Symplectic Transformations And Entanglement In Finite Quantum Systems

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Abstract

Quantum systems with finite Hilbert space are considered. Position and momentum states and their relation through a Fourier transform, displacement in the position-momentum phase-space, and symplectic transformations are introduced and their properties are studied. Symplectic $Sp(2\ell, \mathbb{Z}_{\wp})$ transformations in ℓ-partite finite system are explicit constructed. The general method is applied to bi-partite and tri-partite systems. The effect of these transformations on the correlations is discussed. Entanglement calculations between the subsystems in a bi-partite system and a tri-partite system are presented. The effect of measurements is also studied.

Keywords: Symplectic Transformations, Finite Systems, Phase Space, Entanglement.

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Chapter 1

Introduction

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- Generally About the Dissertation
- The Overview of the Thesis

1.1 Generally About the Dissertation

The subject of this thesis is Quantum systems with finite Hilbert space which were studied originally by Weyl[1] and also by Schwinger[2, 3]. Position and momentum in the context of the harmonic oscillator take values in $\mathcal{R}$(real number). In finite quantum systems with d-dimensional Hilbert space, the position and momentum take values in $\mathbb{Z}_d$(the integers modulo d). The position and momentum phase-space in finite quantum systems is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$.

In our research work, our interest is in the case with d-dimensional Hilbert space where d is a prime number $\wp$. In this case, the position and momentum take values in $\mathbb{Z}_\wp$ and all its nonzero elements have an inverse. Therefore, the position-momentum phase space $\mathbb{Z}_\wp \times \mathbb{Z}_\wp$ have stronger properties. Our research focus on constructing explicitly symplectic transformations in a system comprised of ℓ subsystems and each of them is $\wp$-dimensional(where $\wp$ is an odd prime number). Reference[4] has been presented on this direction in the context of quantum coding. Our research extend these results, and present a numerical technique for the calculation of symplectic transformations. The new numerical technique is simpler than the one used in [4]. We use the new numerical technique to construct explicitly $Sp(4, \mathbb{Z}_\wp)$ transformations in bi-partite systems and $Sp(6, \mathbb{Z}_\wp)$ in tri-patite systems. More aspects of entanglement have been studied by many authors [5]-[13] in the past few years. Here we are interested the effect of symplectic transformations on entanglement in finite multipartite systems. We also consider measurements in this context.

In the next section of this chapter, we give a summary of the thesis through a short overview of each chapter in the text.

1.2 The Overview of the Thesis

The thesis has been organised into six chapters, the first one being this introduction. Chapter 2 gives a brief overview of the basic principles of Quantum Systems and summarizes the basic concepts of quantum information and quantum computation in $\mathbb{R} \times \mathbb{R}$ phase space.

Chapter 3 continues the discussion by formulating the theory of finite quantum systems. There are two main technical points of this chapter. Firstly, to present the Fourier transform in $\mathbb{Z}_d \times \mathbb{Z}_d$ phase space. It is used to define two dual orthonormal bases that is position and momentum, that will be mainly used throughout this thesis. In addition to that, we also discuss the symplectic transformations $Sp(2, \mathbb{Z}_\wp)$. The system operator S depends on three integer parameters(in $\mathbb{Z}_\wp$). Examples show the numerical calculation of the symplectic operator $Sp(2, \mathbb{Z}_\wp)$.

Chapter 4 gives the details of the symplectic $Sp(2\ell, \mathbb{Z}_\wp)$ transformations in general ℓ-partite finite quantum systems in $\mathbb{Z}_\wp \times \mathbb{Z}_\wp$ phase space. Then we present the main numerical technique of that how to calculate the symplectic operators.

Chapter 5 has two parts. In the first, we discuss $Sp(4, \mathbb{Z}_\wp)$ transformations in bi-partite systems. We also study $Sp(6, \mathbb{Z}_\wp)$ transformations in tri-partite systems. We show how factorizable and separable states, after a symplectic transformation become entangled. The entropic quantities show the effect of symplectic transformations on quantum uncertainties, are more complex. In the second part, we give four examples to calculate the generalized measurement and outcomes of measurement.

In Chapter 6 we give some concluding remarks and future lines of research.

Chapter 2

Quantum Systems

Outline:

- Introduction
- Hilbert Space
- Harmonic Oscillator: Phase Space $\mathbb{R} \times \mathbb{R}$
- Entanglement
- Negativity
- Measurement
- Summary

This chapter gives the basic concepts of Quantum systems which will used in forthcoming chapters. The classical definition of the quantum systems could be found in standard textbooks, they are necessary for the Chapters following are introduced and their properties are explored. We will also describe the properties of the entanglement and measurement.

2.1 Introduction

Quantum physics offers correct predictions on the behavior of photons and other elementary particles. The information encoded in quantum systems as a coherent discipline has been studied in 1980's.

2.2 Hilbert Space

In general, the mathematical concept of a Hilbert space was defined by the notion of Euclidean space. It was developed by David Hilbert[1, 14]. It extends the methods of vector algebra from the two-dimensional plane and three-dimensional space to multi-dimensional spaces. In more formal terms, a Hilbert space $\mathcal{H}$ is an inner product space $\langle f, g \rangle$. An inner product combines two vectors and produces a complex number, like

$$|a\rangle = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.1}$$

and,

$$|b\rangle = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \tag{2.2}$$

Now, the inner product can be calculated as:

$$\begin{aligned} \begin{bmatrix} \alpha_1^*, & \ldots, & \alpha_n^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} &= (|a\rangle, |b\rangle) \\ &= \langle a|b\rangle. \end{aligned} \tag{2.3}$$

2.2.1 Eigenvalues and Eigenvectors

Let A be a linear operator and u, $|u\rangle$ a complex number and vector such that

$$A|u\rangle = u|u\rangle \tag{2.4}$$

The complex number u is an eigenvalue of A, and the vector $|u\rangle$ is called an eigenvector of A. If A is an $n \times n$ matrix, there will be n eigenvalues (but some may be the same as others). We can use the *characteristic equation* to find the eigenvalues of A, which is the subspace of the operator A on the vector space.

$$c(\lambda) = \det(\mathrm{A} - \lambda\mathrm{I}) = 0. \tag{2.5}$$

By the fundamental theorem of algebra, each polynomial has at least one complex root. So each operator A has at least one eigenvalue corresponding one eigenvector. A can be written as:

$$A = \sum_i \lambda_i |v_i\rangle\langle v_i| \tag{2.6}$$

The vectors $|v_i\rangle$ form an orthonormal set of eigenvector for operator A, with eigenvalues of λ_i.

2.2.2 Trace

The trace of an matrix A with n-by-n is the sum of its eigenvalues or the sum of the elements on the main diagonal of matrix A, i.e.,

$$\begin{aligned} \mathrm{tr(A)} &= a_{11} + a_{22} + \cdots + a_{nn} \\ &= \sum_{i=1}^{n} a_{ii} \end{aligned} \tag{2.7}$$

2.2.3 Unitary Operators

The matrix U is unitary if:

$$\begin{aligned} U^{-1} &= U^\dagger \\ UU^\dagger &= U^\dagger U = I \end{aligned} \tag{2.8}$$

Unitary operators have the property that they preserve norm, and are invertible.

For our purposes some particularly important operators are the *Pauli operators* which referred to by the letters I, X, Y, and Z. In the case of two-dimensional Hilbert spaces:

$$\begin{aligned} I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{2.9}$$

2.2.4 Linear Operators

A linear operator can be defined as a function $A : V \to W$, where V and W are vector spaces.

$$A(\sum_i a_i|v_i\rangle) = \sum_i a_i A(|v_i\rangle) \tag{2.10}$$

The identity operator $\mathbf{I}_V$ is defined by:

$$\mathbf{I}_V|v\rangle \equiv |v\rangle \tag{2.11}$$

The zero operator maps all vectors to zero vector $\mathbf{0}|v\rangle \equiv \mathbf{0}$. Consider two linear operators $A : V \to W$ and $B : W \to X$. Then

$$(BA)(|v\rangle) \equiv B(A(|v\rangle)) \tag{2.12}$$

Here the notation BA is the composition of B with A.

2.2.5 Tensor Products

The tensor product is a way of putting vector spaces together to form larger vector spaces. It denoted by $\otimes$, combines two smaller vector spaces to a larger one. If $|u\rangle$ and $|v\rangle$ are bases for V and W vector spaces, then $|u\rangle \otimes |v\rangle$ is a basis for $V \otimes W$. The *Kronecker product* of 2×2 matrices is defined as:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} x & y \\ z & v \end{bmatrix} = \begin{bmatrix} a \cdot B & b \cdot B \\ c \cdot B & d \cdot B \end{bmatrix} \\ &= \begin{bmatrix} ax & ay & bx & by \\ az & av & bz & bv \\ cx & cy & dx & dy \\ cz & cv & dz & dv \end{bmatrix} \end{aligned} \tag{2.13}$$

2.2.6 Projectors

An important class of Hermitian operators is the projectors. Suppose W is a subspace of V. W is d-dimensional vector space, V is k-dimensional space. Using the Gram-Schmidt procedure, we consider an orthonormal basis $|1\rangle, \ldots, |k\rangle$ for V, and $|1\rangle, \ldots, |d\rangle$ is an orthonormal basis for $W(d < k)$. The projector onto the subspace W such that

$$\begin{aligned} P &\equiv \sum_{i=1}^{d} |i\rangle\langle i| \\ P^2 &= P \end{aligned} \tag{2.14}$$

From the definition, we know that $|v\rangle\langle v|$ is Hermitian for vector $|v\rangle$, so P also is Hermitian, $P^\dagger = P$.

2.3 Harmonic Oscillator: Phase Space $\mathbb{R} \times \mathbb{R}$

2.3.1 Canonical Commutation Relation

In this section, we consider the phase space methods for quantum particles on a real line $\mathbb{R}$.

In quantum mechanics, we defined a pair of self-adjoint operators $\hat{x}$ and $\hat{p}$

satisfying the canonical commutation relation

$$
\begin{aligned}
[\hat{x}, \hat{p}] &= \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar = i\mathbf{I} \\
[\hat{x}, \hat{x}] &= 0 \\
[\hat{p}, \hat{p}] &= 0
\end{aligned} \tag{2.15}
$$

Here $\hbar$ is the reduced Planck's constant $h/2\pi$, and for simplicity, we use units where $\hbar = \mathbf{I}$. We call $\hat{x}$ the position operator and $\hat{p}$ the momentum operator, which measure the location of the particle on the real line and its momentum respectively. According to $\hat{x}$ and $\hat{p}$ have real continuous spectra, we consider the phase space is a $\mathbb{R} \times \mathbb{R}$ plane. This relation implies the Heisenberg uncertainty principle. We will discuss later.

2.3.2 Position and Momentum Bases

According to the Schrödinger representation of $\hat{x}$ and $\hat{p}$, we now investigate these two operators in more detail. We call $|x\rangle$ and $|p\rangle$ correspondingly the position and momentum eigenstates, they satisfying

$$
\begin{aligned}
\hat{x}|x\rangle &= x|x\rangle \\
\hat{p}|p\rangle &= p|p\rangle
\end{aligned} \tag{2.16}
$$

where $x, p \in \mathbb{R}$.

$|x\rangle$ and $|p\rangle$ are both orthogonal(but not normalizable) bases for the Hilbert space $L^2(\mathbb{R})$ satisfying

$$\begin{aligned} \langle x|x'\rangle &= \delta(x-x') \\ \langle p|p'\rangle &= \delta(p-p') \end{aligned} \tag{2.17}$$

Here $\delta(x)$ is the Dirac delta function; and the eigenstates satisfy the useful relation

$$\int_{-\infty}^{\infty} |x\rangle\langle x|dx = \int_{-\infty}^{\infty} |p\rangle\langle p|dp = \mathbf{1} \tag{2.18}$$

We consider the Fourier transformation for each eigenstate as

$$\begin{aligned} |x\rangle &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp(-ixp)|p\rangle dp \\ |p\rangle &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp(ixp)|x\rangle dx \end{aligned} \tag{2.19}$$

Then we use the property of the delta function

$$\delta(x-x') = 1/2\pi\int_{-\infty}^{\infty} \exp[-i(x-x')p]dp \tag{2.20}$$

to prove that

$$\langle x|p\rangle = (2\pi)^{-1/2}\exp(ixp) \tag{2.21}$$

2.3.3 Fourier Transformation

The position and momentum wave function of an arbitrary pure state $|\psi\rangle$ can be written as:

$$\begin{aligned} \psi(x) &= \langle x|\psi\rangle \\ \psi(p) &= \langle p|\psi\rangle \end{aligned} \tag{2.22}$$

These functions are the position and momentum representation of the state $|\psi\rangle$. It is easily to see that they are related to each other through the Fourier transform

$$\psi(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-ixp)\psi(x)dx \tag{2.23}$$

According to equation(2.23), the Fourier transformation can be represented by an unitary operator F called Fourier operator and defined as

$$F = \int_{-\infty}^{\infty} |\xi\rangle_{p\,x}\langle\xi| d\xi \tag{2.24}$$

Here $|\xi\rangle_x$ and $|\xi\rangle_p$ are the position and momentum eigenstate with the eigenvalue ξ. We can show that,

$$\begin{aligned} F|\xi\rangle_x = |\xi\rangle_p \qquad & F|\xi\rangle_p = |-\xi\rangle_x; \\ F^\dagger \hat{x} F = \hat{p} \qquad & F^\dagger \hat{p} F = -\hat{x} \end{aligned} \tag{2.25}$$

and also that,

$$F^4 = \mathbf{1} \tag{2.26}$$

2.3.4 Uncertainty Principle

The commutation relation implies the Heisenberg uncertainty principle. If a measurement of position is made with accuracy Δx and a measurement of momentum is made simultaneously with accuracy Δp, we define

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \tag{2.27}$$

The position and momentum of a particle cannot be simultaneously measured with arbitrarily high precision. There is a minimum for the product of the uncertainties of these two measurements.

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \tag{2.28}$$

2.3.5 Displacement in Phase Space $\mathbb{R} \times \mathbb{R}$

We can show that

$$\exp(-i\mathcal{X}\hat{p})|x\rangle = |x + \mathcal{X}\rangle$$

$$\exp(i\mathcal{P}\hat{x})|p\rangle = |p+\mathcal{P}\rangle \qquad \mathcal{X},\mathcal{P}\in\mathbb{R} \tag{2.29}$$

We call the unitary operators $\exp(-i\mathcal{X}\hat{p})$ and $\exp(i\mathcal{P}\hat{x})$, displacement operators corresponding to position and momentum. The general displacement operator is defined as:

$$\hat{D}(\mathcal{X},\mathcal{P}) \equiv \exp(i\mathcal{P}\hat{x} - i\mathcal{X}\hat{p}) \tag{2.30}$$

The Baker-hausdorff formula(equation2.31) gives:

$$\begin{aligned} \exp(A+B) &= \exp(-\frac{1}{2}[A,B])\exp(A)\exp(B) \\ &= \exp(\frac{1}{2}[A,B])\exp(B)\exp(A) \end{aligned} \tag{2.31}$$

for the case that $\Big[A,[A,B]\Big] = \Big[B,[A,B]\Big] = 0$. This fundamental operator relation is proven for instance in[15]. Therefore,

$$\begin{aligned} \hat{D}(\mathcal{X},\mathcal{P}) &= \exp(\frac{1}{2}i\mathcal{X}\mathcal{P})\exp(-i\mathcal{X}\hat{p})\exp(i\mathcal{P}\hat{x}) \\ &= \exp(-\frac{1}{2}i\mathcal{X}\mathcal{P})\exp(i\mathcal{P}\hat{x})\exp(-i\mathcal{X}\hat{p}) \end{aligned} \tag{2.32}$$

We can show the following properties

$$\hat{D}^{\dagger}(\mathcal{X},\mathcal{P}) = \hat{D}(-\mathcal{X},-\mathcal{P}) \tag{2.33}$$

2.3.6 Heisenberg-Weyl Group

We can show that

$$\begin{aligned}\hat{D}(\mathcal{X}_1, \mathcal{P}_1)\hat{D}(\mathcal{X}_2, \mathcal{P}_2) &= \hat{D}(\mathcal{X}_1 + \mathcal{X}_2, \mathcal{P}_1 + \mathcal{P}_2) \exp\left[\frac{1}{2}i(\mathcal{P}_1\mathcal{X}_2 - \mathcal{P}_2\mathcal{X}_1)\right] \\ & \mathcal{X}_1, \mathcal{X}_2, \mathcal{P}_1, \mathcal{P}_2 \in \mathbb{R}. \end{aligned} \tag{2.34}$$

From this we can prove that these operators form a group, known as the Heisenberg-Weyl group.

2.4 Entanglement

In the rapidly growing field of quantum information processing, quantum entanglement research has been an active research topic. It[16, 17, 18] is considered the basic concept of quantum physics today.

Quantum entanglement plays an important role in the field of quantum information and quantum computing. It has been widely used in quantum information in all aspects as sending quantum information and processing resources, for instance, quantum teleportation, quantum cryptography, and so on.

In-depth study of the quantum entanglement will provide the basic understanding of the theory of quantum information and quantum mechanics, and contribute to the potential applications of quantum entanglement exploitation.

Entanglement is a fundamental feature of quantum theory. As stated in [19], entanglement is one of the most puzzling properties of quantum physics. The behavior of entangled states has been demonstrated in numerous experiments [20, 21].

The entangled states are widely applied in quantum computation, teleportation etc. In this section and after we will discuss the basic concepts of the entanglement.

2.4.1 Density Operator

The density operator or density matrix is a powerful tool to represent the quantum mechanics. We consider a quantum system in states $|\psi_i\rangle$ with probabilities p_i, where the coefficients p_i are non-negative and add up to one.

$$\begin{aligned} \rho &= \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad 0 \le p_i \le 1 \\ \sum p_i &= 1 \end{aligned} \tag{2.35}$$

In practice, the terms "density matrix" and "density operator" are often used interchangeably. The density operator represent a pure state if it is a rank one projection $\rho = |\psi\rangle\langle\psi|$, otherwise ρ is a mixed state. Equivalently, a density operator ρ is a pure state if and only if

$$\mathrm{tr}(\rho^2) = 1 \tag{2.36}$$

and, a density operator ρ is a mixed state if,

$$\mathrm{tr}(\rho^2) < 1 \tag{2.37}$$

The density operator has two properties:

- $\mathrm{tr}(\rho) = 1$;
- ρ is a non-negative matrix (i.e. it has non-negative eigenvalues).

2.4.2 Reduced Density Operator

Reduced density operators are very important. They are an indispensable tool on the analysis of complex quantum systems. The density matrix can describe any subsystem of a larger quantum system, including mixed subsystems.

Subsystems are described by a reduced density matrix. We consider a quantum mechanical system whose state space is the tensor product of Hilbert spaces, corresponding two subsystem A and B. The overall density matrix is referred to as ρ_{AB}. The partial trace of ρ_{AB} with respect to the system B is denoted by ρ_A and is called the reduced state of ρ_{AB} on system A.

Analogously, ρ_B is called the reduced state of ρ_{AB} on system B. In symbols,

$$\begin{aligned}\rho_A &= \mathrm{tr_B}(\rho_{\mathrm{AB}}) \\ \rho_B &= \mathrm{tr_A}(\rho_{\mathrm{AB}}) \end{aligned} \tag{2.38}$$

$\mathrm{tr_A}$ and $\mathrm{tr_B}$ are called partial traces over subsystems A and B respectively. The partial trace is defined as follows:

$$\begin{aligned}\rho_A &= \mathrm{tr_B}(|\alpha_1\rangle\langle\alpha_2| \otimes |\beta_1\rangle\langle\beta_2|) \\ &= |\alpha_1\rangle\langle\alpha_2|\mathrm{tr}(|\beta_1\rangle\langle\beta_2|) \\ &= \langle\beta_1|\beta_2\rangle|\alpha_1\rangle\langle\alpha_2| \end{aligned} \tag{2.39}$$

2.4.3 Von Neumann Entropy

The comprehensive mathematical formalism of quantum mechanics was first presented in [22] by Johann von Neumann. The von Neumann entropy is the quantum information-theoretic analogue of the Shannon entropy.

Let ρ be a density matrix, the von Neumann entropy of ρ is defined as:

$$S(\rho) = -\mathrm{tr}(\rho \log \rho) \tag{2.40}$$

Here, $\log \rho$ is the Hermitian operator that has exactly the same eigenvectors as ρ, and we take logarithm of the corresponding eigenvalues. $\log \rho$ is only defined for ρ positive definite (implying all eigenvalues are positive).

However, $\rho \log \rho$ may be defined for all positive semidefinite ρ by interpreting $0 \log 0$ as 0. For $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$ (equation (2.35)) we get:

$$S(\rho) = -\sum_i p_i \log p_i \tag{2.41}$$

2.4.3.1 Joint von Neumann Entropy

The joint von Neumann entropy is a measure the total uncertainty or entropy of the joint system, A and B is two subsystems, the terms joint entropy simply refers to the von Neumann entropy of the combined system. This is to distinguish from the entropy of the subsystems. In symbols, the reduced state of the pair (A, B) is $\rho \in (H_A \otimes H_B)$, then

$$S(A, B) = S(\rho_{AB}) = -\mathrm{tr}(\rho_{\mathrm{AB}} \log \rho_{\mathrm{AB}}) \tag{2.42}$$

2.4.3.2 Conditional Entropies

The conditional entropy $I(A|B)$ is:

$$I(A|B) = S(A, B) - S(B)$$

$$S(B) = -\mathrm{Tr}(\rho_B \log \rho_B) \tag{2.43}$$

The negative values of $I(A|B)$ mean that it is acceptable, in quantum information theory, to have $S(A, B) < S(B)$, i.e., the entropy of the entire system A and B can be smaller than the entropy of one of its subparts B. This is not possible in classical physics. This occurs in the case of quantum entanglement between the two subsystems (but not sufficient). The non-negativity of conditional entropies is a necessary condition for separability [23, 24, 25].

