.35), respectively, where the RED results represent the mean of the ten runs.

It is noted in Figure 6.34 that the RED and the analytical models produce similar \(P_L\),
results when the $\alpha_1$ is set to a value lower than the $\beta$ value. However, if the $\alpha_1$ value becomes larger than the $\beta$ value, congestion happens. Thus the RED method loses a greater amount of packets due to overflow at its router buffer than the RED analytical models since it is likely that the differences in the loss probability ($P_L$) for the RED simulation and the $P_L$ results for the RED-Alpha and Linear are due to the different concepts used in detecting the congestion. In RED the average queue length is used. Whereas in the two analytical models the instantaneous queue length is used. For example, in RED the average queue length may be less than the min\(\text{threshold}\). So no probabilistic dropping takes place. If the arrival rate momentarily increases over a short interval, this may cause the buffer to fill and overflow, since the average queue length will take time to build up to counteract this. With the analytical models this will not be the case since the instantaneous queue will pass the min\(\text{threshold}\) and cause probabilistic dropping to occur before the buffer has chance to overflow (see the beginning of Section 6.5). Hence it is logical that the RED simulation should exhibit a higher $P_L$ than the two analytical models.

Both RED-based models offer similar $P_L$ results whether the $\alpha_1$ value is large or not.

Figure 6.35 shows that the RED and RED models drop similar amounts of packets when the $\alpha_1$ value is equal to 0.18. If the $\alpha_1$ is greater than 0.18, RED drops fewer packets than the two models due to the fact that RED router buffer overflows more often than the analytical models, especially when the $\alpha_1$ value is high. The RED-Linear drops marginally fewer packets than the RED-Alpha when the $\alpha_1$ value is set to 0.33, 0.48 or 0.63. This is because the different strategies of the two models in dropping packets and in decreasing the $\alpha_1$ when congestion is present. For instance, when congestion occurs, the
RED-Alpha decreases its $\alpha_1$ value to another smaller one, while the RED-Linear decreases its $\alpha_1$ value linearly when instantaneous queue length is between $\min\text{threshold}$ and $\max\text{threshold}$. Furthermore, in cases of heavy congestion ($\alpha_1 = 0.78$ or 0.93), the RED-Alpha and Linear drop a similar number of packets.

### 6.5.3.3 Evaluation of $\min\text{threshold}$ for the RED, RED-Alpha and RED-Linear

The most satisfactory performance measure results ($mql, T, D, P_{\text{Loss}}, P_L, D_p$) of the RED and the two RED models (Alpha, Linear) with respect to the $\min\text{threshold}$ values are introduced in this subsection. Furthermore, this subsection aims to identify the $\min\text{threshold}$ position from the tested values [4-9] that gives the highest satisfactory performance measure results. The parameters of RED ($\alpha_1$, $\beta$, $K$, $D_{\text{max}}$, $qw$, $\min\text{threshold}$, $\max\text{threshold}$) are set to values as in Subsections 3.1.2 with two differences, these differences are setting $\alpha_1$ to 0.78 and $\min\text{threshold}$ to [4-9]. The $\min\text{threshold}$ parameter is set to [4-9] since we need to examine the $\min\text{threshold}$ with values near and far from the $\max\text{threshold}$ position. As a result, one can infer the optimal $\min\text{threshold}$ position at router buffers of RED and RED models that can offer the most satisfactory performance measure results.

The parameters of the RED models (Alpha, Linear) ($\alpha_1$, $D_{\text{max}}$, $qw$, $\min\text{threshold}$, $\max\text{threshold}$) have been set similar to those of RED. In addition, $\alpha_2$ and $\alpha_3$ are configured to values as in Subsection 6.5.3.1. The $\alpha_1$ parameter was set to 0.78 since this
value can create a congestion situation. The performance measure results are carried out with regard to the tested values for the \(\text{min} \ threshold\) parameter. Figures (6.36-6.41) show the performance measure results \((mql, T, D, P_{Loss}, P_L, D_p)\) versus \(\text{min} \ threshold\) for the RED and RED models. Specifically, \(mql, T, D, P_{Loss}, P_L\) and \(D_p\) versus \(\text{min} \ threshold\) are displayed in Figures (6.36-6.41), respectively.

Figures (6.36-6.39) indicate that the most satisfactory \(mql, T, D\) and \(P_{Loss}\) results for the RED algorithm can be obtained if the \(\text{min} \ threshold\) was set to any tested value, and therefore the \(mql\) and \(D\) results of RED are not influenced with the \(\text{min} \ threshold\) values.

Figure 6.40 indicates that the RED loses the fewest packets when the \(\text{min} \ threshold\) is set a small value (4). Consequently the most satisfactory \(P_L\) result can be obtained for the RED when the \(\text{min} \ threshold\) is set to a value as far as possible from the \(\text{max} \ threshold\). On the other hand, the RED router buffer drops the smallest number of packets when the \(\text{min} \ threshold\) is tuned to the largest tested value (9). Therefore the most satisfactory \(D_p\) result for RED can be achieved when the \(\text{min} \ threshold\) is set to a value near to the \(\text{max} \ threshold\). In the two models based on RED, Figures 6.36 and Figure 6.38 reveal that the most satisfactory \(mql\) and \(D\) results are obtained when the \(\text{min} \ threshold\) is set to a value as far as possible from the \(\text{max} \ threshold\). From Figures 6.37 and (6.39-6.41), the best results for the rest of the performance measures \((T, P_{Loss}, P_L, D_p)\) for the the proposed models are obtained when the \(\text{min} \ threshold\) is set to any tested value; That is, the \(\text{min} \ threshold\) has little effect on these.
Figure 6.36: $mql$ vs. $\text{min\ threshold}$.

Figure 6.37: $T$ vs. $\text{min\ threshold}$.

Figure 6.38: $D$ vs. $\text{min\ threshold}$.

Figure 6.39: $P_{\text{Loss}}$ vs. $\text{min\ threshold}$.

Figure 6.40: $P_Z$ vs. $\text{min\ threshold}$.

Figure 6.41: $D_p$ vs. $\text{min\ threshold}$.
6.5.4 Section 6.5 Summary

In Section 6.5, two discrete-time queue analytical models based on the RED algorithm called RED-Alpha and RED-Linear were developed. When the router buffer is congested, the packet arrival probability value in the Linear model decreases linearly if the $ql$ is between the positions of $\text{min threshold}$ and $\text{max threshold}$. On other hand, the value of packet arrival probability in the Alpha model reduces to another lower value. The performance measure results of RED and the proposed models were obtained by setting the $\alpha_1$ parameter to different values and others by setting the $\text{min threshold}$ parameter to different values.

From the simulation and the numerical results for RED and the RED-based analytical models the following can be inferred:

- When the $\alpha_1$ value was set to 0.18 or 0.33, the RED method and proposed RED-Alpha and Linear models offered similar $mql$ and $D$ results. Whereas, The RED-Alpha and RED-Linear outperform the RED method with reference to $mql$ and $D$ results only when the $\alpha_1$ value was tuned to 0.48 or larger than the $\beta$ value.

- Both the RED and RED models gave similar $T$ results once the $\alpha_1$ becomes 0.18 or 0.33. Moreover, RED slightly outperformed the models concerning to $T$ results when the $\alpha_1$ value was set to 0.48 or 0.63. Also, at these $\alpha_1$ values, the RED-Linear is marginally better than the RED-Alpha. Lastly, the RED and RED models provided similar $T$ results when a high congestion happens ($\alpha_1$ value = 0.78 or 0.93).
• The RED method slightly outperformed the RED models with regard to the $P_{Loss}$ results when the $\alpha_1$ value = 0.33, 0.48 or 0.63. In cases where the $\alpha_1$ was given a large value, i.e. 0.78 or 0.93, the RED and RED models present similar $P_{Loss}$ results.

• The RED method loses more packets due to overflow its router buffer than the proposed RED analytical models when the $\alpha_1$ value is larger than the $\beta$ value. On the other hand, the RED and RED models lose the same amounts of packets due to overflow their routers buffers when the $\alpha_1$ value is smaller than the $\beta$ value. Finally, whether the $\alpha_1$ value is a high or not, the RED analytical models provide similar $P_L$ results.

• Both RED and RED models drop a similar number of packets when the value of $\alpha_1$ was set to 0.18. The RED models drop higher numbers of packets than RED when the $\alpha_1$ was given to a value larger than 0.18. In addition, the RED-Alpha model drops more packets than the RED-Linear when the $\alpha_1$ value = 0.33, 0.48 or 0.63. Finally, the RED models offer similar $D_p$ results in cases of a high congestion.

• The most satisfactory min $\text{threshold}$ position at the RED router buffer that provides the most satisfactory performance measure results ($mql, D$) are given when setting the min $\text{threshold}$ to any tested value [4-9]. Therefore, the $mql$ and $D$ results are not affected with the min $\text{threshold}$ position. Moreover, RED produces the most satisfactory $T$ and $P_{Loss}$ results regardless the position of the min $\text{threshold}$. However, when the min $\text{threshold}$ was set to the farthest value from the max $\text{threshold}$ (4), few number of packets were lost due to buffer overflow than the
other tested values. Whereas the smallest number of dropped packets was obtained when the min threshold was set as close as possible to the max threshold.

- For the RED-Alpha and Linear analytical models the most satisfactory performance measure results (mql, D) with reference to the min threshold parameter were achieved by setting the min threshold to the farthest possible value from the max threshold (4). Lastly, the min threshold position had little effect on the T, P_{loss}, P_L and D_p results.

### 6.6 GRED Based Discrete-time Queue Analytical Models

This section proposes two discrete-time queue analytical models based on the classic GRED method [50]. These models identify the congested router buffers incipiently and aim to offer more satisfactory performance measure results when a high congestion is present. The proposed models are called GRED-Alpha and GRED-Linear, and they use the instantaneous queue length instead of average queue length similar to the RED-based models discussed in this chapter. The GRED models have another aim which is stabilising mql and D at values lower than the GRED values to reduce buffer overflow compared with GRED.

Again, the GRED simulation is not used for validation purposes since the models operate differently, but is used as a basis for comparing the models.
The structure of this section is given as follows: The Alpha and Linear models of GRED are discussed in Subsections 6.6.1, and 6.6.2, respectively. Subsection 6.6.3 gives the simulation and numerical results for the developed models and GRED. Finally, a summary of this section is given in Subsection 6.6.4.