2.4.3.3 Mutual Entropies

The mutual entropies written as $I(A, B)$ is:

$$I(A, B) = S(A) + S(B) - S(A, B) \tag{2.44}$$

$$= S(A) - I(A|B) \tag{2.45}$$

$$= S(B) - I(B|A) \tag{2.46}$$

When $I(A, B)$ is positive, there are correlations (classical and quantum) between the two subsystems A and B.

2.4.4 Linear Entropy

The linear entropy is based on the purity of a state $\mathrm{P} \equiv \mathrm{tr}(\rho^2)$, which is a measure of how pure a quantum state is. The purity is one for pure states. It ranges from 1 (for a pure state) to $1/d$ for maximally mixed states of dimension d. The linear entropy is a measure of mixedness in quantum states. It is defined as:

$$E = 1 - \mathrm{tr}(\rho^2) \tag{2.47}$$

The linear entropy ranges from 0 (for a pure state) to 1 (for a maximally mixed state).

2.5 Negativity

Negativity presented a measure of entanglement, which is defined by Vidal and Werner [27, 28]. It is an entanglement monotone so it does not change under local operations and classical communications (LOCC). It is based on the partial transpose with ρ^T, and related to the trace norm of partial transpose.

2.5.1 Partial Transpose

The partial transpose is a linear algebraic concept, which can be interpreted as a simple generalization of the usual matrix transpose. For this to be possible, the density matrix ρ has to be separable into a sum of direct products,

$$\rho = \sum_A \omega_A \rho'_A \otimes \rho''_A \tag{2.48}$$

Here the positive weights ω_A satisfy $\sum \omega_A = 1$ (they are probabilities), and ρ'_A and ρ''_A are density matrices for the two subsystems. Equation (2.48) can also be written as:

$$\rho_{m\mu,n\nu} = \sum_A \omega_A (\rho'_A)_{mn} (\rho''_A)_{\mu\nu} \tag{2.49}$$

The partial transposition of ρ is given by:

$$\rho^T_{m\mu,n\nu} \equiv \rho_{m\nu,n\mu} = \sum_A \omega_A (\rho'_A)^T \otimes \rho''_A \tag{2.50}$$

2.5.2 Negativity

The negativity can be computed efficiently, and the negativity does not increase under local manipulations of the system. The negativity of a state ρ is defined as:

$$\mathcal{N}(\rho) = \frac{\|\rho^T\| - 1}{2} \tag{2.51}$$

where the trace norm

$$\|A\| = \mathrm{tr}|A| = \mathrm{tr}[(A^\dagger A)^{1/2}] \tag{2.52}$$

then using equation 2.52 to prove equation 2.51 gives:

$$\mathcal{N}(\rho) = \frac{\mathrm{tr}[(\rho^\dagger \rho)^{1/2}] - 1}{2} \tag{2.53}$$

2.6 Measurement

Let $|f\rangle, |g\rangle$ be two orthogonal states. We consider the superposition $\alpha|f\rangle + \beta|g\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. Then we perform a measurement with the orthonormal basis states $|f\rangle, |g\rangle$. We get the result f with probability $|\alpha|^2$ and g with probability $|\beta|^2$. Below we generalize this.

Firstly, we consider a set $\{M_m\}$ of measurement operators. The index number m refers to the measurement outcomes. The measurement operators satisfy the completeness equation $\sum_m M_m^\dagger M_m = \mathbf{I}$ and the orthogonality relation $M_m M_n = M_m \delta_{mn}$. If we perform a measurement, we get the result m with probability.

$$pro(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{2.54}$$

The completeness equation expresses the fact that probabilities sum to one.

$$\sum_m pro(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \mathbf{1} \tag{2.55}$$

The state $|\psi\rangle$ after the measurement is

$$\frac{M_m|\psi\rangle}{(\langle\psi|M_m^\dagger M_m|\psi\rangle)^{1/2}} \tag{2.56}$$

An example is the measurement of a qutrit. The measurement operators defined by $M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|, M_2 = |2\rangle\langle 2|$. The measurement operators are Hermitian and $M_m M_n = M_m \delta_{mn}$. Thus the completeness relations is

$$M_0 + M_1 + M_2 = \mathbf{I} \tag{2.57}$$

We consider the state $|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle$ to be measured. Then the probability of the outcome 0 is:

$$\begin{aligned} pro(0) &= \langle\psi|M_0^\dagger M_0|\psi\rangle \\ &= \langle\psi|M_0|\psi\rangle = |a|^2 \end{aligned} \tag{2.58}$$

Similarly, the probability of the outcome 1 is $pro(1) = |b|^2$, and of the outcome 2 is $pro(2) = |c|^2$. The state after the measurement M_0 becomes $|0\rangle$(and similarly for M_1, M_2).

2.6.1 Measurements on Finite Dimensional Hilbert Space

Our measurements are performed upon a finite dimensional quantum system. We consider a set of orthogonal projectors onto subspaces $\pi_0, \pi_1, \ldots, \pi_{d-1}$, with the property that:

$$\begin{aligned} \sum_{i=0}^{d-1} \pi_i &= \mathbf{1} \\ \pi_i \pi_j &= \pi_i \delta_{ij} \end{aligned} \tag{2.59}$$

π_i are Hermitian matrices, **1** is the identity matrix and d is the dimensionality of Hilbert space. The outcome probabilities Pro_i [32, 33, 34] are.

$$Pro_i = \mathrm{Tr}(\rho\pi_i) \tag{2.60}$$

If the result of the measurement is i, the state after the measurement is

$$\rho_i^{mea} = \frac{\pi_i\rho\pi_i}{\mathrm{Tr}(\rho\pi_i)} \tag{2.61}$$

If we know that the measurement has been performed, but we do not know the result, the state is described by the density matrices.

$$\begin{aligned}\widetilde{\rho} &= \sum_{i=0}^{d-1} \rho_i^{mea}\mathrm{Tr}(\rho\pi_i) \\ &= \sum_{i=0}^{d-1} \frac{\pi_i\rho\pi_i}{\mathrm{Tr}(\rho\pi_i)}\mathrm{Tr}(\rho\pi_i) \\ &= \sum_{i=0}^{d-1} \pi_i\rho\pi_i \end{aligned} \tag{2.62}$$

2.7 Summary

In this chapter we have tried to present basic formulations of quantum systems. From those definitions, we has to deal with: what is the Hilbert space, how to represent physical (pure and mixed) states, how to represent Von Neumann entropy and Negativity and the probability of measurement outcomes. In the next chapter, we therefore give a brief overview of the formal structure of finite quantum system.

Chapter 3

Finite quantum systems

Outline:

- Introduction
- Finite Hilbert Space $L^2(\mathbb{Z}_d)$
- Momentum and Position Bases
- Displacements in Phase Space $\mathbb{Z}_d \times \mathbb{Z}_d$
- Symplectic Transformations
- Summary

In this chapter, we present the basic principles of finite quantum systems. The first section is the introduction. In section 3.2, we review the notion of Fourier transform. Section 3.3, we present the two dual orthonormal bases 'position' and 'momentum'. In section 3.4, we present the formalism of the displacement operators in finite phase space and the Heisenberg-Weyl group for finite quantum systems.

We also discuss the uncertainty principle for the harmonic oscillator. It is presented quantitatively with the entropic uncertainty relation in finite systems. Finally, in the last section of this chapter, we formally define the notation of the symplectic operator S depends on three independent parameters k, λ, μ (in $\mathbb{Z}_{\wp}$).

3.1 Introduction

A quantum system is always associated with a Hilbert space, where a quantum state is represented by a linear vector. A Quantum system on a real line is associated with the Hilbert space $L^2(\mathbb{R})$. This is infinite dimensional Hilbert spaces. In this chapter, we study the quantum system with finite d-dimensional Hilbert space $L^2(\mathbb{Z}_d)$, which is the basic platform of quantum computation[35]. A review of quantum systems with finite Hilbert space has been presented in [36]. In this case, we consider a quantum system with a $d-$dimensional Hilbert space $\mathcal{H}$. An orthonormal basis is the position states in this system, denoted as $|m\rangle_{\mathcal{X}}$. Here, m belong to $\mathbb{Z}_d$ (the integers modulo

d), and then the corresponding phase space is $\mathbb{Z}_d \times \mathbb{Z}_d$. We have to mention that, when the dimension d of the Hilbert space is an odd prime number $\wp$, $\mathbb{Z}_\wp$ is a field. In this case, besides the Heisenberg-Weyl group, a symplectic transformations are also well defined by ([36],[37],[38],[39],[40]).

3.2 Finite Hilbert Space $L^2(\mathbb{Z}_d)$

3.2.1 Fourier Transform

The position states $|m\rangle_{\mathcal{X}}$ obey the following relations:

$$ {}_{\mathcal{X}}\langle m|n\rangle_{\mathcal{X}} = \delta(m,n) \tag{3.1} $$

$$ \sum_m |m\rangle_{\mathcal{X}\mathcal{X}}\langle m| = 1 \tag{3.2} $$

Here $\delta(m,n)$ is the Kronecker delta:

$$ \delta(m,n) = \begin{cases} 1, & if \;\; m = n \\ 0, & if \;\; m \neq n \end{cases} \tag{3.3} $$

The finite Fourier operator acting on the position states is,

$$ \mathcal{F} = d^{-1/2} \sum_{m,n=0}^{d-1} \omega(mn)|m\rangle_{\mathcal{X}\mathcal{X}}\langle n| \tag{3.4} $$

We use the notation

$$\omega(\alpha) \equiv \omega^{\alpha} = \exp\left[i\frac{2\pi\alpha}{d}\right]; \qquad \alpha \in \mathcal{Z}_d \tag{3.5}$$

An identity that is easily proved is,

$$\frac{1}{d}\sum_{n=0}^{d-1} \omega^{n(m-\ell)} = \delta(m,\ell) \tag{3.6}$$

Where $m, \ell \in \mathbb{Z}_d$. Using it, we proved that:

$$\begin{aligned} \mathcal{F}\mathcal{F}^{\dagger} &= \mathcal{F}^{\dagger}\mathcal{F} = 1 \\ \mathcal{F}^{4} &= 1 \end{aligned} \tag{3.7}$$

The fact that $\mathcal{F}^4 = 1$ shows that the Fourier operator has four eigenvalues, which are 1, -1, i, $-i$. The multiplicity of the eigenvalues, is given in Table.(3.1) [41]. The table shows that when $d = 4m$ then $\mathrm{Tr}\mathcal{F} = 1 + \mathrm{i}$; when $d = 4m+1$ then $\mathrm{Tr}\mathcal{F} = 1$; when $d = 4m+2$ then $\mathrm{Tr}\mathcal{F} = 0$; when $d = 4m+3$ then $\mathrm{Tr}\mathcal{F} = \mathrm{i}$.

Table 3.1: The Fourier Operator's Multiplicity of Eigenvalues in d-dimensional Hilbert Space. In This Table, There Have 4 Cases: $d = 4m$, $d = 4m+1$, $d = 4m+2$, $d = 4m+3$.[41]

	1	-1	i	$-i$
$d = 4m$	$m+1$	m	m	$m-1$
$d = 4m+1$	$m+1$	m	m	m
$d = 4m+2$	$m+1$	$m+1$	m	m
$d = 4m+3$	$m+1$	$m+1$	$m+1$	m

The Fourier operator is a very important tool used in signal analysis and fast Fourier transform community. In this thesis, we use the Fourier operator to the theory of finite quantum systems.

3.3 Momentum and Position Bases

Acting with the Fourier operator on position states we get the orthonormal basis "momentum states":

$$\begin{aligned} |m\rangle_{\mathcal{P}} &= \mathcal{F}|m\rangle_{\mathcal{X}} \\ &= d^{-1/2}\sum_{n}\omega^{mn}|n\rangle_{\mathcal{X}} \end{aligned} \tag{3.8}$$

The momentum states $|m\rangle_{\mathcal{P}}$ also obey the relations:

$$_{\mathcal{P}}\langle m|n\rangle_{\mathcal{P}} = \delta(m,n) \tag{3.9}$$

$$\sum_{m}|m\rangle_{\mathcal{P}\,\mathcal{P}}\langle m| = 1 \tag{3.10}$$

Similarly, a finite Fourier operator(equation3.4) also represent as:

$$\mathcal{F} = \sum_{n=0}^{d-1}|n\rangle_{\mathcal{P}\,\mathcal{X}}\langle n| \tag{3.11}$$

Obviously, the Fourier operator satisfies

$$\mathcal{F}|m\rangle_{\mathcal{P}} = |-m\rangle_{\mathcal{X}} \tag{3.12}$$

$|m\rangle_{\mathcal{X}}$ and $|m\rangle_{\mathcal{P}}$ are the eigenstates of the position and momentum operators, which are given by:

$$\begin{aligned}
\hat{x} &= \sum_{n=0}^{d-1} n|n\rangle_{\mathcal{X}\mathcal{X}}\langle n| \\
\hat{p} &= \sum_{n=0}^{d-1} n|n\rangle_{\mathcal{P}\mathcal{P}}\langle n| \\
\hat{p} &= \mathcal{F}\hat{x}\mathcal{F}^{\dagger} \\
-\hat{x} &= \mathcal{F}\hat{p}\mathcal{F}^{\dagger} \qquad (3.13)
\end{aligned}$$

Using the position and momentum bases, we can expand the arbitrary state $|s\rangle$ in Hilbert space $L^2(\mathbb{Z}_d)$ as:

$$\begin{aligned}
|s\rangle &= \sum_{n} \lambda_n |n\rangle_{\mathcal{X}} \\
&= \sum_{m} \mu_m |m\rangle_{\mathcal{P}} \qquad (3.14)
\end{aligned}$$

Then,

$$\lambda_n = d^{-1/2} \sum_{m} \mu_m \omega_{mn} \qquad (3.15)$$

3.4 Displacement in Phase Space $\mathbb{Z}_d \times \mathbb{Z}_d$

3.4.1 Displacement Operators

In section2.3.5, we briefly reviewed the Baker-hausdorff formula(equation2.31). Cause of they are related to commutation relation(equation2.15). It is possible to prove with the operators $\exp(iA\mathcal{X}), \exp(iB\mathcal{P})$ in the harmonic oscillator.

$$\exp(iA\mathcal{X})\exp(iB\mathcal{P}) = \exp(iB\mathcal{P})\exp(iA\mathcal{X})\exp(-iAB) \tag{3.16}$$

In the case of finite quantum systems the phase-space is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$. The displacement operators are defined as

$$\begin{aligned} Z &= \exp\left[i\frac{2\pi}{d}x\right] = \exp\left[i\frac{2\pi}{d}n\right]|n\rangle_{\mathcal{X}\mathcal{X}}\langle n| \\ &= \omega(n)|n\rangle_{\mathcal{X}\mathcal{X}}\langle n| \\ X &= \exp\left[-i\frac{2\pi}{d}p\right] = \exp\left[-i\frac{2\pi}{d}n\right]|n\rangle_{\mathcal{P}\mathcal{P}}\langle n| \\ &= \omega(-n)|n\rangle_{\mathcal{P}\mathcal{P}}\langle n| \end{aligned} \tag{3.17}$$

They perform displacements the operators $\mathcal{P}$ and $\mathcal{X}$ in the $\mathbb{Z}_d \times \mathbb{Z}_d$ phase space

$$Z^\alpha|m\rangle_\mathcal{P} = |m+\alpha\rangle_\mathcal{P}, \qquad Z^\alpha|m\rangle_\mathcal{X} = \omega(\alpha m)|m\rangle_\mathcal{X}$$

$$X^{\beta}|m\rangle_{\mathcal{P}} = \omega(-m\beta)|m\rangle_{\mathcal{P}}, \qquad X^{\beta}|m\rangle_{\mathcal{X}} = |m+\beta\rangle_{\mathcal{X}} \tag{3.18}$$

and they obey the relations

$$X^d = Z^d = 1 \tag{3.19}$$

$$X^{\beta}Z^{\alpha} = Z^{\alpha}X^{\beta}\omega^{-\alpha\beta}, \qquad \alpha, \beta \in \mathcal{Z}_d \tag{3.20}$$

where α, β are integers in $\mathbb{Z}_d$. The equation(3.19) shows the toroidal nature of the phase-space.

3.4.2 Heisenberg-Weyl Group

The structure of the canonical commutation relations 2.15 is related to a group, the so-called Heisenberg-Weyl group [42]-[50]. In the case of finite quantum systems, the following equation introduces the general displacement operators

$$\begin{aligned} D(\alpha,\beta) &= Z^{\alpha}X^{\beta}\omega(-2^{-1}\alpha\beta) \\ &= X^{\beta}Z^{\alpha}\omega(2^{-1}\alpha\beta) \end{aligned} \tag{3.21}$$

which have the following propertices:

$$\begin{aligned} \Big[D(\alpha,\beta)\Big]^{\dagger} &= D(-\alpha,-\beta) \\ D(\alpha,\beta)^{\dagger}D(\alpha,\beta) &= \mathbf{I} \end{aligned} \tag{3.22}$$

We calculate that

$$\begin{aligned} D(\alpha,\beta)|m\rangle_{\mathcal{X}} &= \omega(2^{-1}\alpha\beta+\alpha m)|m+\beta\rangle_{\mathcal{X}} \\ D(\alpha,\beta)|m\rangle_{\mathcal{P}} &= \omega(-2^{-1}\alpha\beta-\beta m)|m+\alpha\rangle_{\mathcal{P}} \end{aligned} \tag{3.23}$$

Using equation 3.19 and 3.20 we can prove the multiplication rule

$$\begin{aligned} D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) &= D(\alpha_1+\alpha_2,\beta_1+\beta_2)\omega\left[2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1)\right] \\ &= \exp\left[i(\alpha_2\beta_1-\alpha_1\beta_2)\right]D(\alpha_2,\beta_2)D(\alpha_1,\beta_1) \end{aligned} \tag{3.24}$$

These displacement operators form a discrete Heisenberg-Weyl group, since they have integer parameters and satisfy the relation(equation(3.24)).

3.4.3 Entropic Uncertainty Relations for Finite Quantum Systems

In the case of finite quantum systems, let λ_n and λ_m corresponding the position and momentum representations in a d-dimensional Hilbert space. For the states $|s\rangle$ of equation (3.14) with these two distributions, the entropies be defined as

$$\begin{aligned} S_{\mathcal{X}} &= -\sum_n |\lambda_n|^2 \ln|\lambda_n|^2 \\ S_{\mathcal{P}} &= -\sum_m |\mu_m|^2 \ln|\mu_m|^2 \end{aligned} \tag{3.25}$$

References [51]-[57] have proved the entropic uncertainty relation

$$S_{\mathcal{X}} + S_{\mathcal{P}} \geq \ln d \tag{3.26}$$

In finite quantum systems. Following the equation (3.25), we note that for the state $|n\rangle_{\mathcal{X}}$, we get $S_{\mathcal{X}} = 0$ and $S_{\mathcal{P}} = \ln d$. For the state $|m\rangle_{\mathcal{P}}$ we get $S_{\mathcal{X}} = \ln d$ and $S_{\mathcal{P}} = 0$.

3.5 Symplectic Transformations

In section 3.2, the position-momentum phase space is the toroidal lattice $\mathbb{Z}_d \times \mathbb{Z}_d$ and the Hilbert space is a d-dimensional. Below we consider the case that d is an odd prime number $\wp$. In this case $\mathbb{Z}_\wp$ is a field and all its nonzero elements have an inverse. Consequently, these systems have stronger properties. Below we study the symplectic transformations $Sp(2, \mathbb{Z}_\wp)$.

3.5.1 The Bogoliubov Transformation for the Harmonic Oscillator

The Bogoliubov transformations is a unitary transformation from a unitary representation of some canonical commutation relation algebra into another unitary representation. A new pair of operators is defined as

$$\mathcal{X}' \quad = \quad k\mathcal{X} + \lambda\mathcal{P}$$

$$\mathcal{P}' = \mu\mathcal{X} + \nu\mathcal{P} \qquad (k, \lambda, \mu, \nu \in \mathbb{R}) \tag{3.27}$$

where $k\nu - \lambda\mu = 1$, and the commutation relation is:

$$[\mathcal{X}', \mathcal{P}'] = [\mathcal{X}, \mathcal{P}] = i\mathbf{1} \tag{3.28}$$

These equations show us they are associated with the symplectic group $Sp(2, R)$ [58]-[61]. In section 3.4, we have given the relations in terms of the displacement in phase space of the position and momentum operator $\mathcal{X}$ and $\mathcal{P}$. They can also be presented in terms of the displacements operators as:

$$\begin{aligned} \exp(i\mathcal{X}') &= \exp\left[i(k\mathcal{X} + \lambda\mathcal{P})\right] \\ &= \exp(ik\mathcal{X})\exp(i\lambda\mathcal{P})\exp(ik\lambda/2) \\ \exp(i\mathcal{P}') &= \exp\left[i(\mu\mathcal{X} + \nu\mathcal{P})\right] \\ &= \exp(i\mu\mathcal{X})\exp(i\nu\mathcal{P})\exp(i\mu\nu/2) \end{aligned} \tag{3.29}$$

These relations with $k\nu - \lambda\mu = 1$, preserve equation(3.16).