6.6.1 GRED-Alpha Analytical Model

This analytical model is implemented at the router buffer as a congestion controller, and it is constructed through the analysis of the single queue node system shown in Figure 6.42. The GRED-Alpha model employs three thresholds \((\text{threshold}_{\text{min}}, \text{threshold}_{\text{max}}, \text{double max threshold})\) [50], where the \text{threshold}_{\text{min}} and the \text{threshold}_{\text{max}} are explained in Subsection 6.5.1, whereas the \text{double max threshold} equals 2 x \text{max threshold}. It should be noted that the implementation of the GRED-Alpha model is similar to that of the RED model (see Subsection 6.5.1) in the following cases a) when the \(ql\) is smaller than the \text{threshold}_{\text{min}}, b) when \(ql\) is between the \text{threshold}_{\text{min}} and the \text{threshold}_{\text{max}}, c) when \(ql\) is larger than or equals the \text{threshold}_{\text{max}} and less than the \text{double max threshold}. The only difference between this model and that of RED is the case where \(ql \geq \text{double max threshold}\), in which the probability of packet arrival of the GRED-Alpha gets reduced from \(\alpha 3\) to \(\alpha 4\), and the \(Dp\) value increases from \(\left(\frac{\alpha 1 - \alpha 3}{\alpha 1}\right)\) to \(\left(\frac{\alpha 1 - \alpha 4}{\alpha 1}\right)\).

On the other hand, the packet arrival probability of the RED-Alpha model remains at \(\alpha 3\)
since RED-Alpha does not have double max threshold, and therefore the \(Dp\) remains at \(\frac{\alpha_1 - \alpha_3}{\alpha_1}\).

![Figure 6.42: The single queue node system for GRED Alpha model.](image)

Since the probability of packet arrival has decreased in the GRED model when congestion occurs, this model is called Alpha. The GRED-Alpha parameters \((\alpha_1, \alpha_2, \alpha_3, \beta, K)\) are similar to those of RED-Alpha and \(\alpha_3\) is active only when double max threshold \(> q_l \geq\) max threshold. The queue is assumed in equilibrium and the queue length process is a Markov chain with finite state space \(\{0,1,2,3,\ldots,\min\text{threshold},\ldots,\max\text{threshold},\ldots,\text{double max threshold},\ldots,K-1,K\}\).

Further, it is assumed that \(\alpha_1 > \alpha_2, \alpha_2 > \alpha_3, \alpha_3 > \alpha_4\) and half of \(\alpha_1\) values are smaller than \(\beta\) \((\alpha_1 < \beta)\), whereas the other half values are larger than \(\beta\) \((\alpha_1 > \beta)\). In addition, \(\beta > \alpha_2, \beta > \alpha_3\) and \(\beta > \alpha_4\). The state transition diagram for the GRED-Alpha is displayed in Figure 6.43. The balance equations (6.94-6.106) are produced from Figure 6.43.
Figure 6.43: The state transition diagram for the GRED Alpha analytical model.

\[ p_0 = (1 - \alpha_1)p_0 + [\beta(1 - \alpha_1)]p_1 \] \hspace{1cm} (6.94)

\[ p_1 = \alpha_1p_0 + [\alpha_1\beta + (1 - \alpha_1)(1 - \beta)]p_1 + [\beta(1 - \alpha_1)]p_2 \] \hspace{1cm} (6.95)

In general:

\[ p_i = [\alpha_1(1 - \beta)]p_{i-1} + [\alpha_1\beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1}, \]

where \( i = 2,3,4,..., \min \text{threshold} - 2 \) \hspace{1cm} (6.96)

\[ p_{\min \text{threshold} - 1} = [\alpha_1(1 - \beta)]p_{\min \text{threshold} - 2} + [\alpha_1\beta + (1 - \alpha_1)(1 - \beta)]p_{\min \text{threshold} - 1} + [\beta(1 - \alpha_2)]p_{\min \text{threshold}} \] \hspace{1cm} (6.97)

\[ p_{\min \text{threshold}} = [\alpha_1(1 - \beta)]p_{\min \text{threshold} - 1} + [\alpha_2\beta + (1 - \alpha_2)(1 - \beta)]p_{\min \text{threshold}} + [\beta(1 - \alpha_2)]p_{\min \text{threshold} + 1} \] \hspace{1cm} (6.98)

\[ p_i = [\alpha_2(1 - \beta)]p_{i-1} + [\alpha_2\beta + (1 - \alpha_2)(1 - \beta)]p_i + [\beta(1 - \alpha_2)]p_{i+1}, \]

where \( i = \min \text{threshold} + 1, \min \text{threshold} + 2, \min \text{threshold} + 3,..., \max \text{threshold} - 2 \). (6.99)
\begin{align}
p_{\text{max threshold }-1} &= \left[\alpha \beta (1-\beta)\right]p_{\text{max threshold }-2} + \left[\alpha \beta (1-\beta)\right]p_{\text{max threshold }-1} + \left[\beta (1-\alpha \beta)\right]p_{\text{max threshold}} \\
p_{\text{max threshold}} &= \left[\alpha \beta (1-\beta)\right]p_{\text{max threshold }-1} + \left[\alpha \beta (1-\beta)\right]p_{\text{max threshold}} + \left[\beta (1-\alpha \beta)\right]p_{\text{max threshold }+1} \\
p_{i} &= \left[\alpha \beta (1-\beta)\right]p_{i-1} + \left[\alpha \beta (1-\beta)\right]p_{i} + \left[\beta (1-\alpha \beta)\right]p_{i+1}, \quad \text{(6.102)}
\end{align}

where $i = \text{max threshold } + 1, \text{max threshold } + 2, \text{max threshold } + 3, \ldots, \text{double max threshold } - 2$