3.5.2 Symplectic $Sp(2, \mathbb{Z}_\wp)$ Transformations

In $\mathbb{Z}_\wp \times \mathbb{Z}_\wp$ phase space, the symplectic transformations are given by [62, 63],

$$X' = SXS^\dagger = X^k Z^\lambda \omega(2^{-1}k\lambda) = D(\lambda, k)$$

$$Z' = SZS^\dagger = X^\mu Z^\nu \omega(2^{-1}\mu\nu) = D(\nu,\mu)$$

$$k\nu - \lambda\mu = 1(\mathrm{mod}(\wp)) \tag{3.30}$$

These transformations ensure equations (3.19) and (3.20) are preserved. The k, λ, μ, ν are integers in $\mathbb{Z}_\wp$, and equation(3.30) shows that there three independent parameters k, λ, μ.

Existence of 'inverses' guarantees that $\nu = k^{-1}(\lambda\mu + 1)$. It is important here that if the variables belong to a ring, in general we can not solve the constraint of equation(3.30). For this reason we require $\wp$ to be a prime number and then $\mathbb{Z}_\wp$ is a field. X and Z are the displacement operators.

3.5.3 Numerical Calculation of the Symplectic Operator $S(k, \lambda, \mu)$

After the introduction of the symplectic transformations $Sp(2, \mathbb{Z}\wp)$, we calculate the operator $S(k, \lambda, \mu)$. Firstly, we calculate the matrix Z' numerically. Its eigenvectors are $|m\rangle_{\mathcal{X}'}$

$$Z'|m\rangle_{\mathcal{X}'} = \omega(m)|m\rangle_{\mathcal{X}'} \tag{3.31}$$

They are defined here, up to a phase factor. To calculate the phases, we note that these eigenvectors must obey the following relations,

$$X^\beta|m\rangle_{\mathcal{X}} = |m+\lambda\rangle_{\mathcal{X}} \tag{3.32}$$

We start from the lowest state $|0\rangle_{\mathcal{X}}$, and we use numerically the equation

$$(X')^m|0\rangle_{\mathcal{X}'} = |m\rangle_{\mathcal{X}'} \tag{3.33}$$

The equation(3.33) gave us a new state $|m\rangle_{\mathcal{X}'}$. So now, we can get the symplectic operator $S(n,m)$ use the equation:

$$S(n,m) \equiv {}_{\mathcal{X}}\langle n|m\rangle_{\mathcal{X}'} \tag{3.34}$$

3.5.4 An Example of Symplectic Operator

We consider a three-dimensional Hilbert space ($d = 3$) and calculate the $S(1,1,-1)$. According to equation(3.30), $\nu = 0$, so

$$\begin{aligned} X' &= SXS^\dagger = XZ\omega(1/2) \\ Z' &= SZS^\dagger = X^{-1} \end{aligned} \tag{3.35}$$

Following the section 3.5.3 about the numerical calculation, we find

$$S(1,1,-1) = \begin{pmatrix} -z_2 & z_2^* & -z_2 \\ -z_1 & z_1 & -z_2 \\ -z_2^* & z_2 & -z_2 \end{pmatrix} \tag{3.36}$$

After the numerical calculation, matrix(3.36) is the symplectic operator $S(1, 1, -1)$ for equation(3.35). Here $z_1 = 0.5774$, $z_2 = z_1\omega^{-1}$ and $\omega = \exp[i2\pi/3]$.

3.6 Summary

In this chapter we brief review the properties of the finite quantum systems. We has to deal with the finite Hilbert space, momentum and position bases, displacements in phase space $\mathbb{Z}_d \times \mathbb{Z}_d$. Finally, we give a numerical calculation of symplectic operator $S(k, \lambda, \mu)$. Next chapter, we will present the symplectic operator $Sp(2\ell, \mathbb{Z}_\wp)$ in multi-mode finite systems.

Chapter 4

Symplectic $Sp(2\ell, \mathbb{Z}_{\wp})$ Transformations in ℓ-partite Finite System

Outline:

- Introduction
- Multi-mode Finite Quantum Systems
- Symplectic $Sp(2\ell, \mathbb{Z}_{\wp})$ Transformaitons
- Numerical Calculation of the Symplectic Operator S in Multi-partite Systems
- Summary

In the previous chapter, through the analysis of the finite quantum systems, we have introduced the symplectic operator $Sp(2, \mathbb{Z}_\wp)$. In this chapter, we present the numerical calculation of the symplectic operator $Sp(2\ell, \mathbb{Z}_\wp)$ in multipartite systems. Although the analysis of two parts finite quantum system with Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we present the multi-mode(ℓ-partite) finite quantum systems with $\wp^\ell$-dimensional Hilbert space in section 4.2. In section 4.3, we study the independent parameters of $Sp(2\ell, \mathbb{Z}_\wp)$. Finally, we present the numerical calculation of the symplectic operator S in ℓ-partite systems.

4.1 Introduction

We consider a finite quantum system composed of two parts with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Therefore the Hilbert space of the total system is $\mathcal{H}_1 \otimes \mathcal{H}_2$. We use an index i $(i = 1, 2)$ to indicate the system,

$$\begin{aligned} Z_1 &= Z \otimes \mathbf{1} \\ Z_2 &= \mathbf{1} \otimes Z \end{aligned} \tag{4.1}$$

Also the position states $|n\rangle_{X_1}$ and $|n\rangle_{X_2}$ belong to the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Then

$$\begin{aligned} Z_i|m_1, m_2\rangle_{\mathcal{X}} &= \omega(m_i)|m_1, m_2\rangle_{\mathcal{X}} \\ |m_1, m_2\rangle_{\mathcal{X}} &\equiv |m_1\rangle_{\mathcal{X}_1}|m_2\rangle_{\mathcal{X}_2} \end{aligned} \tag{4.2}$$

Similarly:

$$\begin{aligned} X_i|m_1, m_2\rangle_{\mathcal{P}} &= \omega(-m_i)|m_1, m_2\rangle_{\mathcal{P}} \\ |m_1, m_2\rangle_{\mathcal{P}} &\equiv |m_1\rangle_{\mathcal{P}_1}|m_2\rangle_{\mathcal{P}_2} \end{aligned} \tag{4.3}$$

The position space is $\mathbb{Z}_d \otimes \mathbb{Z}_d$, the momentum space is also $\mathbb{Z}_d \otimes \mathbb{Z}_d$, so the phase-space is $[\mathbb{Z}_d]^4$. Using the same notation as above, we get the displacement operators $D_1(\alpha_1, \beta_1)$ and $D_2(\alpha_2, \beta_2)$ corresponding to each Hilbert space $\mathcal{H}_1, \mathcal{H}_2$,

$$\begin{aligned} D_1(\alpha_1, \beta_1) &\equiv D(\alpha_1, \beta_1) \otimes \mathbf{1} \\ D_2(\alpha_2, \beta_2) &\equiv \mathbf{1} \otimes D(\alpha_2, \beta_2) \end{aligned} \tag{4.4}$$

We also consider the displacement operators in phase-space in both systems:

$$D_1(\alpha_1, \beta_1)D_1(\alpha_2, \beta_2) \equiv D(\alpha_1, \beta_1) \otimes D(\alpha_2, \beta_2) \tag{4.5}$$

Based on the brief introduction of the transformations in composite finite systems, below we will give some notation in the multi-mode finite quantum systems.

4.2 Multi-mode Finite Quantum Systems

Generalization to a system with more components, i.e. a multipartite system is obvious [64]. We consider a system composed of ℓ component subsystems each of which has $\wp$-dimensional in Hilbert space $\mathcal{H}$. Its Hilbert space H is $\wp^\ell$ dimensional:

$$H = \mathcal{H}_1 \otimes \mathcal{H}_2 \ldots \otimes \mathcal{H}_a \otimes \ldots \otimes \mathcal{H}_\ell \tag{4.6}$$

The position states are:

$$|m_1, \ldots, m_\ell\rangle_{\mathcal{X}} \equiv |m_1\rangle_{\mathcal{X}} \otimes \ldots \otimes |m_\ell\rangle_{\mathcal{X}} \tag{4.7}$$

where $m_i \in \mathbb{Z}_\wp$.The momentum states in the ℓ-partite system are:

$$|m_1, \ldots, m_\ell\rangle_{\mathcal{P}} \equiv |m_1\rangle_{\mathcal{P}} \otimes \ldots \otimes |m_\ell\rangle_{\mathcal{P}} \tag{4.8}$$

4.2.1 Fourier Transformation in ℓ-partite systems

The Fourier transform is given by:

$$F = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_\ell \tag{4.9}$$

Using equation(3.4) and (4.9) we get

$$\mathcal{F} = \wp^{\frac{-\ell}{2}} \sum \omega(m_1 n_1 + \ldots + m_\ell n_\ell)|m_1\rangle_{\mathcal{X}\mathcal{X}}\langle n_1| \otimes \ldots \otimes |m_\ell\rangle_{\mathcal{X}\mathcal{X}}\langle n_\ell| \quad (4.10)$$

The summation is over all $m_1, n_1, \ldots, m_\ell, n_\ell \in \mathbb{Z}_\wp$.

4.2.2 Displacement Operator

Let

$$\begin{aligned}(\alpha_\lambda) &= (\alpha_1, \ldots, \alpha_\ell) \\ (\beta_\lambda) &= (\beta_1, \ldots, \beta_\ell)\end{aligned} \quad (4.11)$$

In ℓ-partite finite systems, where $\alpha_\lambda, \beta_\lambda$ in $\mathbb{Z}_\wp$. The displacement operators are defined by:

$$\begin{aligned}Z(\alpha_\lambda) &= Z^{\alpha_1} \otimes \ldots \otimes Z^{\alpha_\ell} \\ X(\beta_\lambda) &= X^{\beta_1} \otimes \ldots \otimes X^{\beta_\ell}\end{aligned} \quad (4.12)$$

They also obey the relation of

$$X(\beta_\lambda)Z(\alpha_\lambda) = Z(\alpha_\lambda)X(\beta_\lambda)\omega(-\sum_\lambda \alpha_\lambda\beta_\lambda) \quad (4.13)$$

And the general displacement in multi-partite finite systems are defined as

$$\begin{aligned} D(\alpha_\lambda, \beta_\lambda) &= Z(\alpha_\lambda)X(\beta_\lambda)\omega(-2^{-1}\sum_\lambda \alpha_\lambda\beta_\lambda) \\ &= D(\alpha_1, \beta_1) \otimes \ldots \otimes D(\alpha_\ell, \beta_\ell) \end{aligned} \tag{4.14}$$

4.3 Symplectic $Sp(2\ell, \mathbb{Z}_\wp)$ Transformations

Section 3 already introduced the Symplectic transformation $Sp(2, \mathbb{Z}_\wp)$. Here we will give the details about Symplectic transformations $Sp(2\ell, \mathbb{Z}_\wp)$ in ℓ-partite finite systems. Generalizing equation(3.30) we consider the representations of the symplectic $Sp(2\ell, \mathbb{Z}_\wp)$ transformations

$$\begin{aligned} Z'(\alpha_\lambda) &= SZ(\alpha_\lambda)S^\dagger = D(\sum t_{\lambda k}\alpha_k, \sum s_{\lambda k}\alpha_k) \\ X'(\beta_\lambda) &= SX(\beta_\lambda)S^\dagger = D(\sum r_{\lambda\mu}\beta_\mu, \sum q_{\lambda\mu}\beta_\mu) \end{aligned} \tag{4.15}$$

Here all exponents belong in $\mathcal{Z}_\wp$. S is a unitary symplectic operator which we will discuss below. The new displacement operator X'_i and Z'_i displace in different directions in phase space in comparison with X_i and Z_i. Equations (4.15) need to satisfy equations (3.19, 3.20). Requirement (3.19, 3.20) leads to the constraints

$$\sum_\lambda (s_{\lambda k}t_{\lambda\mu} - t_{\lambda k}s_{\lambda\mu}) = 0$$

$$\begin{aligned}\sum_\lambda (q_{\lambda k} r_{\lambda\mu} - r_{\lambda k} q_{\lambda\mu}) &= 0 \\ \sum_\lambda (q_{\lambda k} t_{\lambda\mu} - r_{\lambda k} s_{\lambda\mu}) &= \delta(k, \mu)\end{aligned} \tag{4.16}$$

The operators $X(\beta_\lambda)$ and $Z(\alpha_\lambda)$ satisfied equation(4.13). The transformed operators also satisfy this relation:

$$X'(\beta_\lambda)Z'(\alpha_\lambda) = Z'(\alpha_\lambda)X'(\beta_\lambda)\omega(-\sum_\lambda \alpha_\lambda\beta_\lambda) \tag{4.17}$$

$Sp(2\ell, \mathbb{Z}_\wp)$ transformations have $4\ell^2$ parameters (which belongs to $\mathbb{Z}_\wp$) and $2\ell^2 - \ell$ constraints. The constricts can be solved because the parameters belong to a field. Therefore we have $2\ell^2+\ell$ independent parameters. A subgroup of $Sp(2\ell, \mathbb{Z}_\wp)$ is the group of local symplectic transformation $Sp(2, \mathbb{Z}_\wp) \times \ldots \times Sp(2, \mathbb{Z}_\wp)$.

4.4 Numerical Calculation of the Symplectic Operator S in Multi-partite Systems

Let

$$\begin{aligned}&Z_i \equiv Z(0, \ldots, 0, 1, 0, \ldots, 0) \\ &Z_i^\wp = \mathbf{1} \\ &[Z_i, Z_j] = 0\end{aligned} \tag{4.18}$$

where the 1 is in the i-th position.

The operators Z_i commute and their common eigenvectors are $|m_1, \ldots, m_\ell\rangle_{\mathcal{X}}$:

$$\begin{aligned} Z_i|m_1, \ldots, m_\ell\rangle_{\mathcal{X}} &= \omega(m_i)|m_1, \ldots, m_\ell\rangle_{\mathcal{X}} \\ i &= 1, \ldots, \ell \end{aligned} \tag{4.19}$$

To each set of eigenvalues $m_1, \ldots, m_\ell$ corresponds (up to a phase factor) one normalized eigenvector which we calculate numerically. Given Z_i and a given eigenvalue $\omega(m_i)$, there is a $\wp^{\ell-1}$-dimensional space of corresponding eigenvectors.

We consider the projector $\hat{\pi}_i(m_i)$ to this eigenspace. This projector belongs in the null space of $[Z_i - \omega(m_i)\mathbf{1}]$. This implies that

$$[Z_i - \omega(m)\mathbf{1}]\hat{\pi}_i(m) = 0 \tag{4.20}$$

where $i = 1, \ldots, \ell$; and $m \in \mathbb{Z}_\wp$. The fact that $Z_i^\wp = \mathbf{1}$ implies that:

$$\begin{aligned} Z_i &= \hat{\pi}_i(0) + \omega(1)\hat{\pi}_i(1) + \ldots + \omega(\wp - 1)\hat{\pi}_i(\wp - 1) \\ \sum_{m \in \mathbb{Z}_\wp} \hat{\pi}_i(m) &= \mathbf{1} \\ \hat{\pi}_i(m)\hat{\pi}_i(n) &= \hat{\pi}_i(m)\delta(m, n) \end{aligned} \tag{4.21}$$

where $m, n \in \mathbb{Z}_\wp$. The $\hat{\pi}_i(m)$ are orthogonal projectors to the eigenspaces corresponding to the various eigenvalues of Z_i. These can be expressed in terms of powers of Z_i as

$$\hat{\pi}_i(m) = \frac{1}{\wp}\Big\{\mathbf{1} + \omega(-m)Z_i + \Big[\omega(-m)Z_i\Big]^2 + \ldots + \Big[\omega(-m)Z_i\Big]^{\wp-1}\Big\} \quad (4.22)$$

We have seen in equation(4.18), that $[Z_i, Z_j] = 0$. Consequently we conclude

$$[\hat{\pi}_i(m), \hat{\pi}_j(n)] = 0$$

$$m, n \in \mathbb{Z}_\wp \qquad i, j = 1, \ldots, \ell \quad (4.23)$$

More generally we introduce the projector expressed as

$$\hat{\pi}_{i_1,i_2,\ldots,i_k}(m_1, m_2, \ldots, m_k) \equiv \hat{\pi}_{i_1}(m_1)\hat{\pi}_{i_2}(m_2)\ldots\hat{\pi}_{i_k}(m_k) \quad (4.24)$$

This projector is the common null space of the k commuting matrices $\Big[Z_{i_r} - \omega(m_{i_r})\mathbf{1}\Big]$ (where we consider $r = 1, \ldots, k$),

$$\Big[Z_{i_r} - \omega(m_{i_r})\mathbf{1}\Big]\hat{\pi}(m_{i_1}, m_{i_2}, \ldots, m_{i_k}) = 0 \quad (4.25)$$

This common null space is $\wp^{\ell-k}$-dimensional.

Multiplying the above operators with the symplectic operators S we get the 'prime counterparts' of these operators(the transformed operators). For the numerical calculation we first evaluate the $\wp^\ell \times \wp^\ell$ matrices Z_i'. We then use a numerical library (MATLAB) to calculate the projector to the null space

$\hat{\pi}_i'(m)$ and then using equation(4.24) gives

$$\hat{\pi}_{1,\ldots,\ell}(m_1, \ldots, m_\ell) \equiv \hat{\pi}_1(m_1) \ldots \hat{\pi}_\ell(m_\ell) \tag{4.26}$$

Here, the projector is one-dimensional, and gives the common eigenvector $|m_1, \ldots, m_\ell\rangle_{\mathcal{X}'}$ of all Z_i'. To each set of eigenvalues $m_1, \ldots, m_\ell$ corresponds one normalized eigenvector which we calculate numerically in equation(4.19). This common eigenvector is defined up to a phase factor.

In order to calculate the phases, we start from the lowest state $|0, \ldots, 0\rangle_{\mathcal{X}'}$ (whose phase we choose arbitrarily), and use numerically the equation to get the eigenvectors

$$(X_1')^{m_1} \ldots (X_\ell')^{m_\ell} |0, \ldots, 0\rangle_{\mathcal{X}'} = |m_1, \ldots, m_\ell\rangle_{\mathcal{X}'} \tag{4.27}$$

The eigenvectors $|m_1, \ldots, m_\ell\rangle_{\mathcal{X}'}$ calculated through this equation(4.27) differ from the corresponding vectors calculated above as common eigenvectors of the matrices of all Z_i', only by a phase factor. This step tests that the numerical work is correct and at the same time it provides the phases.

Therefore we stress that the phases are very important for the self-consistency of the formalism. We can calculate the matrix elements of the operator S using the following equation,

$$S(m_1, \ldots, m_\ell | n_1, \ldots, n_\ell) = {}_{\mathcal{X}}\langle m_1, \ldots, m_\ell | n_1, \ldots, n_\ell\rangle_{\mathcal{X}'} \tag{4.28}$$

4.5 Summary

In this chapter we presented a general method for the numerical calculation of Symplectic $Sp(2\ell, \mathbb{Z}_\wp)$ transformation in ℓ-partite systems. At next chapter, we will use this numerical calculation to $Sp(4, \mathbb{Z}_\wp)$ transformations in bi-partite systems and $Sp(6, \mathbb{Z}_\wp)$ in tri-partite systems.

Chapter 5

Examples

Outline:

- $Sp(4, \mathbb{Z}_\wp)$ Transformations in Bi-partite Systems
- Entanglement in Bi-partite Systems
- $Sp(6, \mathbb{Z}_\wp)$ Transformations in Tri-partite Systems
- Entanglement in Tri-partite Systems
- Measurement in First Subsystem of Bi-partite Systems
- Summary

In this chapter, using the method of numerical calculation of symplectic operator, we apply the general formalism to $Sp(4,\mathbb{Z}_\wp)$ transformations in bi-partite systems, and to $Sp(6,\mathbb{Z}_\wp)$ transformations in tri-partite systems. We present two examples of $Sp(4,\mathbb{Z}_\wp)$ transformation, and show how factorizable and separable states, after a symplectic transformation become entangled. We also calculate similar quantities in tri-partite systems. In section 5.5, we focus on the measurements. We also give four examples. The source code of the symplectic transformations and measurements are available in Appendix B and C.

5.1 $Sp(4,\mathbb{Z}_\wp)$ Transformations in Bi-partite Systems

In this section, we consider a bi-partite system with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Both of them are 5-dimensional Hilbert spaces. Here, we calculated two examples which we call A and B. They must satisfy equations (4.15) , (4.16).

Firstly, we study the transformations of the example A that is:

$$
\begin{aligned}
X_1' &= S_A X_1 S_A^\dagger = (\mathcal{Z}^{-1}) \otimes (\mathcal{X}^{-2}\mathcal{Z}) \\
Z_1' &= S_A Z_1 S_A^\dagger = (\mathcal{X}^{2}\mathcal{Z}) \otimes (\mathcal{X}) \\
X_2' &= S_A X_2 S_A^\dagger = (\mathcal{X}^{2}\mathcal{Z}) \otimes (\mathcal{X}^{2})
\end{aligned}
$$

$$Z_2' \;=\; S_A Z_2 S_A^{\dagger} = (\mathcal{X}) \otimes (\mathcal{X}^{-1}\mathcal{Z}) \tag{5.1}$$

In chapter 4, we have shown how to calculate the matrix elements of the operator S. We follow that numerical technique, to calculate the matrix elements S (equation(4.28)) for example A. The operators S_A have 5^4 matrix elements.