\begin{align}
p_{\text{double max threshold }-1} &= \left[\alpha \beta (1-\beta)\right]p_{\text{double max threshold }-2} + \left[\alpha \beta (1-\beta)\right]p_{\text{double max threshold }-1} + \left[\beta (1-\alpha \beta)\right]p_{\text{double max threshold}} \\
p_{\text{double max threshold}} &= \left[\alpha \beta (1-\beta)\right]p_{\text{double max threshold }-1} + \left[\alpha \beta (1-\beta)\right]p_{\text{double max threshold}} + \left[\beta (1-\alpha \beta)\right]p_{\text{double max threshold }+1} \\
p_{i} &= \left[\alpha \beta (1-\beta)\right]p_{i-1} + \left[\alpha \beta (1-\beta)\right]p_{i} + \left[\beta (1-\alpha \beta)\right]p_{i+1}, \quad \text{(6.105)}
\end{align}

where $i = \text{double max threshold } + 1, \text{double max threshold } + 2, \ldots, K - 1$

Finally,

\begin{align}
p_{K} &= \left[\alpha \beta (1-\beta)\right]p_{K-1} + \left[\alpha \beta (1-\beta)\right]p_{K} + \left[\beta (1-\alpha \beta)\right]p_{K+1}, \quad \text{(6.106)}
\end{align}

where $K = \text{double max threshold } + X, \text{double max threshold } = \text{max threshold } + J$

and $\text{max threshold } = \text{min threshold } + I$. Therefore, $K$ can also be defined as below:

\begin{align}
K &= \text{min threshold } + I + J + X \quad \text{(6.107)}
\end{align}

Let $\gamma_{i} = \frac{\alpha i (1-\beta)}{\beta (1-\alpha i)}, i = 1, 2, 3, 4$. \quad \text{(6.108)}
Applying equation (6.108) in equations (6.94-6.106) to generate the equilibrium probabilities, the following are obtained.

In general:

\[ p_i = \frac{\alpha^i (1 - \beta)^{i-1}}{\beta^i (1 - \alpha)^i} p_0 = \frac{\gamma^i}{(1 - \beta)} p_0, \text{ where } i = 1, 2, 3, \ldots, \min \text{ threshold} - 1 \]  

\[ p_{\min \text{ threshold} + i} = \frac{\alpha^i}{\beta^i (1 - \alpha)^{i-1}} \frac{\alpha^{i+1} (1 - \beta)^{i-1}}{(1 - \alpha^{i+1}) (1 - \alpha^{i+1})} p_0 = \frac{\gamma^i}{(1 - \alpha^{i+1}) (1 - \beta)} p_0 \]

where \( i = 0, 1, 2, 3, \ldots, I - 1 \)  

\[ p_{\min \text{ threshold} + I + i} = \frac{\alpha^i}{\beta^i (1 - \alpha)^{i-1}} \frac{\alpha^{i+1} (1 - \beta)^{i-1}}{(1 - \alpha^{i+1}) (1 - \alpha^{i+1})} p_0 \]

\[ = \frac{\gamma^i}{(1 - \alpha^{i+1}) (1 - \beta)} p_0 \]

where \( i = 0, 1, 2, 3, \ldots, J - 1 \)  

\[ p_{\min \text{ threshold} + I + J + i} = \frac{\alpha^i}{\beta^i (1 - \alpha)^{i-1}} \frac{\alpha^{i+1} (1 - \beta)^{i-1}}{(1 - \alpha^{i+1}) (1 - \alpha^{i+1})} p_0 \]

\[ = \frac{\gamma^i}{(1 - \alpha^{i+1}) (1 - \beta)} p_0 \]

where \( i = 0, 1, 2, 3, \ldots, X \)  

By applying equation (6.12) in equations (6.109-6.112), the following is obtained:
\[
p_0 = \left[ 1 - \gamma_1^{\text{min threshold}} - \beta(1 - \gamma_1) + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \right] \left[ \frac{(1 - \gamma_2')}{(1 - \alpha_2)(1 - \gamma_2)} + \gamma_2' \frac{(1 - \gamma_3')}{(1 - \alpha_3)(1 - \gamma_3)} \right]^{-1} \\
\]

\[\gamma_1 - \gamma_1^{\text{min threshold}} \left[ \gamma_1 + \min \text{threshold} (1 - \gamma_1) \right] + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \]

\[
\min \text{threshold} (1 - \gamma_2) \left[ (1 - \gamma_2') \right] + \gamma_2' \left[ (1 - \gamma_2) \right] + \gamma_2' \left[ \gamma_2 + (1 - \gamma_2) \right] \\
\frac{(1 - \alpha_2)(1 - \gamma_2)}{1 - \alpha_2(1 - \gamma_2)} + \gamma_3' \left[ \gamma_3 + J(1 - \gamma_3) \right] \\
\frac{(1 - \alpha_3)(1 - \gamma_3)}{1 - \alpha_3(1 - \gamma_3)} + \gamma_4' \left[ \gamma_4 + J(1 - \gamma_4) \right] \\
\frac{(1 - \alpha_4)(1 - \gamma_4)}{1 - \alpha_4(1 - \gamma_4)} \\
\]

The performance measures for the GRED-Alpha can then be computed. Firstly, the \(mql\) can be obtained as:

\[
mql = \frac{p^{(i)}(1)}{1 - \beta} = \left[ \frac{\gamma_1 - \gamma_1^{\text{min threshold}} \left[ \gamma_1 + \min \text{threshold} (1 - \gamma_1) \right] + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \left[ \gamma_1 + \min \text{threshold} (1 - \gamma_1) \right] + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \right] \\
\left[ \frac{(1 - \gamma_2')}{(1 - \alpha_2)(1 - \gamma_2)} + \gamma_2' \frac{(1 - \gamma_3')}{(1 - \alpha_3)(1 - \gamma_3)} \right]^{-1} \\
\]