The element $S_A(m_1, m_2|0, 1)$ is represented as:

$$S_A(m_1, m_2|0, 1) = \begin{pmatrix} z_5 & z_3 & z_2 & z_2 & z_3 \\ z_4 & z_2 & z_1 & z_2 & z_4 \\ z_3 & z_2 & z_2 & z_3 & z_5 \\ z_4 & z_3 & z_4 & z_5 & z_5 \\ z_5 & z_5 & z_4 & z_3 & z_4 \end{pmatrix} \tag{5.2}$$

Here $z_1 = 0.0433 + 0.1953i$, $z_2 = z_1\omega(-2)$, $z_3 = z_1\omega(-1)$, $z_4 = z_1\omega(2)$,$z_5 = z_1\omega(1)$, where $\omega(1) = \exp(i2\pi/5)$.

In Table.5.1, there is the pseudo codes of the algorithm for Numerical Calculation of the Symplectic Transformation.

Table 5.1: The Algorithm for Numerical Calculation of the Symplectic Transformation

```
INPUT: 2 Position State & 2 Moment State in Bi-partite System;
METHOD: Numerical Calculation of the Symplectic Transformation;
OUTPUT: A Symplectic Operator;
```

```
GET Z1, Z2, X1, X2 from INPUT();
// Generate an Indentity Matrix;
SET I5 to 5-by-5 Identity Matrix;
// Returns the Kronecker Tensor Product (KTP);
FOR each column of I5
 FOR each column of I5
 GET KT[ ] from KTP BETWEEN every column of I5;
ENDFOR
ENDFOR
// Generate an Eigenvalue (Fourier Parameter's coefficient)
// by odd prime number 5;
SET W to exp(2*pi*i/5);
// Generate another Indentity Matrix I25;
SET I25 to 25-by-25 Identity Matrix;
FOR each integer in the interval [-2, 2]
 // Create a Null Space for Input Position State Z1;
 SET NS1 to NullSpace(Z1 - W*I25);
 FOR each integer in the interval [-2, 2]
 // Create another Null Space for Input Position State Z2;
  SET NS2 to NullSpace(Z2 - W*I25)*NS1;
 // Get the Common Eigenvector in position Phase Space;
 GET CEp[ ] from CommonOperator(NS1, NS2)/Norm(NS1, NS2);
 ENDFOR
ENDFOR
// Compute the Common Eigenvector in position and
// momentum Phase Space from X1 and X2 at interval [0,4]
// while CEp at lowest state;
GET CEpm[ ] from X1^[0,4]*X2^[0,4]*CEp[0,0];
// Get the Symplectic Operator;
GET SO = KT[ ].transpose() * CEpm[ ];
OUTPUT(SO);
```

Similarly, we considered example B where

$$\begin{aligned}
X_1' &= S_B X_1 S_B^\dagger = (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}^2) \\
Z_1' &= S_B Z_1 S_B^\dagger = (\mathcal{X}^{-2}) \otimes (\mathcal{X}\mathcal{Z}) \\
X_2' &= S_B X_2 S_B^\dagger = (\mathcal{X}^{-1}) \otimes (\mathcal{X}\mathcal{Z}) \\
Z_2' &= S_B Z_2 S_B^\dagger = (\mathcal{X}^{-1}\mathcal{Z}) \otimes (\mathcal{X}^{-1}\mathcal{Z}) \qquad (5.3)
\end{aligned}$$

Then using the equation(4.28), we calculated the matrix elements of the operator S_B for example B. The operators S_B have 5^4 matrix elements.

The elements $S_B(m_1, m_2|0, 1)$ is computed as:

$$S_B(m_1, m_2|0,1) = \begin{pmatrix} z_5 & z_5 & z_3 & z_1 & z_3 \\ z_2 & z_3 & z_2 & z_1 & z_1 \\ z_2 & z_1 & z_1 & z_2 & z_3 \\ z_5 & z_3 & z_1 & z_3 & z_5 \\ z_4 & z_5 & z_2 & z_2 & z_5 \end{pmatrix} \qquad (5.4)$$

Here $z_1 = 0.0724 + 0.1864i$, $z_2 = z_1\omega(-2)$, $z_3 = z_1\omega(-1)$, $z_4 = z_1\omega(2)$,$z_5 = z_1\omega(1)$, $\omega(1) = \exp(i2\pi/5)$.

Below we use the matrix elements S_A and S_B to produce the entangled states.

5.2 Entanglement in Bi-partite Systems

Here we use the symplectic operator S_A and S_B to produce entangled states. We consider the normalized factorizable pure states $|a\rangle$ and $|b\rangle$

$$|a\rangle = \begin{pmatrix} 0.2500 - 0.2300i \\ 0.0500 - 0.0700i \\ 0.2300 - 0.0400i \\ 0.3300 - 0.4100i \\ 0.7387 \end{pmatrix} \otimes \begin{pmatrix} 0.5400 + 0.3100i \\ 0.1100 + 0.2100i \\ 0.0900 + 0.4100i \\ 0.0200 + 0.1400i \\ 0.5999 \end{pmatrix} \tag{5.5}$$

$$|b\rangle = \begin{pmatrix} 0.1300 + 0.1500i \\ 0.2500 + 0.3200i \\ 0.6900 + 0.0800i \\ 0.0600 + 0.4000i \\ 0.3868 \end{pmatrix} \otimes \begin{pmatrix} 0.1250 + 0.0350i \\ 0.2500 + 0.0610i \\ 0.1340 + 0.2350i \\ 0.1500 + 0.0150i \\ 0.9061 \end{pmatrix} \tag{5.6}$$

We also consider the mixed state described with the density operator:

$$\rho_{12}(p) = p|a\rangle\langle a| + (1-p)|b\rangle\langle b| \qquad 0 \le p \le 1 \tag{5.7}$$

We use the symplectic operator S_A to calculated the new transformed states $|a'\rangle$ and $|b'\rangle$,

$$
\begin{aligned}
|a'\rangle &= S_A|a\rangle \\
|b'\rangle &= S_A|b\rangle
\end{aligned}
\tag{5.8}
$$

and the new mixed state after the symplectic S_A transformation described by

$$
\begin{aligned}
\rho'_{12}(p) &= S_A\rho_{12}(p)S_A^\dagger \\
&= p|a'\rangle\langle a'| + (1-p)|b'\rangle\langle b'|
\end{aligned}
\tag{5.9}
$$

We repeat the same calculations with S_B and denote the corresponding quantities as $|a''\rangle$, $|b''\rangle$ and $\rho''_{12}(p)$.

5.2.1 Numerical Calculation of Measures of Entanglement

In this section, we calculate various quantities which quantify entanglement of the density matrices before and after the transformation with S_A and S_B. The purpose is to show that the symplectic transformations generate entanglement between the subsystems. See Appendix B for the source codes (The MathWorks™ MATLAB).

We have briefly introduced the concepts of the entanglement in Chapter 2. We consider the reduced density matrices of ρ_{12} which before the symplectic transformations are

$$\begin{aligned}\rho_1(p) &= \mathrm{Tr}_2\rho_{12}(p) \\ \rho_2(p) &= \mathrm{Tr}_1\rho_{12}(p)\end{aligned} \tag{5.10}$$

Then the new reduced density matrices after the transformations S_A and S_B are $\rho_1'(p)$, $\rho_2'(p)$ and $\rho_1''(p)$, $\rho_2''(p)$ given by:

$$\begin{aligned}\rho_1'(p) &= \mathrm{Tr}_2\rho_{12}'(p) \\ \rho_2'(p) &= \mathrm{Tr}_1\rho_{12}'(p) \\ \rho_1''(p) &= \mathrm{Tr}_2\rho_{12}''(p) \\ \rho_2''(p) &= \mathrm{Tr}_1\rho_{12}''(p)\end{aligned} \tag{5.11}$$

We have calculated the linear entropy using equation(2.47).

$$E(p) = 1 - \mathrm{Tr}\Big[\{\rho_1(\mathrm{p})\}^2\Big] = 1 - \mathrm{Tr}\Big[\{\rho_2(\mathrm{p})\}^2\Big] \tag{5.12}$$

If the value of the quantity is zero, the reduced density matrices are pure states, otherwise the non-zero value of the equation(5.12) indicate that the reduced density matrices are mixed states.

The mutual information can be described by,

$$I(\rho_1, \rho_2) = \mathfrak{S}(\rho_1(p)) + \mathfrak{S}(\rho_2(p)) - \mathfrak{S}(\rho_{12}(p)) \geq 0 \tag{5.13}$$

Where $\mathfrak{S}(\rho)$ are ,

$$\begin{aligned} \mathfrak{S}(\rho_{12}) &= -\mathrm{Tr}\rho_{12} \ln \rho_{12} \\ \mathfrak{S}(\rho_1) &= -\mathrm{Tr}\rho_1 \ln \rho_1 \\ \mathfrak{S}(\rho_2) &= -\mathrm{Tr}\rho_2 \ln \rho_2 \end{aligned} \tag{5.14}$$

When $I(\rho_1, \rho_2)$ is non-negative, there are correlations (classical and quantum) between the two subsystems described with the density matrices $\rho_1(p)$ and $\rho_2(p)$. Throughout the thesis we use logarithms with base e and all entropic results are in natural units (nats).

We have also calculated the conditional entropies using the equation(2.43),

$$\begin{aligned} I(\rho_2|\rho_1) &= \mathfrak{S}(\rho_{12}(p)) - \mathfrak{S}(\rho_1(p)) \\ I(\rho_1|\rho_2) &= \mathfrak{S}(\rho_{12}(p)) - \mathfrak{S}(\rho_2(p)) \end{aligned} \tag{5.15}$$

The negative value of these quantities indicates that two subsystems are entangled. Positive value does not lead to any conclusion.

Following equation(2.51), we have also calculated the negativity of the matrix $\rho_{12}(p)$. We consider $\sigma_{12}(p)$ to be the partial transpose of $\rho_{12}(p)$ with respect to the first subsystem. Then we calculate the sum of the singular values of $\sigma_{12}(p)$, (i.e., the $\mathrm{Tr}[\sigma_{12}(\mathrm{p})^{\dagger}\sigma_{12}(\mathrm{p})]^{1/2}$). The negativity is defined by:

$$N\Big[\rho_{12}(p)\Big] = \frac{\mathrm{Tr}[\sigma_{12}(p)^{\dagger}\sigma_{12}(p)]^{1/2} - 1}{2} \tag{5.16}$$

5.2.2 Numerical Results in Bi-partite

In Table(5.2), we calculate the pure states $|a'\rangle = S_A|a\rangle$, $|a''\rangle = S_B|a\rangle$ and present the numerical results of linear entropy for the various quantities before and after the symplectic transformations S_A and S_B. The non-zero values of the linear entropies show that after the transformation S_A and S_B, the factorizable state $|a\rangle$ is entangled.

Table 5.2: Various quantities which characterize the entanglement before and after the transformations S_A and S_B.

	bef. transf.	aft. transf. S_A	aft. transf. S_B
$E(1)$ for the states $\|a>$ of Eq(5.5)	0	0.6539	0.65
$I(\rho_1,\rho_2)(1)$ for the states $\|a>$ of Eq(5.5)	0	2.3669	2.4063
$I(\rho_1,\rho_2)(0.6)$ for the mixed states of Eq(5.7)	0.2968	2.2269	2.3120
$I(\rho_2\|\rho_1)(0.6)$ for the mixed states of Eq(5.7)	0.0340	-0.8089	-0.8457
$I(\rho_1\|\rho_2)(0.6)$ for the mixed states of Eq(5.7)	0.2892	-0.7979	-0.8462
$N[\rho_{12}(0.6)]$ for the mixed states of Eq(5.7)	0	1.0163	1.0786

We also calculated the mutual information $I(\rho_1, \rho_2)$ for the pure state $|a\rangle$. The positive numbers confirmed after the transformations S_A and S_B, the factorizable state $|a\rangle$ becomes entangled.

We also show results for the mixed state $\rho(0.6)$. The positive values of the mutual information indicate correlations (both classical and quantum) between the two subsystems which increase after the transformations S_A and S_B. The negative values of conditional entropies $I(\rho_2'|\rho_1')$, $I(\rho_1'|\rho_2')$ and $I(\rho_2''|\rho_1'')$, $I(\rho_1''|\rho_2'')$ indicate entanglement between the two subsystems, after the transformations S_A and S_B. The negativity results also lead to the same conclusion.

Now we turn to describing the exact details of our applications. Figure(5.1,5.2) presents the linear entropy $E(p)(0 \leq p \leq 1)$ for equation(5.12). In Figure(5.1), we plot the $E(p)$ for the mixed state $\rho_{12}(p)$ of equation(5.7). When $p = 0$ and also $p = 1$, the linear entropy is zero. In this case the linear entropy $1 - \mathrm{Tr}[{\rho_1}^2] = 1 - \mathrm{Tr}[{\rho_2}^2] = 0$. Otherwise the value of the quantity $1 - \mathrm{Tr}[{\rho_1}^2] \neq 1 - \mathrm{Tr}[{\rho_2}^2]$.

Similarly, Figure(5.2) plot the $E(p)$ of density matrix ρ_{12}' and ρ_{12}''. We notice that the linear entropy is $1 - \mathrm{Tr}[{\rho_1}^2] = 1 - \mathrm{Tr}[{\rho_2}^2] > 0$ when $p = 0$ or $p = 1$. Also the nonnegative numbers present that the factorizable state is entangled after the transformations. Approximately, we observed the value of $1 - \mathrm{Tr}[{\rho_1}^2] \neq 1 - \mathrm{Tr}[{\rho_2}^2]$, when $0 < p < 1$.

In Figure(5.3) we consider the mixed state $\rho_{12}(p)$ of equation(5.7), and plot the mutual information $I(\rho_1, \rho_2)$ as a function of p before and after the transformations A and B. The results show that the symplectic transformations create strong correlations (classical and quantum) between the two subsystems.

In Figure(5.4), we plot the conditional informations $I(\rho_2|\rho_1)$ as a function of p before and after the transformation S_A and S_B. The negative values of these quantities after the transformations, show that the symplectic transformations entangle the two subsystems.

In Figure(5.5), we plot the conditional informations $I(\rho_1|\rho_2)$ as a function of p. The negative values lead to the same conclusion. They show the fact that the two subsystems are entangled after the symplectic transformations.

Results for the negativity of the matrix $\rho_{12}(p)$ as a function of p, before and after the transformations A and B, are shown in Figure(5.6). Before the transformation the state is separable and the negativity is zero. These results also show that the symplectic transformations entangle the two subsystems.

We might use the values of the conditional entropies $I(\rho'_2|\rho'_1)$ and $I(\rho'_1|\rho'_2)$ and of the negativity N as measures of entanglement. In this case, which of two bipartite states is more entangled, depends on the quantity that we use. It might be that according to one measure the first state is more entangled, and according to another measure the second state is more entangled.

Figures (5.4, 5.5, 5.6) show that, in the examples that we have considered. In the region $0 < p < 1$, according to the conditional entropies the state ρ'_{12} of example A is more entangled than the state ρ''_{12} of example B, but according to the negativity the state ρ'_{12} of example A is less entangled than the state ρ''_{12} of example B.

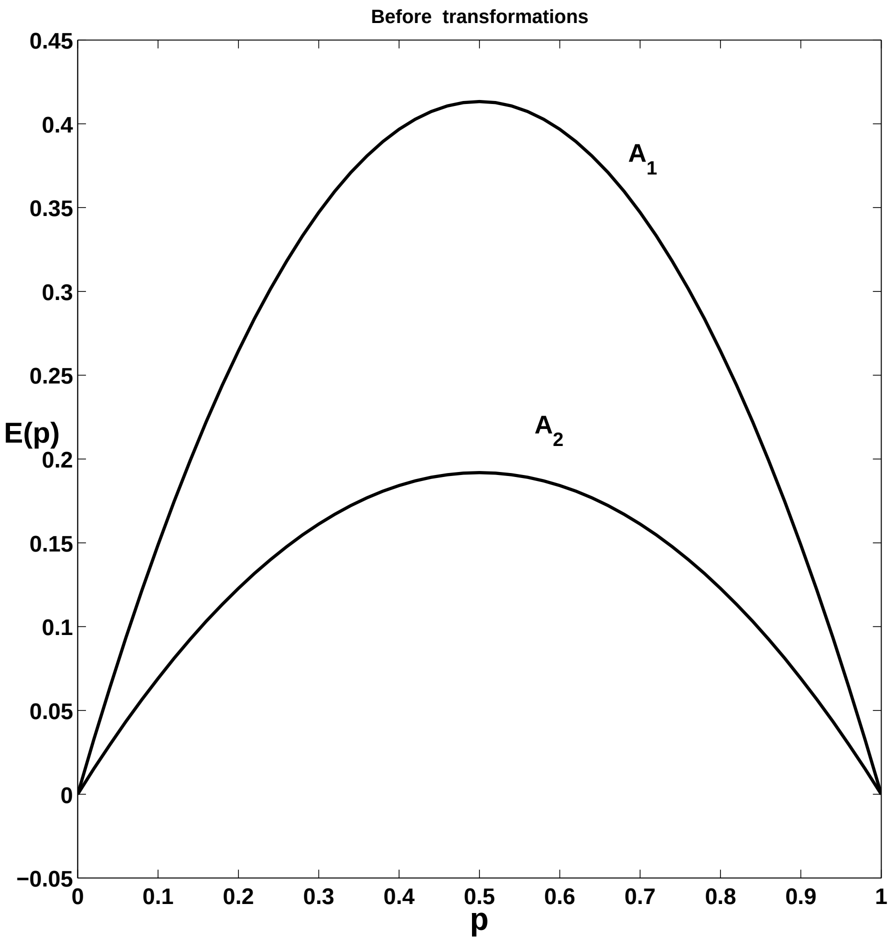

Figure 5.1: Linear entropy $E(p)$ for the states $\rho_{12}(p)$ of equation(5.7) . Curve A_1 shows $E(p) = 1 - \mathrm{Tr}[\rho_1(\mathrm{p})^2]$; curve A_2 shows $E(p) = 1 - \mathrm{Tr}[\rho_2(\mathrm{p})^2]$.

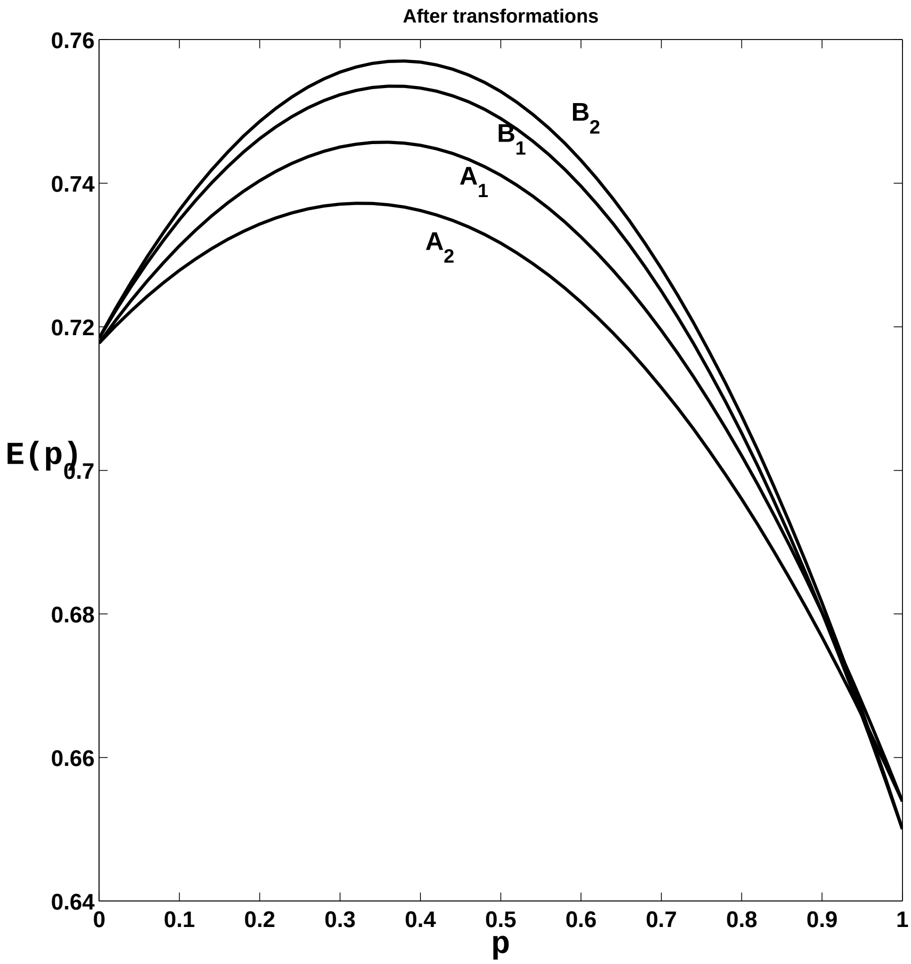

Figure 5.2: Linear entropy $E(p)$ for the states $\rho'_{12}(p) = S_A\rho_{12}(p)S_A^\dagger$ and $\rho''_{12}(p) = S_B\rho_{12}(p)S_B^\dagger$. Curve A_1 shows $E(p) = 1 - \mathrm{Tr}[\rho'_1(\mathrm{p})^2]$; curve A_2 shows $E(p) = 1 - \mathrm{Tr}[\rho'_2(\mathrm{p})^2]$. Curve B_1 shows $E(p) = 1 - \mathrm{Tr}[\rho''_1(\mathrm{p})^2]$; curve B_2 shows $E(p) = 1 - \mathrm{Tr}[\rho''_2(\mathrm{p})^2]$.

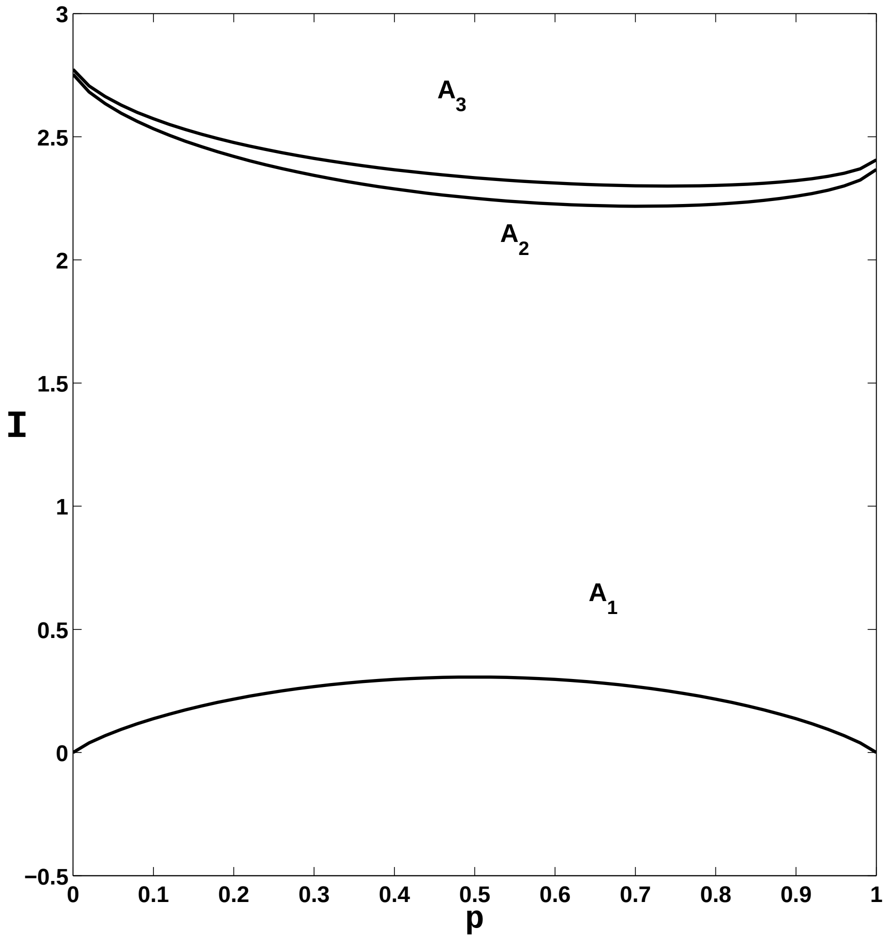

Figure 5.3: Mutual informations $I(\rho_1, \rho_2)$ for the states $\rho_{12}(p)$ (curve $A1$); $I(\rho'_1, \rho'_2)$ for the states $\rho'_{12}(p) = S_A \rho_{12}(p) S_A^\dagger$ (curve $A2$); $I(\rho''_1, \rho''_2)$ for the states $\rho''_{12}(p) = S_B \rho_{12}(p) S_B^\dagger$ (curve $A3$).

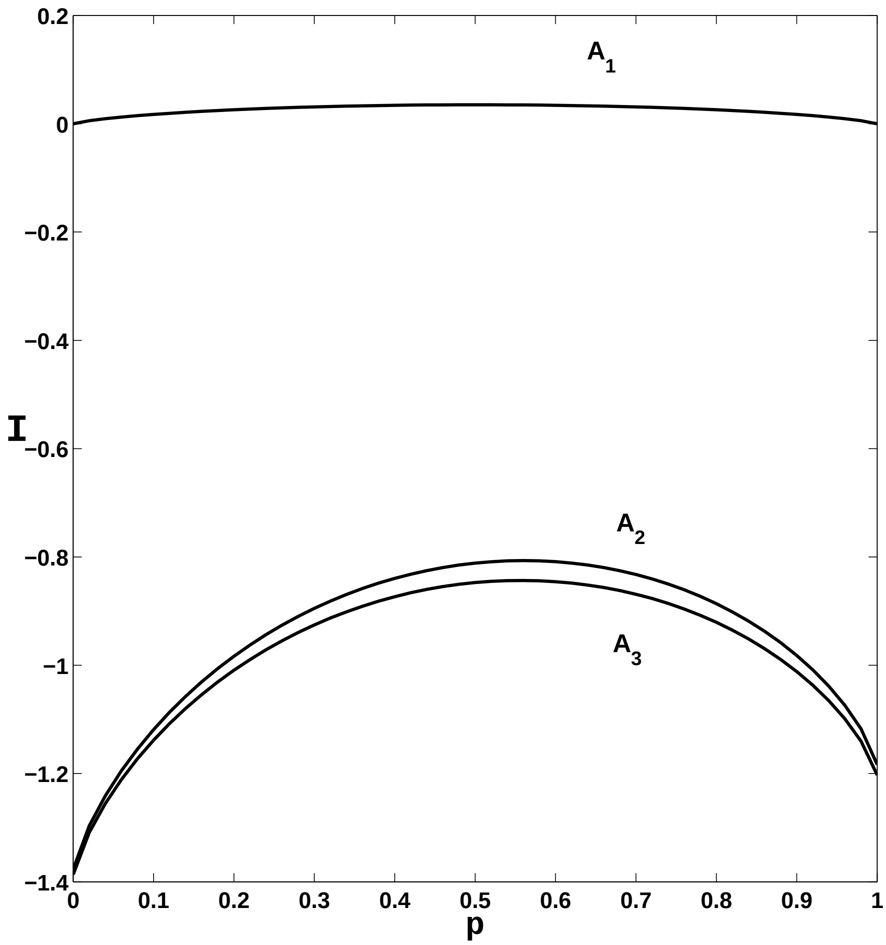

Figure 5.4: Conditional entropies $I(\rho_2|\rho_1)$ for the states $\rho_{12}(p)$ (curve $A1$); $I(\rho_2'|\rho_1')$ for the states $\rho_{12}'(p) = S_A\rho_{12}(p)S_A^\dagger$ (curve $A2$); $I(\rho_2''|\rho_1'')$ for the states $\rho_{12}''(p) = S_B\rho_{12}(p)S_B^\dagger$ (curve $A3$).

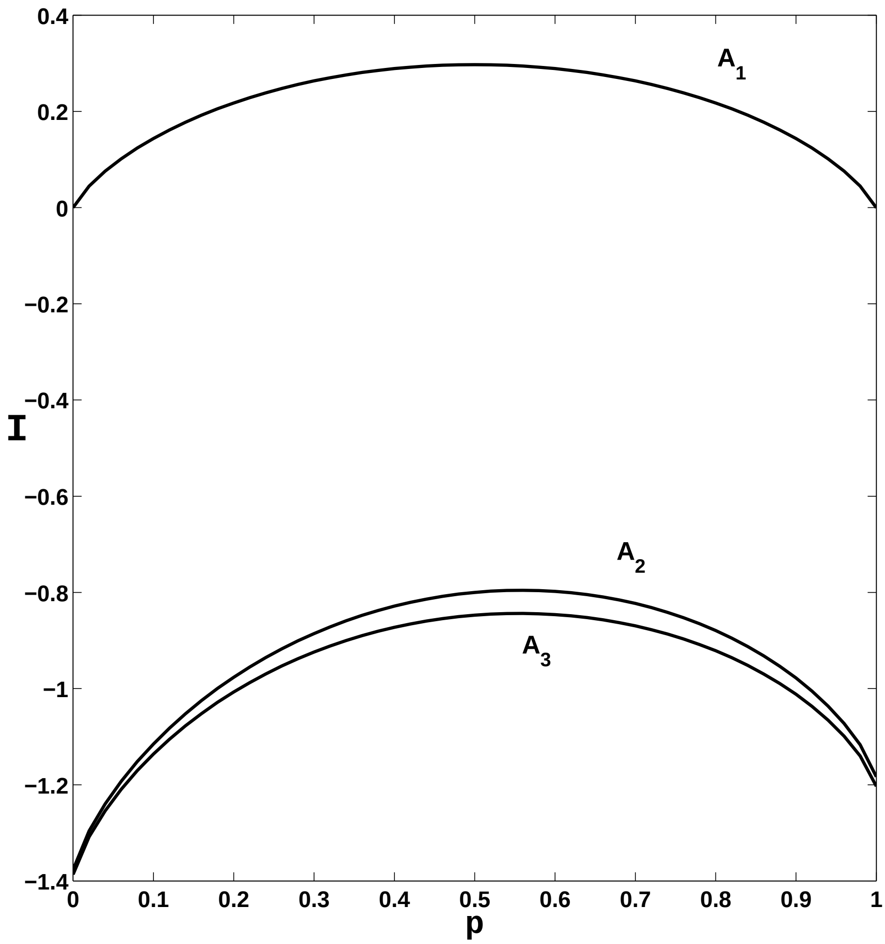

Figure 5.5: Conditional entropies $I(\rho_1|\rho_2)$ for the states $\rho_{12}(p)$ (curve $A1$); $I(\rho_1'|\rho_2')$ for the states $\rho_{12}'(p) = S_A\rho_{12}(\lambda)S_A^\dagger$ (curve $A2$); $I(\rho_1''|\rho_2'')$ for the states $\rho_{12}''(p) = S_B\rho_{12}(p)S_B^\dagger$ (curve $A3$).

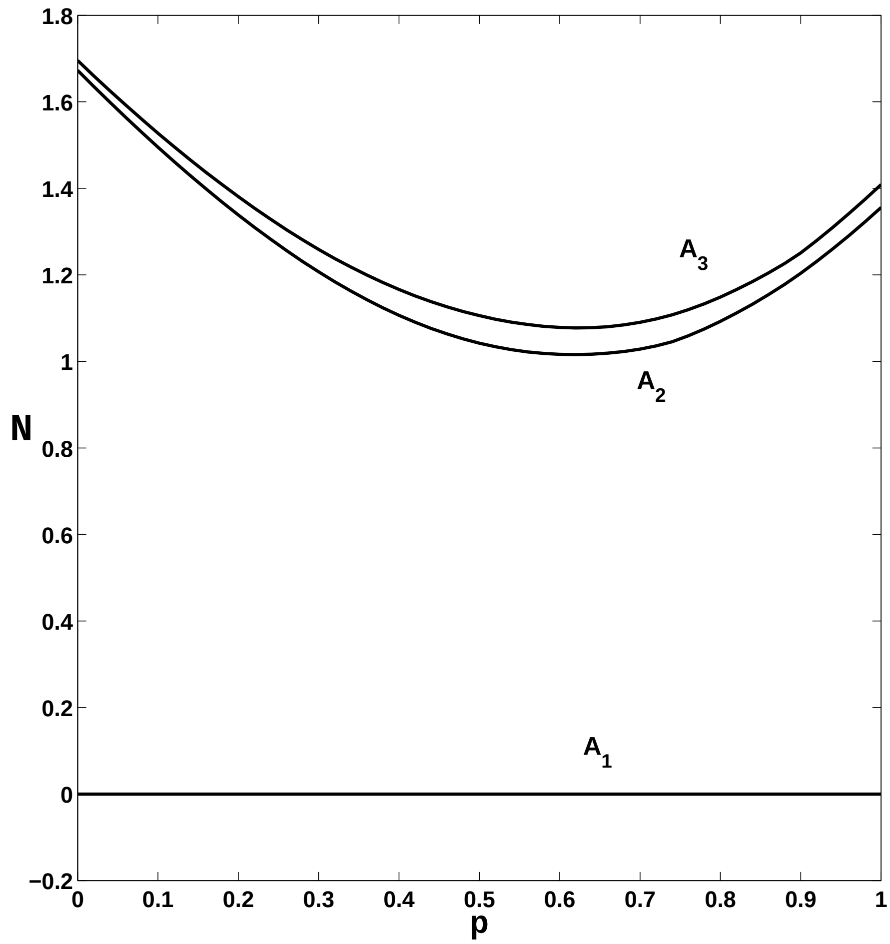

Figure 5.6: Negativities $N[\rho_{12}(p)]$ for the states $\rho_{12}(p)$ (curve $A1$); $N[\rho'_{12}(p)]$ for the states $\rho'_{12}(p) = S_A\rho_{12}(p)S_A^\dagger$ (curve $A2$); $N[\rho''_{12}(p)]$ for the states $\rho''_{12}(p) = S_B\rho_{12}(p)S_B^\dagger$ (curve $A3$).

5.3 $Sp(6, \mathbb{Z}_\wp)$ Transformations in Tri-partite Systems

Here we consider tri-partite systems comprised of subsystems with 3-dimensional Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$. We call this example C. The Hilbert space of the whole system is $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

The transformation of the example C is defined by

$$\begin{aligned}
X_1' &= S_C X_1 S_C^\dagger = (\mathcal{X}^{-1}\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}^{-1}) \\
Z_1' &= S_C Z_1 S_C^\dagger = (\mathcal{X}^{-1}\mathcal{Z}^{-1}) \otimes (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{X}^{-1}) \\
X_2' &= S_C X_2 S_C^\dagger = (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}^{-1}) \otimes (\mathcal{Z}) \\
Z_2' &= S_C Z_2 S_C^\dagger = (\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{X}^{-1}\mathcal{Z}^{-1}) \\
X_3' &= S_C X_3 S_C^\dagger = (\mathcal{X}\mathcal{Z}^{-1}) \otimes (\mathcal{Z}^{-1}) \otimes (\mathcal{Z}) \\
Z_3' &= S_C Z_3 S_C^\dagger = (\mathcal{X}\mathcal{Z}) \otimes (\mathcal{Z}) \otimes (\mathcal{X}\mathcal{Z}) \qquad (5.17)
\end{aligned}$$

All exponents of the transformations obey the equation(4.16). We calculated the matrix elements of symplectic operator S_C which has 3^6 matrix elements.

Below we give the matrix elements $S_C(m_1, m_2, 2|0, 1, 0)$:

$$S_C(m_1, m_2, 2|0, 1, 0) = \begin{pmatrix} z_2 & z_3 & z_2 \\ z_1 & z_2 & z_1 \\ z_1 & z_2 & z_1 \end{pmatrix} \tag{5.18}$$

Here we consider $z_1 = -0.0527 - 0.1851i$, $z_2 = z_1\omega(-1)$, $z_3 = z_1\omega(1)$, $\omega(1) = \exp(i2\pi/3)$

5.4 Entanglement in Tri-partite Systems

We quantify the entanglement generation by $Sp(6, \mathbb{Z}_\wp)$ transformations in example C. Firstly we consider the state,

$$\begin{aligned} \rho_{123}(p) &= p|c\rangle\langle c| + (1-p)|d\rangle\langle d| \\ 0 \leq & \; p \; \leq 1 \end{aligned} \tag{5.19}$$

The normalized factorizable pure states $|c\rangle$ and $|d\rangle$ are defined by:

$$|c\rangle = \begin{pmatrix} 0.4500 - 0.1240i \\ -0.5700 + 0.0120i \\ 0.6761 \end{pmatrix} \otimes \begin{pmatrix} 0.0120 - 0.4100i \\ 0.0700 + 0.0100i \\ 0.9093 \end{pmatrix}$$

$$\otimes \begin{pmatrix} 0.0425 - 0.2750i \\ -0.3650 - 0.0810i \\ 0.8848 \end{pmatrix} \tag{5.20}$$

$$|d\rangle = \begin{pmatrix} 0.0730 - 0.1340i \\ 0.4120 + 0.3710i \\ 0.8181 \end{pmatrix} \otimes \begin{pmatrix} -0.1850 - 0.0300i \\ -0.2630 + 0.5150i \\ 0.7940 \end{pmatrix}$$

$$\otimes \begin{pmatrix} 0.3990 - 0.4120i \\ 0.0200 + 0.0900i \\ 0.8140 \end{pmatrix} \tag{5.21}$$

The new state $|c'\rangle$ and $|d'\rangle$ after the symplectic transformation S_C are:

$$\begin{aligned} |c'\rangle &= S_C|c\rangle \\ |d'\rangle &= S_C|d\rangle \end{aligned} \tag{5.22}$$

Therefore the new mixed state $\rho'_{123}(p)$ is:

$$\begin{aligned} \rho'_{123}(p) &= S_C \rho_{123}(p) S_C^\dagger \\ &= p|c'\rangle\langle c'| + (1-p)|d'\rangle\langle d'| \end{aligned} \tag{5.23}$$

5.4.1 Numerical Calculation of Entanglement

We compare measures of entanglement before and after the symplectic transformations. We present various quantities for both cases. The reduced density matrices are given by:

$$\begin{aligned}\rho_i &= \mathrm{Tr_{jk}}\rho_{123} \\ \rho_{ij} &= \mathrm{Tr_{k}}\rho_{123} \\ i,j,k &= 1,2,3; \quad i \neq j \neq k \end{aligned} \tag{5.24}$$

The linear entropies corresponding to the pure state $|c\rangle$ are:

$$\begin{aligned}E_i &= 1 - \mathrm{Tr}[\rho_i^2] \\ &= 1 - \mathrm{Tr}[\rho_{jk}^2]\end{aligned} \tag{5.25}$$

If the quantity of E_i is non-zero, the subsystem S_i is entangled with the joint subsystems S_{jk}.

Next, we regard the tri-partite system as being two sub-systems which is the S_i and the joint subsystem S_{jk}. The mutual information is described by:

$$I(\rho_{jk}, \rho_i) = \mathfrak{S}(\rho_i) + \mathfrak{S}(\rho_{jk}) - \mathfrak{S}(\rho_{123}) \geq 0 \tag{5.26}$$

The conditional informations are given by:

$$
\begin{aligned}
I(\rho_{jk}|\rho_i) &= \mathfrak{S}(\rho_{123}) - \mathfrak{S}(\rho_i) \\
I(\rho_i|\rho_{jk}) &= \mathfrak{S}(\rho_{123}) - \mathfrak{S}(\rho_{jk})
\end{aligned} \tag{5.27}
$$

As with examples A and B, the non-zero values of $I(\rho_{jk}, \rho_i)$ indicate that the subsystem S_i is correlated with the subsystems S_{jk} (and if ρ_{123} describes a pure state, then these two subsystems are entangled). Also in the general case that ρ_{123} describes a mixed state, negative values of $I(\rho_{jk}|\rho_i)$ or $I(\rho_i|\rho_{jk})$ indicate that the subsystems S_i and S_{jk} are entangled, while positive values of both of these quantities do not lead to any conclusion.

In tri-partite systems, we can present another entropic quantity, which we call the quantum conditional mutual information. It is given by:

$$
I(\rho_j;\rho_k|\rho_i) = \mathfrak{S}(\rho_{ij}) + \mathfrak{S}(\rho_{ik}) - \mathfrak{S}(\rho_{123}) - \mathfrak{S}(\rho_i) \tag{5.28}
$$

According to the strong subadditive property [65, 66, 67], this quantity is non-negative. Zero values of $I(\rho_j;\rho_k|\rho_i)$ indicate that the reduced density matrix ρ_{jk} is separable[68].

5.4.2 Numerical Results in Tri-partite System

In Table(5.3), we consider the pure states $|c'\rangle = S_c|c\rangle$, and calculate numerical results for the various quantities before and after the symplectic transformation S_C. The zero-value of the linear entropies show that before the transformation the subsystem S_i with S_{jk} is separable. After the transformations, the values of the linear entropies are non-zero and the subsystem S_i is entangled with the joint subsystems S_{jk}.

We also calculated the mutual informations $I(\rho_i, \rho_{jk})$. The results show that these are correlations (classical and quantum) between the subsystem S_i and S_{jk}, which increase after the transformation S_C. We also calculate the conditional informations $I(\rho_i|\rho_{jk})$. The negative values indicated that the subsystems S_i and S_{jk} are entangled after the transformations. Table(5.3) also shows the values for the quantum conditional mutual information $I(\rho_i; \rho_j|\rho_k)$. Figures (5.7, 5.8) present the results of linear entropies E_i of equation(5.25).