The performance measures with respect to \((T, D)\) can be obtained using equations (6.16-6.17). While \(P_L\) and \(P_{Loss}\) of GRED-Alpha are computed as before and the \(D_p\) is evaluated as:

\[
D_p = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \times \sum_{i=\text{min threshold}}^{\text{max threshold}} p_i + \left( \frac{\alpha_1 - \alpha_3}{\alpha_1} \right) \times \sum_{i=\text{max threshold}}^{\text{double max threshold}} p_i + \left( \frac{\alpha_1 - \alpha_4}{\alpha_1} \right) \times \sum_{i=\text{double max threshold}}^{K-1} p_i \\
\]

The performance measures for the GRED-Alpha can then be computed. Firstly, the \(mql\) can be obtained as:

\[
mql = \frac{p^{(i)}(1)}{1 - \beta} = \left[ \frac{\gamma_1 - \gamma_1^{\text{min threshold}} \left[ \gamma_1 + \min \text{threshold} (1 - \gamma_1) \right] + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \left[ \gamma_1 + \min \text{threshold} (1 - \gamma_1) \right] + \gamma_1^{\text{min threshold}} (1 - \alpha_1) \right] \\
\left[ \frac{(1 - \gamma_2')}{(1 - \alpha_2)(1 - \gamma_2)} + \gamma_2' \frac{(1 - \gamma_3')}{(1 - \alpha_3)(1 - \gamma_3)} \right]^{-1} \\
\]

The performance measures with respect to \((T, D)\) can be obtained using equations (6.16-6.17). While \(P_L\) and \(P_{Loss}\) of GRED-Alpha are computed as before and the \(D_p\) is evaluated as:

\[
D_p = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \times \sum_{i=\text{min threshold}}^{\text{max threshold}} p_i + \left( \frac{\alpha_1 - \alpha_3}{\alpha_1} \right) \times \sum_{i=\text{max threshold}}^{\text{double max threshold}} p_i + \left( \frac{\alpha_1 - \alpha_4}{\alpha_1} \right) \times \sum_{i=\text{double max threshold}}^{K-1} p_i \\
\]
6.6.2 GRED-Linear Analytical Model

This model [4] is similar to the RED-Linear model (see Subsection 6.5.2) with some exceptions. These exceptions are when the $ql$ at the router buffer is either between the $max\ threshold$ and $double\ max\ threshold$ positions or larger than the $double\ max\ threshold$ position. The single queue node used in the GRED Linear model is depicted in Figure 6.44.

![Figure 6.44: The single queue node system for GRED Linear model.](image)

It can be observed in Figure 6.44 that when the $ql$ is smaller than the $min\ threshold$ or between the $min\ threshold$ and the $max\ threshold$, this model behaves similar to the RED-Linear model. However, if the $ql$ is between $max\ threshold$ and $double\ max\ threshold$, then the packet arrival probability of this model keeps decreasing linearly on $\alpha_i$ value. This linear decreasing remains as long as the $ql$ increases from the $min\ threshold$ to the $double\ max\ threshold$ − 1. The $\alpha_i$ value is calculated as follows:
\[ \alpha_i = \alpha_1 - \left(1 + i - \min \text{ threshold}\right) \frac{(\alpha_1 - \alpha_2)}{(1 + \text{ double max threshold} - \min \text{ threshold})}, \]  

if \( \min \text{ threshold} \leq i < \text{ double max threshold} \), and the \( \alpha_i \) depends on the state transition diagram (see Figure 6.45). In addition, the \( D_p \) value increases linearly.

Furthermore, if the \( ql \) reaches the \( \text{ double max threshold} \) position, the packet arrival probability in this model reduces from \( \alpha_i \) to \( \alpha_2 \) in order to manage the heavy congestion, and the router buffer will increase its packet dropping (\( D_p \)) from \( \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \) to \( \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \). The \( \alpha_i \) and \( \alpha_2 \) parameters represent the packet arrival probabilities in a slot when the \( ql \) is between the \( \min \text{ threshold} \) and the \( \text{ double max threshold} \) and equal to or above the \( \text{ double max threshold} \), respectively. The rest of the GRED Linear model parameters (arrival process, \( a_n \), queuing discipline, the state space, \( \alpha_1, \beta, \min \text{ threshold}, \max \text{ threshold}, \text{ double max threshold}, K \)) are similar to those of the GRED-Alpha discussed earlier. It is also assumed that the single queue is in equilibrium, and the queue length process is a Markov chain with finite state space. It is also assumed that \( \alpha_1 > \alpha_i, \alpha_i > \alpha_2 \), and half of \( \alpha_1 \) values are lower than \( \beta \) (\( \alpha_1 < \beta \)), while the rest of the values are higher than \( \beta \) (\( \alpha_1 > \beta \)). Also, \( \beta > \alpha_i \) and \( \beta > \alpha_2 \). The state transition diagram for the proposed GRED Linear model is shown in Figure 6.45.
Based on Figure 6.45, the balance equations (6.116-6.125) for the GRED Linear model are produced.

\[ p_0 = (1 - \alpha_1)p_0 + [\beta(1 - \alpha_1)]p_1 \] ................................. (6.116)

\[ p_1 = \alpha_1 p_0 + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_1 + [\beta(1 - \alpha_1)]p_2 \] ................................. (6.117)