Table 5.3: Various quantities which characterize the entanglement before and after the transformation S_C.

	before transf.	after transf. S_C
E_1 for the state $\|c\rangle$ of Eq(5.20)	0	0.6115
E_2 for the state $\|c\rangle$ of Eq(5.20)	0	0.5374
E_3 for the state $\|c\rangle$ of Eq(5.20)	0	0.6070
$I(\rho_1, \rho_{23})$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0 .4149	1.8780
$I(\rho_2, \rho_{13})$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.3569	1.7812
$I(\rho_3, \rho_{12})$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.2562	1.9876
$I(\rho_1\|\rho_{23})$for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.1520	-0.8320
$I(\rho_2\|\rho_{13})$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.0360	-0.8563
$I(\rho_3\|\rho_{12})$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.0163	-0.9128
$I(\rho_2; \rho_3\|\rho_1)$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.0162	1.3585
$I(\rho_1; \rho_3\|\rho_2)$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.0741	1.4552
$I(\rho_1; \rho_2\|\rho_3)$ for the mixed states $\rho_{123}(0.6)$ of Eq(5.19)	0.1749	1.2488

In Figure(5.7), we plot the value of quantities of $E_i(p)(0 \leq p \leq 1)$ for density matrix ρ_{123} of equation(5.19) which is before the transformations. We consider that pure state ρ_{123}(when p=0,and also p=1), the linear entropy

$E_i = 1 - \text{Tr}[\rho_i^2] = 1 - \text{Tr}[\rho_{jk}^2] = 0$, otherwise the positive value of linear entropy show that $1 - \text{Tr}[\rho_i^2] \neq 1 - \text{Tr}[\rho_{jk}^2]$. Similarly, Figure(5.8) plots the value of linear entropy for density matrix $\rho'_{123} = S_C \rho_{123} S_C^\dagger$ (eg. after symplectic transformation S_C). We notice the results of linear entropy $E_i = 1 - \text{Tr}[\rho_i^2] = 1 - \text{Tr}[\rho_{jk}^2] > 0$(when p=0, or p=1). That means, the pure state ρ_{123} after transformation is entangled. Figure(5.8) also plot the results for mixed state $\rho'_{123}(0 < p < 1)$, and we notice for this case, the linear entropy $1 - \text{Tr}[\rho_i^2] \neq 1 - \text{Tr}[\rho_{jk}^2]$.

In Figure(5.9) we present the mutual information $I(\rho_{jk}, \rho_i)$ as functions of p, before and after the transformation. The mutual information show that the correlations (classical and quantum)between S_{jk} and S_i, increase drastically after the transformation. In Figure(5.10) we present the conditional information $I(\rho_{jk}|\rho_i)$ as function p before and after the transformations. After the transformations, we noticed the values of the conditional information are all negative indicated that the S_i and S_{jk} become entangled.

The conditional information $I(\rho_i|\rho_{jk})$ as function p presented in Figure(5.11). The same conclusion with Figure(5.10), that after the transformations S_C, the conditional information show that the subsystem S_i are entangled with the joint subsystem S_{jk}.

Also the quantum conditional mutual information $I(\rho_i; \rho_j|\rho_k)$ as function p presented in Figure(5.12). This figure plots the values of the conditional mutual information before and after the transformations. We see that it takes non-negative values as required by the strong subadditivity property.

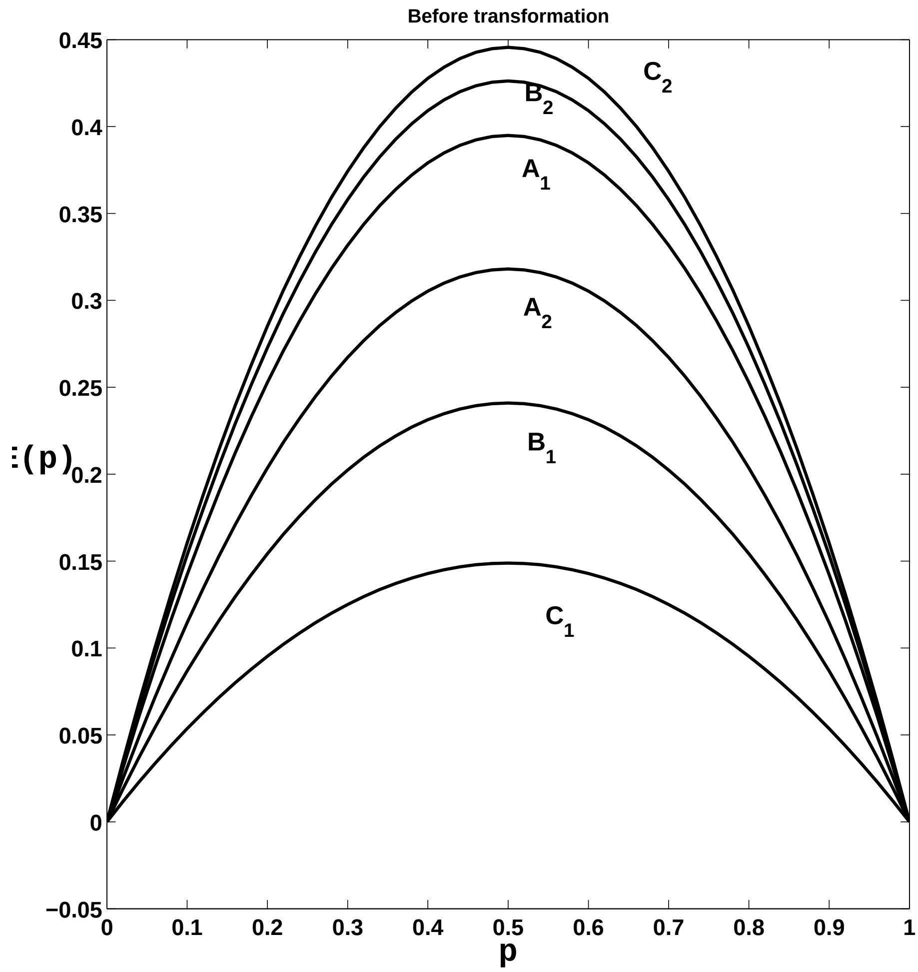

Figure 5.7: Linear entropy $E(p)$ for the states ρ_{123} of equation(5.19). Curve A_1 shows $E_1(p) = 1 - \mathrm{Tr}[\rho_1(\mathrm{p})^2]$; Curve A_2 shows $E_{23}(p) = 1 - \mathrm{Tr}[\rho_{23}(\mathrm{p})^2]$. Curve B_1 shows $E_2(p) = 1 - \mathrm{Tr}[\rho_2(\mathrm{p})^2]$; Curve B_2 shows $E_{13}(p) = 1 - \mathrm{Tr}[\rho_{13}(\mathrm{p})^2]$. Curve C_1 shows $E_3(p) = 1 - \mathrm{Tr}[\rho_3(\mathrm{p})^2]$; Curve C_2 shows $E_{12}(p) = 1 - \mathrm{Tr}[\rho_{12}(\mathrm{p})^2]$.

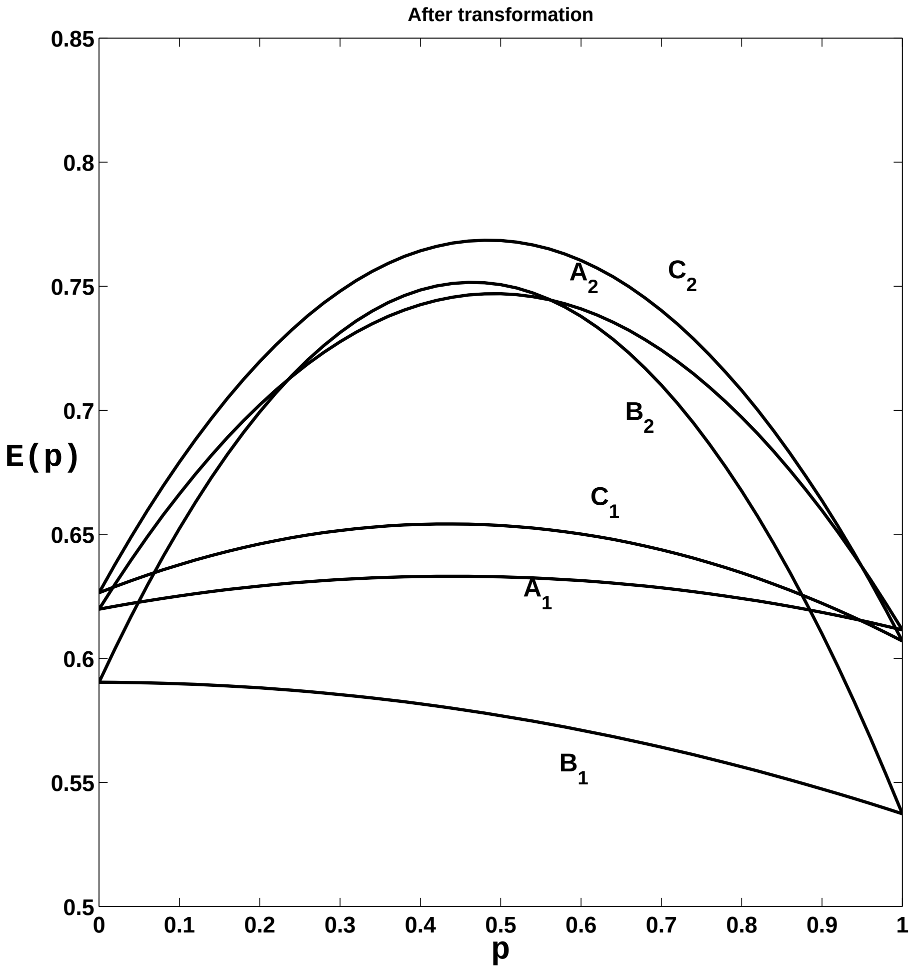

Figure 5.8: Linear entropy $E(p)$ for the states ρ'_{123} of equation(5.23). Curve A_1 shows $E_1(p) = 1 - \mathrm{Tr}[\rho'_1(\mathrm{p})^2]$; Curve A_2 shows $E_{23}(p) = 1 - \mathrm{Tr}[\rho'_{23}(\mathrm{p})^2]$. Curve B_1 shows $E_2(p) = 1 - \mathrm{Tr}[\rho'_2(\mathrm{p})^2]$; Curve B_2 shows $E_{13}(p) = 1 - \mathrm{Tr}[\rho'_{13}(\mathrm{p})^2]$. Curve C_1 shows $E_3(p) = 1 - \mathrm{Tr}[\rho'_3(\mathrm{p})^2]$; Curve C_2 shows $E_{12}(p) = 1 - \mathrm{Tr}[\rho'_{12}(\mathrm{p})^2]$.

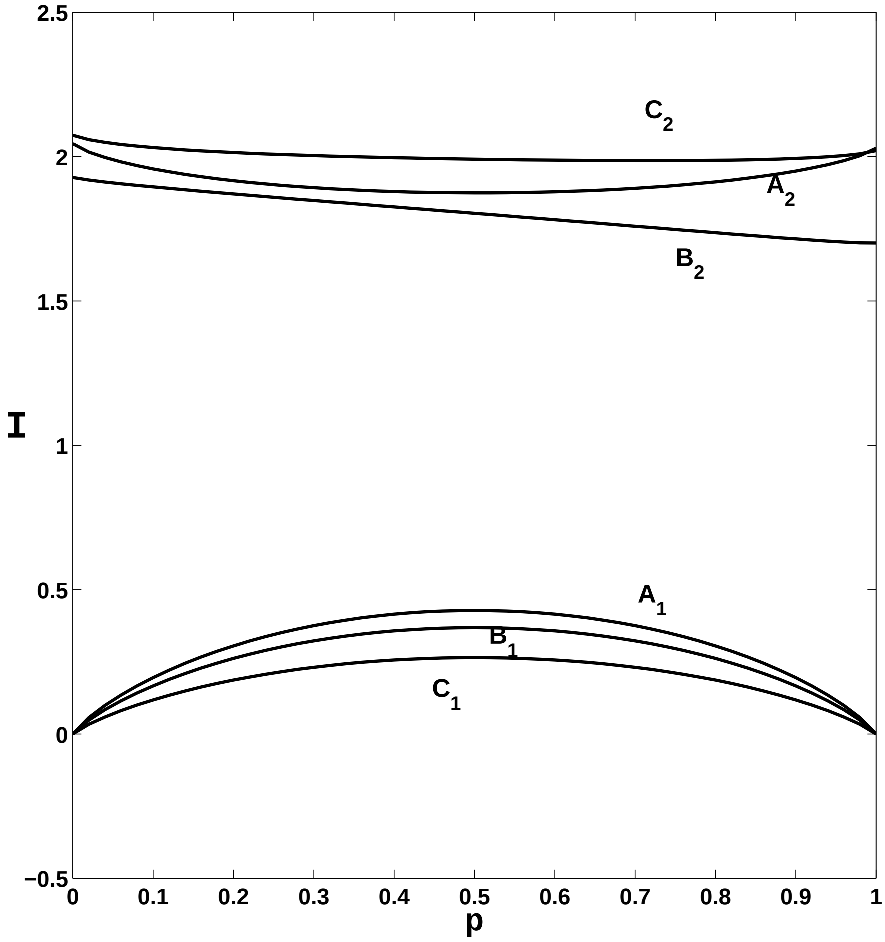

Figure 5.9: Mutual information $I(\rho_1, \rho_{23})$ (curve A1); $I(\rho_2, \rho_{13})$ (curve B1); $I(\rho_3, \rho_{12})$ (curve C1). The same quantities for the transformed states $\rho'_{123} = S_C \rho_{123} S_C^\dagger$ are also presented (curves A2,B2,C2).

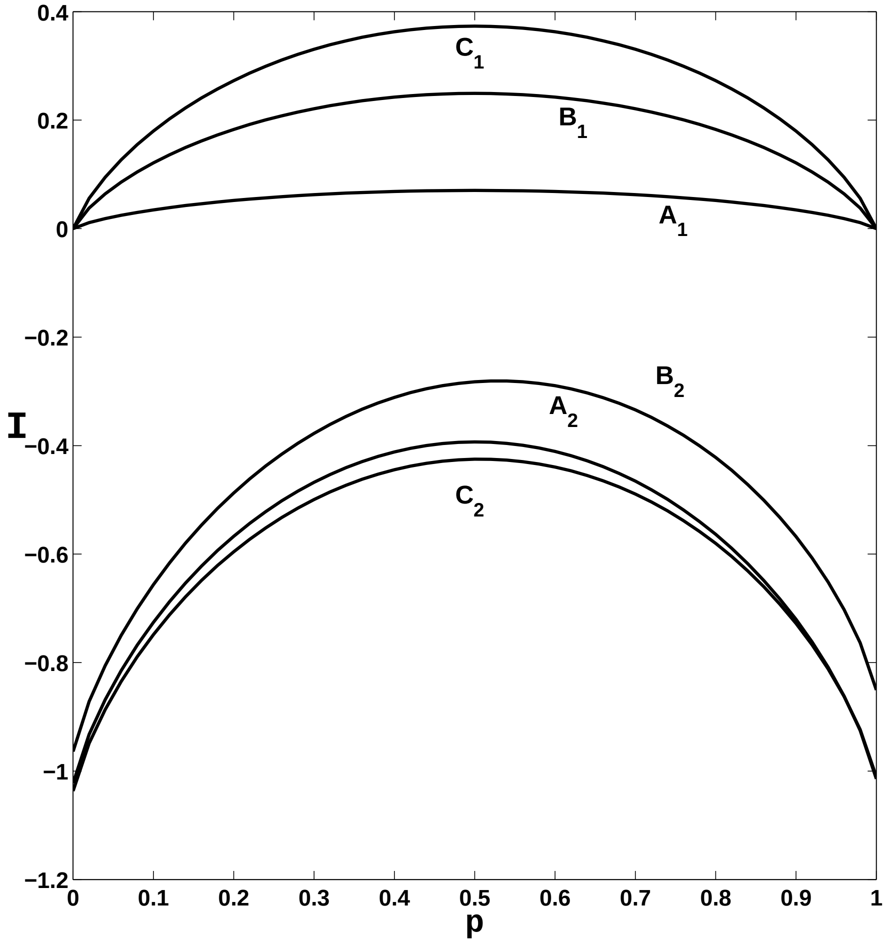

Figure 5.10: The conditional information $I(\rho_{23}|\rho_1)$ (curve A1); $I(\rho_{13}|\rho_2)$ (curve B1); $I(\rho_{12}|\rho_3)$ (curve C1). The same quantities for the transformed states $\rho'_{123} = S_C \rho_{123} S_C^{\dagger}$ are also presented (curves A2,B2,C2).

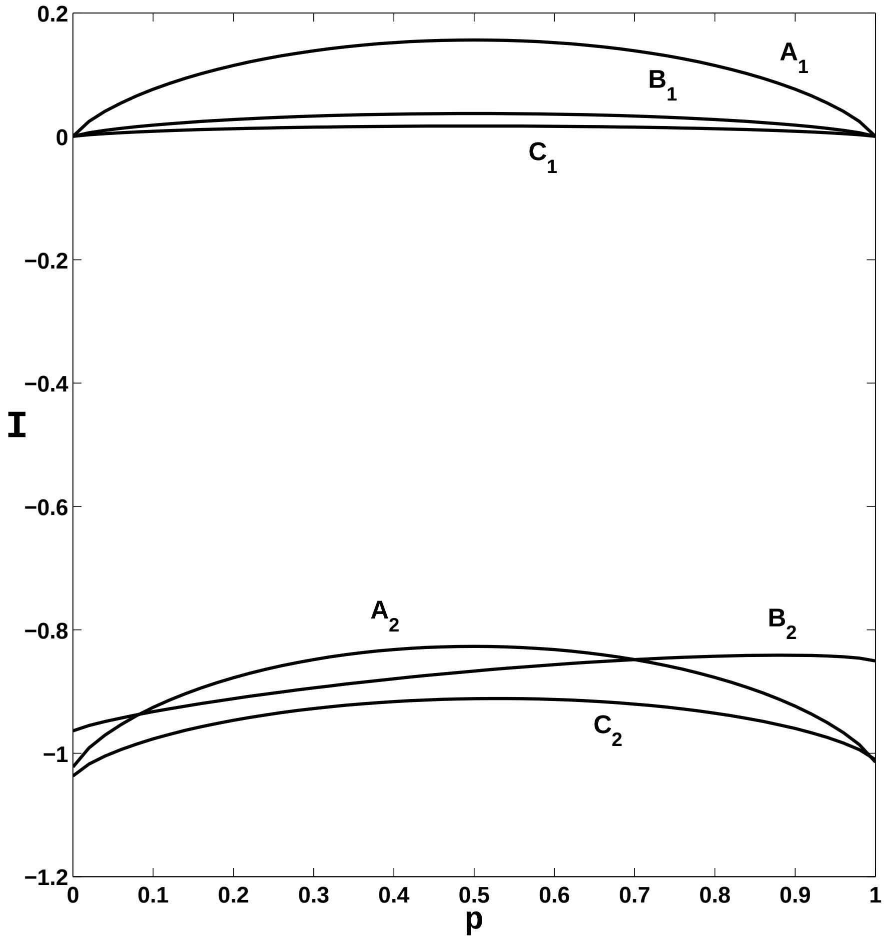

Figure 5.11: The conditional information $I(\rho_1|\rho_{23})$ (curve A1); $I(\rho_2|\rho_{13})$ (curve B1); $I(\rho_3|\rho_{12})$ (curve C1). The same quantities for the transformed states $\rho'_{123} = S_C \rho_{123} S_C^{\dagger}$ are also presented (curves A2,B2,C2).

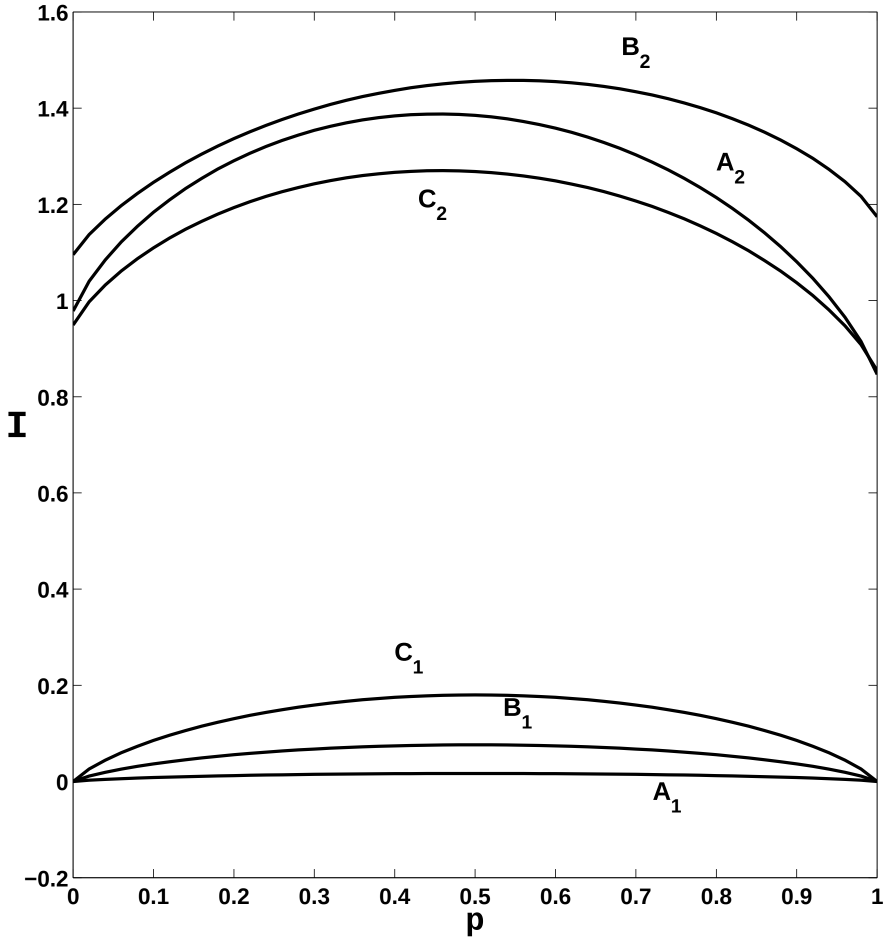

Figure 5.12: Conditional mutual information $I(\rho_2;\rho_3|\rho_1)$ (curve A1); $I(\rho_1;\rho_3|\rho_2)$ (curve B1); $I(\rho_1;\rho_2|\rho_3)$ (curve C1). The same quantities for the transformed states $\rho'_{123} = S_C\rho_{123}S_C^{\dagger}$ are also presented (curves A2,B2,C2).

5.5 Measurement in First Subsystem of Bi-partite Systems

We next consider a measurement in the first subsystem of a bi-partite system. We consider the projection operators is $\Pi_N = \pi_N \otimes \mathbf{1}$($\pi_N$ is in equation(2.59)). In this section, we use four different examples to calculate the entropic values after a measurement on a density matrix ρ. We present examples where ρ is separable state, entangled state. See Appendix C for the source codes (The MathWorks™ MATLAB). In this case, the new density matrix is

$$\rho_N^{mea} = \frac{\Pi_N \rho \Pi_N}{\mathrm{Tr} \rho \Pi_N} \tag{5.29}$$

where $N = 0, 1, \ldots, d-1$.

If we do a measurement without looking at the outcome then,

$$\begin{aligned} \widetilde{\rho} &= \sum_{N=0}^{d-1} \Pi_N \rho \Pi_N \\ &= \sum_{N=0}^{d-1} \mathrm{Tr}(\rho \Pi_N) \rho_N^{mea} \end{aligned} \tag{5.30}$$

where $N = 0, 1, \ldots, d-1$.

Calculation of the reduced density matrix of second subsystem gives

$$\rho_{2,N}^{mea} = \mathrm{Tr}_1 \rho_{12,N}^{mea} \tag{5.31}$$

for the case that the outcome is N, and

$$\widetilde{\rho_2} = \mathrm{Tr}_1 \widetilde{\rho_{12}} \tag{5.32}$$

for the case that we do not know the outcome.

The entropy of ρ_2, $\rho_{2,N}^{mea}$ and $\widetilde{\rho_2}$ are:

$$\begin{aligned} \mathfrak{S}(\rho_2) &= -\mathrm{Tr}(\rho_2 \ln \rho_2) \\ \mathfrak{S}(\rho_{2,N}^{mea}) &= -\mathrm{Tr}(\rho_{2,N}^{mea} \ln \rho_{2,N}^{mea}) \\ \mathfrak{S}(\widetilde{\rho_2}) &= -\mathrm{Tr}(\widetilde{\rho_2} \ln \widetilde{\rho_2}) \end{aligned} \tag{5.33}$$

5.5.1 Separable States

In this case, the density matrix is defined as

$$\rho_{12} = p(|a\rangle\langle a|) + (1-p)|b\rangle\langle b|, \quad \text{where} \ \ 0 \le p \le 1; \tag{5.34}$$

State $|a\rangle$ and $|b\rangle$ both same with equation(5.5). We calculate the value of quantities of the entropy $\mathfrak{S}(\rho_2)$, $\mathfrak{S}(\widetilde{\rho_2})$ and $\mathfrak{S}(\rho_{2,N}^{mea})$(equation(5.33)) with $p = 0.3$, the results show in Table(5.4).

Table 5.4: Measurement for separable states. Results of entropy for density matrix ρ_2, $\widetilde{\rho_2}$ and $\rho_{2,N}^{mea}$.

		The Entropy of the Second Subsystem
ρ_2 for Equation(5.33)		0.2988
$\widetilde{\rho_2}$ for Equation(5.33)		0.2988
$\rho_{2,N}^{mea}$ for equation (5.33)	Π_0	0.3379
	Π_1	0.0425
	Π_2	0.0869
	Π_3	0.3347
	Π_4	0.3287

We note that $\mathfrak{S}(\widetilde{\rho_2}) \leq \mathfrak{S}(\rho_2)$. This means that the measurement on the first subsystem increases on average, the knowledge about the second subsystem. However, we note that for $N = 0, \mathfrak{S}(\rho_{2,0}^{mea}) > \mathfrak{S}(\rho_2)$. So the previous statement is true on average. Figure(5.13) shows these quantities for $0 \leq p \leq 1$.

In Figure(5.13) we see that, for example, when $N = 0$, and $0 \leq p < 0.38$, $\mathfrak{S}(\rho_{2,0}^{mea}) > \mathfrak{S}(\rho_2)$, otherwise $\mathfrak{S}(\rho_{2,0}^{mea}) < \mathfrak{S}(\rho_2)$.

5.5.2 Entangled States

In this subsection, we consider the density matrix,

$$\rho_{12} \quad = \quad n(|a\rangle\langle a| + |a\rangle\langle b| + |b\rangle\langle a| + |b\rangle\langle b|) \tag{5.35}$$

where n is a normalization factor. States $|a\rangle$ and $|b\rangle$ are defined in equation(5.5). We calculate the values of the entropies $\mathfrak{S}(\rho_2)$, $\mathfrak{S}(\widetilde{\rho_2})$ and $\mathfrak{S}(\rho_{2,N}^{mea})$ in Table(5.5).

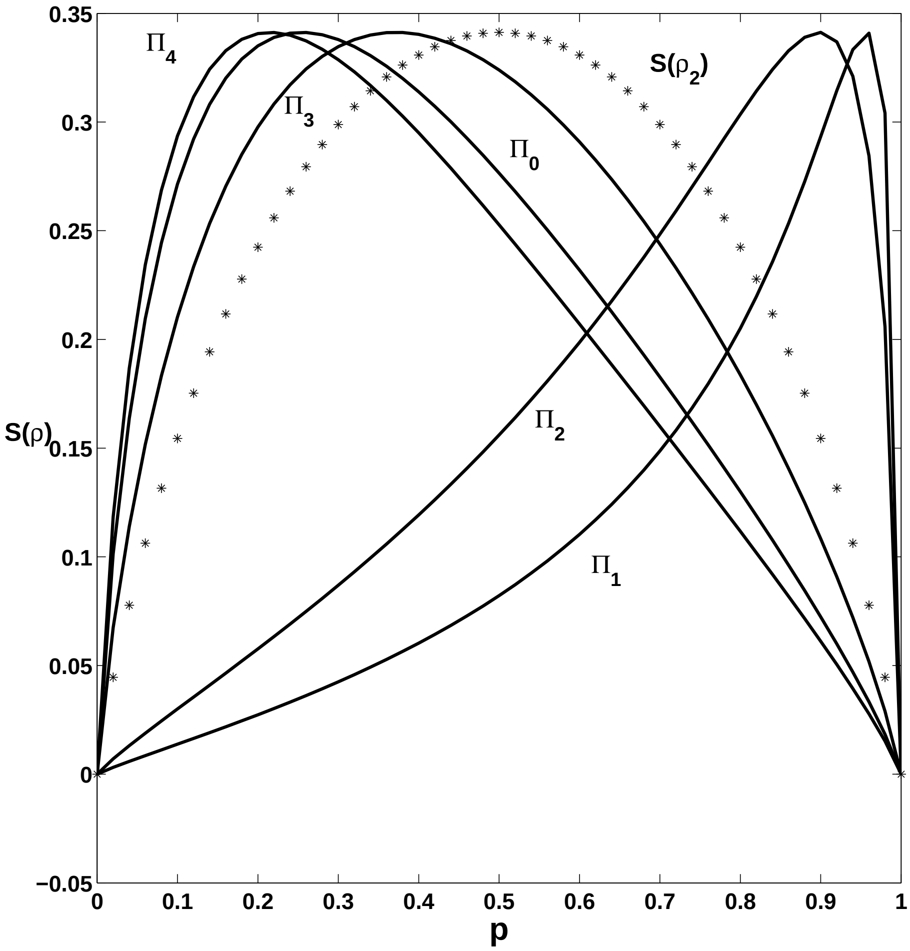

Figure 5.13: Entropy for the separate states ρ. Curve $S(\rho_2)$ shows the entropies of ρ_2 which is before the measurement. Curve $\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ shows the entropy results of ρ_2 which is after the measurements

Table 5.5: Measurement for entangled states. Results of entropy for density matrix ρ_2, $\widetilde{\rho_2}$ and $\rho_{2,N}^{mea}$.

<table>
<tr><td colspan="2"></td><td>The entropy of the second subsystem.</td></tr>
<tr><td colspan="2">ρ_2 for equation(5.33)</td><td>0.2071</td></tr>
<tr><td colspan="2">$\widetilde{\rho_2}$ for equation(5.33)</td><td>0.2071</td></tr>
<tr><td rowspan="5">$\rho_{2,N}^{mea}$ for equation (5.33)</td><td>Π_0</td><td>0</td></tr>
<tr><td>Π_1</td><td>0</td></tr>
<tr><td>Π_2</td><td>0</td></tr>
<tr><td>Π_3</td><td>0</td></tr>
<tr><td>Π_4</td><td>0</td></tr>
</table>

5.5.3 Measurement

Here we consider the states $\rho'_{12}(p) = S_A \rho_{12} S_A^\dagger$ and $\rho''_{12}(p) = S_B \rho_{12} S_B^\dagger$ defined in equation(5.9). The various of quantities $\mathfrak{S}(\rho_2)$, $\mathfrak{S}(\widetilde{\rho_2})$ and $\mathfrak{S}(\rho_{2,N}^{mea})$ are shown in Tables(5.6) and (5.7) correspondingly. Also the Figure(5.14) and (5.15) show the quantities of $\mathfrak{S}(\rho_2)$ and $\mathfrak{S}(\rho_{2,N}^{mea})$ for $0 \leq p \leq 1$.

Table 5.6: Measurement for example $\rho'_{12}(p) = S_A \rho_{12} S_A^\dagger$. Results of entropy for density matrix ρ_2, $\widetilde{\rho_2}$ and $\rho_{2,N}^{mea}$ with $p = 0.3$.

<table>
<tr><td colspan="2"></td><td>The entropy of the second subsystem</td></tr>
<tr><td colspan="2">ρ_2 for example $\rho'_{12}(0.3)$(sec.5.2)</td><td>1.4480</td></tr>
<tr><td colspan="2">$\widetilde{\rho_2}$ for equation(5.33)</td><td>1.4480</td></tr>
<tr><td rowspan="5">$\rho_{2,N}^{mea}$ for equation (5.33)</td><td>Π_0</td><td>0.4215</td></tr>
<tr><td>Π_1</td><td>0.3542</td></tr>
<tr><td>Π_2</td><td>0.5153</td></tr>
<tr><td>Π_3</td><td>0.3146</td></tr>
<tr><td>Π_4</td><td>0.5285</td></tr>
</table>

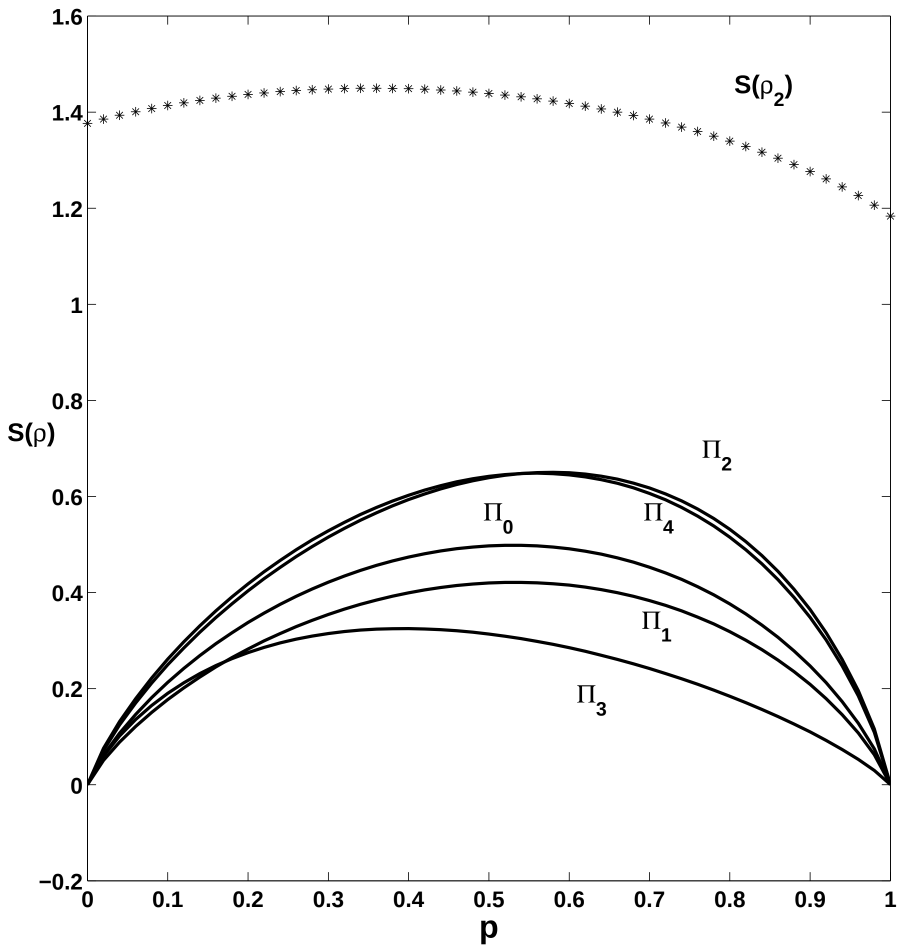

Figure 5.14: Entropy for example S_A. Curve $S(\rho_2)$ shows the entropies of ρ_2 which is before the measurement. Curve $\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ shows the entropy results of ρ_2 which is after the measurement.

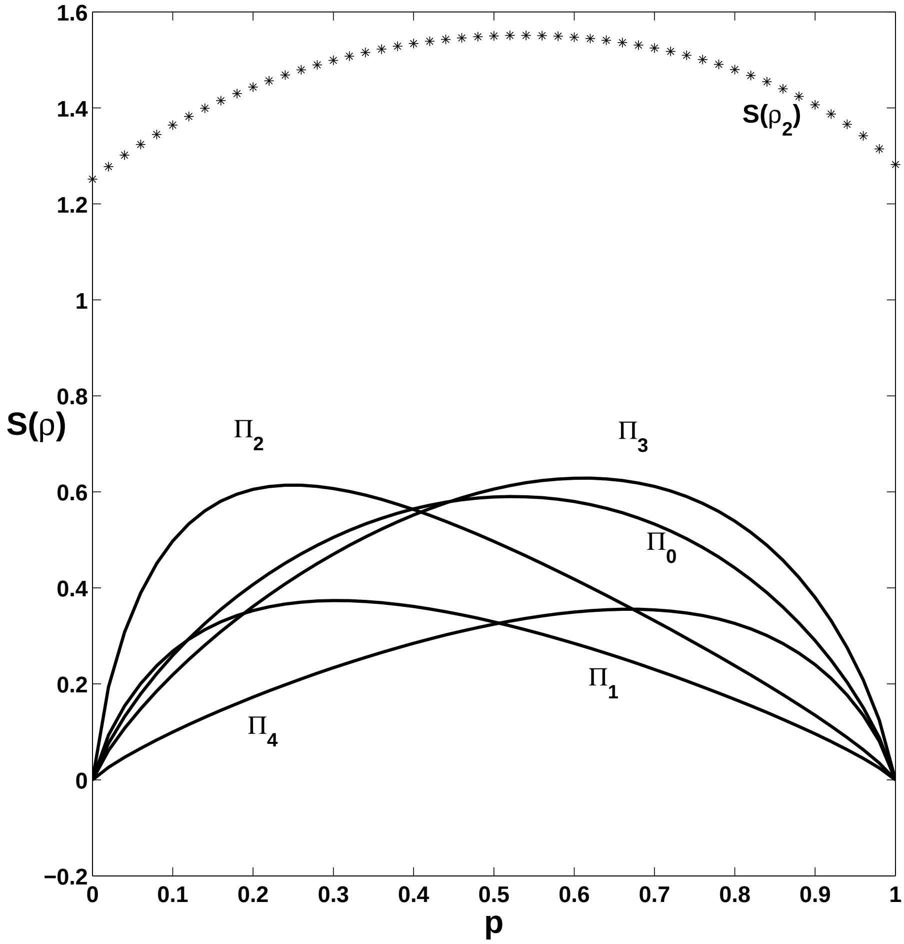

Figure 5.15: Entropy for example S_B. Curve $S(\rho_2)$ shows the entropies of ρ_2 which is before the measurement. Curve $\Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ shows the entropy results of ρ_2 which is after the measurement.

Table 5.7: Measurement for example $\rho''_{12}(p) = S_B \rho_{12} S_B^\dagger$. Results of entropy for density matrix ρ_2, $\widetilde{\rho_{2,N}}$ and $\rho_{2,N}^{mea}$ with $p = 0.3$.

		The Entropy of the Second Subsystem
ρ_2 for example $\rho''_{12}(0.3)$(sec.5.2)		1.4865
$\widetilde{\rho}_2$ for equation(5.33)		1.4865
$\rho_{2,N}^{mea}$ for equation (5.33)	Π_0	0.5095
	Π_1	0.6274
	Π_2	0.4727
	Π_3	0.4991
	Π_4	0.5674

5.6 Summary

In this chapter, we have seen how the measurement on the first subsystem affects our knowledge about the second subsystem. We have given various entropic quantities to demonstrate this.

Chapter 6

Conclusion

Outline:

- Conclusion

The symplectic $Sp(2, \mathbb{Z}_\wp)$ transformations play an important role in both classical and quantum mechanics. In classical mechanics they preserve the Poisson brackets . In quantum mechanics they preserve the commutation relations.

In the present context of finite quantum systems, they preserve equation(4.13) and this leads to the constraints of equation(4.16). For this reason, although all global unitary transformations can turn a product state into an entangled state, the symplectic ones are of special importance.

The main subject of this thesis is the study of the symplectic transformation in multi-partite finite quantum systems. Based on the commutativity of the projectors $\pi_i(m)$ and $\pi_i(n)$ in equation(4.23), we constructed a relatively simple method for the calculation of the symplectic operator S.

Using the symplectic operators S, we apply the symplectic transformation $Sp(4, \mathbb{Z}_\wp)$ in bi-partite systems. We also consider $Sp(6, \mathbb{Z}_\wp)$ transformations in tri-partite systems. We calculate various of numerical results in Table(5.2), and plot Figures(5.1-5.6). These proved that how the symplectic transformations affect the correlations and the entanglement between the two subsystems.

We also plot Figures(5.7-5.12) and present various entropic quantities in Table(5.3) for tri-partite systems. We consider in tri-partite systems, the correlations and entanglement are more complex between this three subsystems. We also consider measurement on one of the subsystems, and study its effect on the other subsystem.