In general:

\[ p_i = [\alpha_1(1 - \beta)]p_{i-1} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_i + [\beta(1 - \alpha_1)]p_{i+1}, \]

where \( i = 2,3,4,\ldots, \min\text{ threshold} - 2 \) ................................. (6.118)

\[ p_{\min\text{ threshold} - 1} = [\alpha_1(1 - \beta)]p_{\min\text{ threshold} - 2} + [\alpha_1 \beta + (1 - \alpha_1)(1 - \beta)]p_{\min\text{ threshold} - 1} + [\beta(1 - \alpha_{\min\text{ threshold}})]p_{\min\text{ threshold}} \] ................................. (6.119)

\[ p_{\min\text{ threshold}} = [\alpha_1(1 - \beta)]p_{\min\text{ threshold} - 1} + [\alpha_{\min\text{ threshold}} \beta + (1 - \alpha_{\min\text{ threshold}})(1 - \beta)]p_{\min\text{ threshold}} + [\beta(1 - \alpha_{\min\text{ threshold} + 1})]p_{\min\text{ threshold} + 1} \] ................................. (6.120)
\[ p_i = [\alpha_{i-1}(1 - \beta)]p_{i-1} + [\alpha_i, \beta + (1 - \alpha_i)(1 - \beta)]p_i + [\beta(1 - \alpha_{i+1})]p_{i+1}, \]  

\[ (6.121) \]

where \( i = \text{min threshold} + 1, \text{min threshold} + 2, \text{min threshold} + 3, \ldots, \text{double max threshold} - 2 \)

\[ p_{\text{double max threshold} - 1} = [\alpha_{\text{double max threshold} - 2}(1 - \beta)]p_{\text{double max threshold} - 2} + \left[ \alpha_{\text{double max threshold} - 1}\beta + (1 - \alpha_{\text{double max threshold} - 2})(1 - \beta) \right]p_{\text{double max threshold} - 1} + [\beta(1 - \alpha_2)]p_{\text{double max threshold}} \]

\[ (6.122) \]

\[ p_{\text{double max threshold}} = [\alpha_{\text{double max threshold} - 1}(1 - \beta)]p_{\text{double max threshold} - 1} + [\alpha_2\beta + (1 - \alpha_2)(1 - \beta)]p_{\text{double max threshold} + 1} + [\beta(1 - \alpha_2)]p_{\text{double max threshold} + 1} \]

\[ (6.123) \]

\[ p_i = [\alpha_2(1 - \beta)]p_{i-1} + [\alpha_2\beta + (1 - \alpha_2)(1 - \beta)]p_i + [\beta(1 - \alpha_2)]p_{i+1}, \]

where \( i = \text{double max threshold} + 1, \text{double max threshold} + 2, \ldots, K - 1 \)  

\[ (6.124) \]

Finally,

\[ p_K = [\alpha_2(1 - \beta)]p_{K-1} + [\alpha_2\beta + (1 - \beta)]p_K \]

\[ (6.125) \]

where \( K \) is shown in equation (6.107).

Assume \( \gamma_i = \frac{\alpha_i(1 - \beta)}{\beta(1 - \alpha)} \), where \( \text{min threshold} \leq i < \text{double max threshold} \)  

\[ (6.126) \]

Using equations (6.9, 6.126) and the balance equations (6.116-6.125) the equilibrium probabilities can be defined.

In general:

\[ p_i = \frac{\alpha_1(1 - \beta)^{-i}}{\beta(1 - \alpha_1)} p_0 = \frac{\gamma_1^i}{(1 - \beta)} p_0, \text{ where } i = 1, 2, 3, \ldots, \text{min threshold} - 1 \]  

\[ (6.127) \]
\[ p_i = \frac{\alpha_i^{\min \text{threshold}}(1-\beta)^{i-1}}{\beta^i(1-\alpha_i)^{\min \text{threshold}-1}} \prod_{j=\min \text{threshold}}^{i-1} \alpha_j \quad p_0 = \frac{\gamma_i^{\min \text{threshold}}(1-\alpha_i)}{(1-\alpha_i)(1-\beta)} \prod_{j=\min \text{threshold}}^{\text{double max threshold}-1} \gamma_j \]

where \( i = \min \text{threshold}, \min \text{threshold} + 1, \min \text{threshold} + 2, \ldots, \text{double max threshold} - 1 \)

and \( \alpha_j = 1 \), if \( j < \min \text{threshold} \).

\[ p_{\text{double max threshold}+1} = \frac{\alpha_i^{\min \text{threshold}}(1-\beta)^{i-1}}{\beta^{\text{double max threshold}+1}(1-\alpha_i)^{\min \text{threshold}-1}} \prod_{j=\min \text{threshold}}^{\text{double max threshold}-1} \alpha_j \quad p_0 \]

\[ = \frac{\gamma_i^{\min \text{threshold}}(1-\alpha_i)}{(1-\alpha_i)(1-\beta)} \prod_{j=\min \text{threshold}}^{\text{double max threshold}-1} \gamma_j \]

where \( i = 0, 1, 2, \ldots, X \) ………………………………………………………………………………………….. (6.129)

Using the equilibrium probabilities (6.127-6.129), \( p_0 \) can be calculated by substituting them in equation (6.12). Then \( p_0 \) is given as follows:

\[ p_0 = \left[ \frac{1 - \gamma^{\min \text{threshold}}}{(1-\beta)(1-\gamma^{\min \text{threshold}})} + \gamma^{\min \text{threshold}}(1-\alpha_i) \sum_{i=\min \text{threshold}}^{\text{double max threshold}-1} \prod_{j=\min \text{threshold}}^{i-1} \gamma_j \right]^{-1} \]

……………………………………………………………………………………….. (6.130)

Next, the performance measures for the proposed Linear model can be obtained. Firstly, the \( mql \) can be found as before and the rest of the performance measures \( (T, D) \) can be obtained using equations (6.16-6.17). The \( mql \) is obtained as:
Both $P_L$ and $P_{Loss}$ results of the GRED-Linear are computed as before and the $D_p$ result of GRED-Linear is given as follows:

$$D_p = \sum_{i=\text{min}\,\text{threshold}}^{\text{double max threshold}-1} \left( \frac{\alpha_1 - \alpha_i}{\alpha_1} \right) \times p_i + \left( \frac{\alpha_1 - \alpha_2}{\alpha_1} \right) \times \sum_{i=\text{double max threshold}}^{K-1} p_i \quad \text{........... (6.132)}$$

### 6.6.3 GRED Simulation and Analytical Model Numerical Results

In Subsection 6.6.3.1, the proposed analytical models (Alpha, Linear) are compared with the GRED algorithm with respect to different performance measures ($mql$, $T$, $D$, $P_{Loss}$) in order to identify the one that gains more satisfactory performance measure results. Also, Subsection 6.6.3.2 introduces a comparison between the GRED models and GRED algorithm with reference to ($P_L$, $D_p$) to determine the method that loses fewer packets due to overflow and packet drops.
6.6.3.1 Performance Evaluation of the GRED, GRED-Alpha and GRED-Linear

In this subsection, the GRED algorithm has been compared with the GRED-Alpha and Linear models concerning to \((mql, T, D, P_{Loss})\) to identify the one that produces more enhanced performance measure results. The GRED parameters \((\alpha_1, \beta, K, D_{\text{max}}, qw, \text{min\ threshold}, \text{max\ threshold})\) are set to values as in Subsection 3.1.2. The parameters of GRED-Alpha and Linear models \((\alpha_1, D_{\text{max}}, qw, \text{min\ threshold}, \text{max\ threshold})\) are set similar to the GRED values in this subsection. In addition, the \(\alpha_2, \alpha_3\) and \(\alpha_4\) are set to 0.1, 0.05 and 0.025, respectively.

The performance measure results \((mql, T, D, P_{Loss})\) versus the \(\alpha_1\) of the GRED and GRED based models are displayed in Figures (6.46-6.49), respectively.

It is clear in Figures 6.46 and 6.48 that the GRED and the analytical models provide similar \(mql\) and \(D\) results when the \(\alpha_1\) value is equal to 0.18 or 0.33. On the other hand, if the \(\alpha_1\) value increases to be equal to or larger than 0.48, a congestion situation occurs and the models become better than GRED with reference to \(mql\) and \(D\) since they stabilise their \(mql\) at smaller values than GRED, and provide lower packets average queueing delay than GRED. Moreover, smaller \(mql\) and \(D\) results are obtained for the GRED models since they drop more packets than GRED when the \(\alpha_1\) value is larger than 0.33 (see Subsection 6.6.3.2 for further details).

Figure 6.47 indicates that the GRED models and GRED generate similar \(T\) results.
when the $\alpha 1$ value is set to 0.18 or 0.33. However, when the $\alpha 1$ becomes 0.48 (close to the $\beta$ value), and this leads to slightly better $T$ results for the GRED than those of the proposed GRED models. Also, at these $\alpha 1$ values the GRED-Linear model marginally outperforms the GRED-Alpha model in terms of $T$ results. Moreover, when the $\alpha 1$ is set to 0.63, the GRED and GRED-Linear achieve similar $T$ results and their results are marginally better than those of the GRED-Alpha. Lastly, similar $T$ results can be obtained for the GRED and GRED models in cases of high congestion, i.e. $\alpha 1$ value = 0.78 or 0.93.

It is noted also in Figure 6.47 that the $T$ results for the GRED and its models increase as long as the $\alpha 1$ value increases when $\alpha 1 < \beta$. On the other hand, the $T$ results for the two models and GRED are sustained at the value of $\beta$ when $\alpha 1 > \beta$.

After analysing Figure 6.49, it is observed that all values of the $\alpha 1$ except 0.48 lead to similar $P_{Loss}$ results for the GRED and developed GRED-Linear model. This makes us infer that the total overflow lost and dropped packets at the router buffers of the GRED and GRED-Linear model are similar excluding when $\alpha 1$ value is very close to $\beta$ value. regardless of whether there is congestion or not. Nonetheless, when the $\alpha 1$ is set to a value very close to $\beta$, the GRED provides slightly better $P_{Loss}$ result than the GRED-Linear. When the $\alpha 1$ value is tuned to 0.33, 0.48 or 0.63, a congestion situation may occur and at these values the total number of lost and dropped packets ($P_{Loss}$) of GRED and GRED-Linear are less than those of the GRED-Alpha. The GRED-Alpha presents similar $P_{Loss}$ results of the GRED and GRED-Linear when the $\alpha 1$ value is equal to 0.18, 0.78 or 0.93.
6.6.3.2 The Overflow Loss and Dropping Probability Results for the GRED, GRED-Alpha and GRED-Linear

The GRED algorithm and GRED-Alpha and Linear models have been compared with respect to $P_L$ and $D_P$ results to determine which loses and drops the fewest packets. The
GRED’s parameters ($\alpha_1$, $\beta$, $K$, $D_{\text{max}}$, $q_w$, $\text{min threshold}$, $\text{max threshold}$) are tuned to values as in Subsection 3.1.2. While the parameters of the GRED models have been set to values similar to those in Subsection 6.6.3.1. The performance measure results ($P_L$, $D_P$) versus $\alpha_1$ are shown in Figures (6.50-6.51), respectively, where $P_L$ and $D_P$ results of GRED are denoted by the mean results of ten runs.

Figure 6.50: $P_L$ vs. probability of packet arrival ($\alpha_1$).

Figure 6.51: $D_P$ vs. probability of packet arrival ($\alpha_1$).

Figure 6.50 shows that the GRED-Alpha and Linear lose similar numbers of packets due to their routers buffers overflowing whether the $\alpha_1$ is high or not. In addition, this figure reveals that the models and GRED lose similar amounts of packets at their routers buffers ($P_L$) when the $\alpha_1$ value is smaller than the $\beta$. In contrast, if the $\alpha_1$ value is greater than $\beta$, congestion occurs, and the proposed GRED models lose fewer packets ($P_L$) than GRED because of the different mechanisms used in detecting the congestion by the considered methods. Specifically, GRED exploits an average queue length as a congestion measure, whereas in the models the instantaneous queue length is used.
Figure 6.51 depicts that the GRED and the two models based on it drop similar numbers of packets at their routers buffers when the $\alpha_1$ value is equal to 0.18. If the $\alpha_1$ value increases to 0.33, the GRED and GRED-Linear continue dropping similar numbers of packets, and their numbers are fewer than the number of dropped packets for the GRED-Alpha. Furthermore, the GRED analytical models drop higher numbers of packets than GRED when the $\alpha_1$ value is equal to or greater than 0.48. This is since the GRED models produce lower $mql$ results than GRED (see Subsection 6.6.3.1). A larger number of dropped packets for the models is the price of achieving smaller $mql$ results. When the $\alpha_1$ is set to either 0.48 or 0.63, the GRED-Linear drops fewer packets than the GRED-Alpha because of the different methods used in controlling the congestion. Finally, the two GRED models drop similar amounts of packets when a high congestion is existed ($\alpha_1 = 0.78$ or 0.93).

### 6.6.4 Chapter Summary

In Section 6.6, two discrete-time queue analytical models based on the traditional GRED algorithm but using instantaneous queue length as a congestion measure were proposed. The proposed models were compared with the GRED method aiming to evaluate their relative performances with reference to different performance measures. The results of the GRED and proposed GRED models concerning the performance measures were accomplished by setting the $\alpha_1$ parameter to different values. From the simulation and the numerical results for GRED and the analytical models the following can be concluded:
• When the $\alpha$ value was set to 0.18 or 0.33, the proposed GRED-Alpha and GRED-Linear and GRED offer similar $mql$ and $D$ results. Alternatively, the GRED-Alpha and Linear models outperformed the GRED when the $\alpha$ was set to 0.48 or larger than the $\beta$. In addition, the GRED-Alpha outperformed the GRED-Linear according to $mql$ and $D$ results when the value of $\alpha \geq 0.48$.

• Both the GRED algorithm and the models present similar $T$ results when the $\alpha$ value $= 0.18$ or 0.33. Whereas if the $\alpha$ value was set to 0.48, then the GRED marginally outperformed the two models with respect to $T$ results. Both GRED and GRED-Linear obtain similar $T$ results when the $\alpha$ value was set to 0.63, and these results are better than the GRED-Alpha $T$ result. The models and GRED generate similar $T$ results when high congestion exists ($\alpha$ value $= 0.78$ or 0.93).

• When the $\alpha$ was not set to a very close value to $\beta$, i.e. $\alpha = 0.48$, the GRED and GRED-Linear offered similar $P_{Loss}$ results. On contrary, if the $\alpha$ value was set to a value very close to the $\beta$, the $P_{Loss}$ result of GRED is slightly better than that of the GRED-Linear. The GRED-Linear and GRED outperformed the GRED-Alpha in terms of the $P_{Loss}$ results when the $\alpha$ value was set to 0.33, 0.48 or 0.63. The GRED-Alpha achieved similar $P_{Loss}$ results to both GRED and GRED-Linear when the $\alpha$ was set to any of the following values: 0.18, 0.78 or 0.93.

• The proposed GRED models (Alpha, Linear) achieved similar $P_L$ results regardless the value of the $\alpha$.
• The GRED presented similar $P_L$ results to those of the proposed GRED models when the $\alpha_1$ value is smaller than the $\beta$ value. However, if the $\alpha_1$ value was set to a value larger than the value of $\beta$, then the GRED models lose fewer packets than GRED due to buffer overflow.

• The two GRED models and GRED drop similar number of packets when the $\alpha_1$ value was set to 0.18.

The GRED and GRED-Linear provide similar $D_p$ results when the $\alpha_1$ was set to 0.33 and these results were superior to that of the GRED-Alpha. When the $\alpha_1$ was set to a value equal to or greater than 0.48, the GRED dropped fewer packets than the two GRED models. Additionally, the GRED-Linear gives better $D_p$ results than the GRED-Alpha when $\alpha_1$ value = 0.48 or 0.63. Finally, if a high congestion occurs ($\alpha_1 = 0.78$ or 0.93), both GRED models drop similar amounts of packets.