Appendix A - Heisenberg-Weyl Group

For a position state is $|m\rangle_{\mathcal{X}}$, momentum state is $|m\rangle_{\mathcal{P}}$,

$$X^{\beta}Z^{\alpha} = Z^{\alpha}X^{\beta}\omega^{-\alpha\beta}, \qquad \alpha, \beta \in \mathcal{Z}_d \tag{6.1}$$

$$D(\alpha, \beta) = Z^{\alpha}X^{\beta}\omega(-2^{-1}\alpha\beta) \tag{6.2}$$

Proof equation $D(\alpha, \beta)|m\rangle_{\mathcal{X}} = \omega(2^{-1}\alpha\beta + \alpha m)|m + \beta\rangle_{\mathcal{X}}$

$$\begin{aligned}
D(\alpha, \beta)|m\rangle_{\mathcal{X}} &= Z^{\alpha}X^{\beta}\omega(-\frac{1}{2}\alpha\beta)|m\rangle_{\mathcal{X}} \\
&= X^{\beta}Z^{\alpha}\omega(\alpha\beta)\omega(-\frac{1}{2}\alpha\beta)|m\rangle_{\mathcal{X}} \\
&= X^{\beta}\omega(\alpha\beta)\omega(-\frac{1}{2}\alpha\beta)\omega(\alpha m)|m\rangle_{\mathcal{X}} \\
&= X^{\beta}\omega(\frac{1}{2}\alpha\beta + \alpha m)|m\rangle_{\mathcal{X}}
\end{aligned}$$

$$= \omega(\frac{1}{2}\alpha\beta + \alpha m)|m+\beta\rangle_{\mathcal{X}} \qquad (6.3)$$

Proof equation $D(\alpha, \beta)|m\rangle_{\mathcal{P}} = \omega(-2^{-1}\alpha\beta - \beta m)|m+\alpha\rangle_{\mathcal{P}}$

$$\begin{aligned}
D(\alpha, \beta)|m\rangle_{\mathcal{P}} &= Z^{\alpha}X^{\beta}\omega(-\frac{1}{2}\alpha\beta)|m\rangle_{\mathcal{P}} \\
&= Z^{\alpha}\omega(-\frac{1}{2}\alpha\beta)\omega(-m\beta)|m\rangle_{\mathcal{P}} \\
&= Z^{\alpha}\omega(-\frac{1}{2}\alpha\beta - \beta m)|m\rangle_{\mathcal{P}} \\
&= \omega(-\frac{1}{2}\alpha\beta - \beta m)|m+\alpha\rangle_{\mathcal{P}} \qquad (6.4)
\end{aligned}$$

Appendix B - Source Codes for Symplectic Transformation

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% ST.m                                                     %
% programmed by Lina Wang (email: L.WANG4@bradford.ac.uk)  %
%                                                          %
% Copyright (c) 2008-2009                                  %
% All Rights Reserved                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [  ] = ST( a11, a12, b11, b12, S, Sc )

% Get 'Kronecker Tensor Product' of the pure state |a> and |b>
% from inputs;
KTPa = kron(a11,a12);
KTPb = kron(b11,b12);
KTPaS = S*KTPa;
KTPbS = S*KTPb;
KTPaSc = Sc*KTPa;
KTPbSc = Sc*KTPb;

% Get Density Matrix of the pure states;
DMa = KTPa*KTPa';
DMb = KTPb*KTPb';
DMaS = KTPaS*KTPaS';
DMbS = KTPbS*KTPbS';
DMaSc = KTPaSc*KTPaSc';
DMbSc = KTPbSc*KTPbSc';

% Get (Partial) Transpose matrix;
Tm = [];
PTm = [];
TmS = [];
PTmS = [];
TmSc = [];
PTmSc = [];
for i = 1:5
    tempTm = [];
    tempPTm = [];
    tempTmS = [];
    tempPTmS = [];
```

```
    tempTmSc = [];
    tempPTmSc = [];
    for j = 1:5
        tempTm = [tempTm, (DMa((5*i-4):(5*i),(5*j-4):(5*j))).'];
        tempPTm = [tempPTm, (DMb((5*i-4):(5*i),(5*j-4):(5*j))).'];
        tempTmS = [tempTmS, (DMaS((5*i-4):(5*i),(5*j-4):(5*j))).'];
        tempPTmS = [tempPTmS, (DMbS((5*i-4):(5*i),(5*j-4):(5*j))).'];
        tempTmSc = [tempTmSc, (DMaSc((5*i-4):(5*i),(5*j-4):(5*j))).'];
        tempPTmSc = [tempPTmSc, (DMbSc((5*i-4):(5*i),(5*j-4):(5*j))).'];
    end
Tm = [Tm; tempTm];
  PTm = [PTm; tempPTm];
    TmS = [TmS; tempTmS];
  PTmS = [PTmS; tempPTmS];
    TmSc = [TmSc; tempTmSc];
  PTmSc = [PTmSc; tempPTmSc];
end

% Compute Reduce Density matrix;
counter = 1;
for lambda = 0:0.02:1
    rho = lambda*DMa + (1-lambda)*DMb;
    RDm_1 = [];
    RDm_2 = zeros(5);
    rhoS = lambda*DMaS + (1-lambda)*DMbS;
    RDmS_1 = [];
    RDmS_2 = zeros(5);
    rhoSc = lambda*DMaSc + (1-lambda)*DMbSc;
    RDmSc_1 = [];
    RDmSc_2 = zeros(5);
    for i = 1:5
        tempRho = [];
        tempTrace = [];
        tempRhoS = [];
        tempTraceS = [];
        tempRhoSc = [];
        tempTraceSc = [];
        for j = 1:5
            tempTrace = [tempTrace, trace(rho((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceS = [tempTraceS, trace(rhoS((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSc = [tempTraceSc, trace(rhoSc((5*i-4):(5*i),(5*j-4):(5*j)))];
            if(j==i)
                RDm_2 = RDm_2 + rho((5*i-4):(5*i),(5*j-4):(5*j));
                RDmS_2 = RDmS_2 + rhoS((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSc_2 = RDmSc_2 + rhoSc((5*i-4):(5*i),(5*j-4):(5*j));
            end
        end
        RDm_1 = [RDm_1; tempTrace];
        RDmS_1 = [RDmS_1; tempTraceS];
        RDmSc_1 = [RDmSc_1; tempTraceSc];
    end

    % Calculate Linear Entropics;
    LE_1(counter) = 1 - trace(RDm_1^2);
    LE_2(counter) = 1 - trace(RDm_2^2);
    LES_1(counter) = 1 - trace(RDmS_1^2);
    LES_2(counter) = 1 - trace(RDmS_2^2);
    LESc_1(counter) = 1 - trace(RDmSc_1^2);
    LESc_2(counter) = 1 - trace(RDmSc_2^2);

    % Calculate Entropics;
    E_1(counter) = trace(RDm_1*logmm(RDm_1));
```

```
    E_2(counter) = trace(RDm_2*logmm(RDm_2));
    E(counter) = trace(rho*logmm(rho));
    ES_1(counter) = trace(RDmS_1*logmm(RDmS_1));
    ES_2(counter) = trace(RDmS_2*logmm(RDmS_2));
    ES(counter) = trace(rhoS*logmm(rhoS));
    ESc_1(counter) = trace(RDmSc_1*logmm(RDmSc_1));
    ESc_2(counter) = trace(RDmSc_2*logmm(RDmSc_2));
    ESc(counter) = trace(rhoSc*logmm(rhoSc));

    % Calculate Negativities based on Density matrix;
    Dm = lambda*Tm + (1-lambda)*PTm;
    N(counter) = 0.5*(trace((Dm'*Dm)^(0.5))-1);
    DmS = lambda*TmS + (1-lambda)*PTmS;
    NS(counter) = 0.5*(trace((DmS'*DmS)^(0.5))-1);
    DmSc = lambda*TmSc + (1-lambda)*PTmSc;
    NSc(counter) = 0.5*(trace((DmSc'*DmSc)^(0.5))-1);

    % Calculate Mutual Information;
    MI(counter) = E(counter) - E_1(counter) - E_2(counter);
    MIS(counter) = ES(counter) - ES_1(counter) - ES_2(counter);
    MISc(counter) = ESc(counter) - ESc_1(counter) - ESc_2(counter);

    % Compute Conditional Entropies;
    CE_1(counter) = E_1(counter) - E(counter);
    CE_2(counter) = E_2(counter) - E(counter);
    CES_1(counter) = ES_1(counter) - ES(counter);
    CES_2(counter) = ES_2(counter) - ES(counter);
    CESc_1(counter) = ESc_1(counter) - ESc(counter);
    CESc_2(counter) = ESc_2(counter) - ESc(counter);

    counter = counter + 1;
end

% Plot the result figures;
figure(1);
plot(0:0.02:1, MI, 'k', 0:0.02:1, MIS, 'k', 0:0.02:1, MISc, 'k');
axis([0, 1, -0.5, 3.0]);

figure(2);
plot(0:0.02:1, CE_1, 'k', 0:0.02:1, CES_1, 'k', 0:0.02:1, CESc_1, 'k');
axis([0, 1, -1.4, 0.2]);

figure(3);
plot(0:0.02:1, CE_2, 'k', 0:0.02:1, CES_2, 'k', 0:0.02:1, CESc_2, 'k');
axis([0, 1, -1.4, 0.4]);

figure(4);
plot(0:0.02:1, N, 'k', 0:0.02:1, NS, 'k', 0:0.02:1, NSc, 'k');
axis([0, 1, -0.2, 1.8]);

figure(5);
plot(0:0.02:1, LE_1, '-k', 0:0.02:1, LE_2, '*k');
axis([0, 1, -0.05, 0.45]);

figure(6);
plot(0:0.02:1, LES_1, '-k', 0:0.02:1, LES_2, '*k', 0:0.02:1, LESc_1, '*k', 0:0.02:1, LESc_2, '*k');
axis([0, 1, 0.64, 0.76]);

end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% STTestDriver.m                                           %
% programmed by Lina Wang (email: L.WANG4@bradford.ac.uk)  %
%                                                          %
% Copyright (c) 2008-2009                                  %
% All Rights Reserved                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Initialize Matlab Workspace;
close all;
clear;
clc;

% Generate testing data for pure states
% |a> = a1 \otimes a2;
% and
% |b> = b1 \otimes b2;
a1 = [
    0.25 - 0.23i;
    0.05 - 0.07i;
    0.23 - 0.04i;
    0.33 - 0.41i;
    sqrt(1 - abs(0.25 - 0.23i)^2 - abs(0.05 - 0.07i)^2 ...
            - abs(0.23 - 0.04i)^2 - abs(0.33 - 0.41i)^2)
    ];

a2 = [
    0.54 + 0.31i;
    0.11 + 0.21i;
    0.09 + 0.41i;
    0.02 + 0.14i;
    sqrt(1 - abs(0.54 + 0.31i)^2 - abs(0.11 + 0.21i)^2 ...
            - abs(0.09 + 0.41i)^2 - abs(0.02 + 0.14i)^2)
    ];

b1 = [
    0.13 + 0.15i;
    0.25 + 0.32i;
    0.69 + 0.08i;
    0.06 + 0.4i;
    sqrt(1 - abs(0.13 + 0.15i)^2 - abs(0.25 + 0.32i)^2 ...
            - abs(0.69 + 0.08i)^2 - abs(0.06 + 0.4i)^2)
    ];

b2 = [
    0.125 + 0.035i;
    0.25 + 0.061i;
    0.134 + 0.235i;
    0.15 + 0.015i;
    sqrt(1 - abs(0.125 + 0.035i)^2 - abs(0.25 + 0.061i)^2 ...
            - abs(0.134 + 0.235i)^2 - abs(0.15 + 0.015i)^2)
    ];

% Generate parameter 'w';
w = exp(i*2*pi/5);

% Generate Pauli Operators for testing;
X = [
    0, 0, 0, 0, 1;
    1, 0, 0, 0, 0;
    0, 1, 0, 0, 0;
    0, 0, 1, 0, 0;
```

```
    0, 0, 0, 1, 0
    ];

Z = [
    w^(-2), 0,      0,      0,      0;
    0,      w^(-1), 0,      0,      0;
    0,      0,      w^(0),  0,      0;
    0,      0,      0,      w^1,    0;
    0,      0,      0,      0,      w^2
    ];

% Generate Symplectic Operator for testing;
X1 = kron(Z^(-1), X^(-2)*Z);
Z1 = kron(X^(2)*Z, X);
X2 = kron(X^(2)*Z, X^(2));
Z2 = kron(X, X^(-1)*Z);
S = Symplectic(X1, Z1, X2, Z2);

% Generate another Symplectic Operator for testing;
X1c = kron(X*Z, X*Z^(2));
Z1c = kron(X^(-2), X*Z);
X2c = kron(X^(-1), X*Z);
Z2c = kron(X^(-1)*Z, X^(-1)*Z);
Sc = Symplectic(X1c, Z1c, X2c, Z2c);

% Run Symplectic Transformation with generated testing data;
% 'ST' provides the Entanglement results;
ST(a1, a2, b1, b2, S, Sc);
```

Appendix C - Source Codes for Measurement

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Mea.m                                                    %
% programmed by Lina Wang (email: L.WANG4@bradford.ac.uk)  %
%                                                          %
% Copyright (c) 2008-2009                                  %
% All Rights Reserved                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [ SP, En, SA,SB ] = ...
    Mea( a, b, Proj1, Proj2, Proj3, Proj4, Proj5, SyA, SyB )
% Mea function goes to calculation the measurement results;
% Inputs:
% 'a' and 'b' is the pure states;
% Outputs:
% 'SP' is measurement results for Seperate state;
% 'En' is measurement results for Entangled state;
% 'SA' is measurement results for seperate state after
% Symplectic operater SyA;
% 'SB' is measurement results for seperate state after
% Symplectic operater SyB;

% Compute Reduce Density matrix for seperate state;
counter = 1;
for lambda = 0:0.02:1
    % Original density matrix;
    rho_Sp = lambda*(a*a') + (1-lambda)*b*b';
    RDmSp_2 = zeros(5);
    % The density matrix for unknown the outcome yet;
    rho_Sp1 = Proj1*rho_Sp*Proj1;
    rho_Sp2 = Proj2*rho_Sp*Proj2;
    rho_Sp3 = Proj3*rho_Sp*Proj3;
    rho_Sp4 = Proj4*rho_Sp*Proj4;
    rho_Sp5 = Proj5*rho_Sp*Proj5;
    rho_Sp_Sum = rho_Sp1 + rho_Sp2 + rho_Sp3 + rho_Sp4 + rho_Sp5;
    RDmSpSum_2 = zeros(5);
    % Get the measurement density matrix;
    rho_Sp_Mea1 = Proj1*rho_Sp*Proj1/(trace(rho_Sp*Proj1));
    RDmSpMea1_2 = zeros(5);
```

```
    rho_Sp_Mea2 = Proj2*rho_Sp*Proj2/(trace(rho_Sp*Proj2));
    RDmSpMea2_2 = zeros(5);
    rho_Sp_Mea3 = Proj3*rho_Sp*Proj3/(trace(rho_Sp*Proj3));
    RDmSpMea3_2 = zeros(5);
    rho_Sp_Mea4 = Proj4*rho_Sp*Proj4/(trace(rho_Sp*Proj4));
    RDmSpMea4_2 = zeros(5);
    rho_Sp_Mea5 = Proj5*rho_Sp*Proj5/(trace(rho_Sp*Proj5));
    RDmSpMea5_2 = zeros(5);
    for i = 1:5
        tempTrace = [];
        tempTraceSum = [];
        tempTraceMea1 = [];
        tempTraceMea2 = [];
        tempTraceMea3 = [];
        tempTraceMea4 = [];
        tempTraceMea5 = [];
        for j = 1:5
            tempTrace = [tempTrace, trace(rho_Sp((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSum = [tempTraceSum, trace(rho_Sp_Sum((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceMea1 = [tempTraceMea1, trace(rho_Sp_Mea1((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceMea2 = [tempTraceMea2, trace(rho_Sp_Mea2((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceMea3 = [tempTraceMea3, trace(rho_Sp_Mea3((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceMea4 = [tempTraceMea4, trace(rho_Sp_Mea4((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceMea5 = [tempTraceMea5, trace(rho_Sp_Mea5((5*i-4):(5*i),(5*j-4):(5*j)))];
            if(j==i)
                RDmSp_2 = RDmSp_2 + rho_Sp((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpSum_2 = RDmSpSum_2 + rho_Sp_Sum((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpMea1_2 = RDmSpMea1_2 + rho_Sp_Mea1((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpMea2_2 = RDmSpMea2_2 + rho_Sp_Mea2((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpMea3_2 = RDmSpMea3_2 + rho_Sp_Mea3((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpMea4_2 = RDmSpMea4_2 + rho_Sp_Mea4((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSpMea5_2 = RDmSpMea5_2 + rho_Sp_Mea5((5*i-4):(5*i),(5*j-4):(5*j));
            end
        end
    end
    E2(counter) = -trace(RDmSp_2*logm(RDmSp_2));
    ESum2(counter) = -trace(RDmSpSum_2*logm(RDmSpSum_2));
    EMea1_2(counter) = -trace(RDmSpMea1_2*logm(RDmSpMea1_2));
    EMea2_2(counter) = -trace(RDmSpMea2_2*logm(RDmSpMea2_2));
    EMea3_2(counter) = -trace(RDmSpMea3_2*logm(RDmSpMea3_2));
    EMea4_2(counter) = -trace(RDmSpMea4_2*logm(RDmSpMea4_2));
    EMea5_2(counter) = -trace(RDmSpMea5_2*logm(RDmSpMea5_2));

    counter=counter+1;
end

% When lambda = 0.3, get the entropy of \rho_2,
% \widetilde{{\rho_{2}}, and \rho_{2,N}^{mea}
% N is equal 0,1,...,4 for Seperate state.
SP = [E2(16); ESum2(16); EMea1_2(16); EMea2_2(16); EMea3_2(16); EMea4_2(16); EMea5_2(16)]

% Compute Reduce Density matrix for Entangled state;
% Original density matrix.
temp_rho_En =a*a' + a*b' + b*a' +b*b';
rho_En = temp_rho_En/trace(temp_rho_En);
RDmEn_2 = zeros(5);

% The density matrix for un-known the outcome yet;
rho_En1 = Proj1*rho_En*Proj1;
rho_En2 = Proj2*rho_En*Proj2;
rho_En3 = Proj3*rho_En*Proj3;
rho_En4 = Proj4*rho_En*Proj4;
```

```
rho_En5 = Proj5*rho_En*Proj5;
rho_En_Sum = rho_En1 + rho_En2 + rho_En3 + rho_En4 + rho_En5;
RDmEnSum_2 = zeros(5);

% Get the measurement density matrix;
rho_En_Mea1 = Proj1*rho_En*Proj1/(trace(rho_En*Proj1));
RDmEnMea1_2 = zeros(5);
rho_En_Mea2 = Proj2*rho_En*Proj2/(trace(rho_En*Proj2));
RDmEnMea2_2 = zeros(5);
rho_En_Mea3 = Proj3*rho_En*Proj3/(trace(rho_En*Proj3));
RDmEnMea3_2 = zeros(5);
rho_En_Mea4 = Proj4*rho_En*Proj4/(trace(rho_En*Proj4));
RDmEnMea4_2 = zeros(5);
rho_En_Mea5 = Proj5*rho_En*Proj5/(trace(rho_En*Proj5));
RDmEnMea5_2 = zeros(5);
for i = 1:5
    tempTraceEn = [];
    tempTraceEnSum = [];
    tempTraceEnMea1 = [];
    tempTraceEnMea2 = [];
    tempTraceEnMea3 = [];
    tempTraceEnMea4 = [];
    tempTraceEnMea5 = [];
    for j = 1:5
        tempTraceEn = [tempTraceEn, trace(rho_En((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnSum = [tempTraceEnSum, trace(rho_En_Sum((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnMea1 = [tempTraceEnMea1, trace(rho_En_Mea1((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnMea2 = [tempTraceEnMea2, trace(rho_En_Mea2((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnMea3 = [tempTraceEnMea3, trace(rho_En_Mea3((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnMea4 = [tempTraceEnMea4, trace(rho_En_Mea4((5*i-4):(5*i),(5*j-4):(5*j)))];
        tempTraceEnMea5 = [tempTraceEnMea5, trace(rho_En_Mea5((5*i-4):(5*i),(5*j-4):(5*j)))];
        if(j==i)
            RDmEn_2 = RDmEn_2 + rho_En((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnSum_2 = RDmEnSum_2 + rho_En_Sum((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnMea1_2 = RDmEnMea1_2 + rho_En_Mea1((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnMea2_2 = RDmEnMea2_2 + rho_En_Mea2((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnMea3_2 = RDmEnMea3_2 + rho_En_Mea3((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnMea4_2 = RDmEnMea4_2 + rho_En_Mea4((5*i-4):(5*i),(5*j-4):(5*j));
            RDmEnMea5_2 = RDmEnMea5_2 + rho_En_Mea5((5*i-4):(5*i),(5*j-4):(5*j));
        end
    end
end
EEn2 = -trace(RDmEn_2*logm(RDmEn_2));
EEnSum2 = -trace(RDmEnSum_2*logm(RDmEnSum_2));
EEnMea1_2 = -trace(RDmEnMea1_2*logm(RDmEnMea1_2));
EEnMea2_2 = -trace(RDmEnMea2_2*logm(RDmEnMea2_2));
EEnMea3_2 = -trace(RDmEnMea3_2*logm(RDmEnMea3_2));
EEnMea4_2 = -trace(RDmEnMea4_2*logm(RDmEnMea4_2));
EEnMea5_2 = -trace(RDmEnMea5_2*logm(RDmEnMea5_2));

%When p=0.3, get the entropy of \rho_2,
% \widetilde{{\rho_{2}}, and \rho_{2,N}^{mea}
% N is equal 0,1,...,4 for Entangled state.
En = [EEn2; EEnSum2; EEnMea1_2; EEnMea2_2; EEnMea3_2; EEnMea4_2; EEnMea5_2]

% Compute Reduce Density matrix for seperate state
% after the Symplecitc operator SyA;
counter = 1;
for lambda = 0:0.02:1
    % Original density matrix.
    rho_SA = lambda*((SyA*a)*(SyA*a)') + (1-lambda)*((SyA*b)*(SyA*b)');
    RDmSA_2 = zeros(5);
```

```
    % The density matrix for un-known the outcome yet;
    rho_SA1 = Proj1*rho_SA*Proj1;
    rho_SA2 = Proj2*rho_SA*Proj2;
    rho_SA3 = Proj3*rho_SA*Proj3;
    rho_SA4 = Proj4*rho_SA*Proj4;
    rho_SA5 = Proj5*rho_SA*Proj5;
    rho_SA_Sum = rho_SA1 + rho_SA2 + rho_SA3 + rho_SA4 + rho_SA5;
    RDmSASum_2 = zeros(5);
    % Get the measurement density matrix.
    rho_SA_Mea1 = Proj1*rho_SA*Proj1/(trace(rho_SA*Proj1));
    RDmSAMea1_2 = zeros(5);
    rho_SA_Mea2 = Proj2*rho_SA*Proj2/(trace(rho_SA*Proj2));
    RDmSAMea2_2 = zeros(5);
    rho_SA_Mea3 = Proj3*rho_SA*Proj3/(trace(rho_SA*Proj3));
    RDmSAMea3_2 = zeros(5);
    rho_SA_Mea4 = Proj4*rho_SA*Proj4/(trace(rho_SA*Proj4));
    RDmSAMea4_2 = zeros(5);
    rho_SA_Mea5 = Proj5*rho_SA*Proj5/(trace(rho_SA*Proj5));
    RDmSAMea5_2 = zeros(5);
    for i = 1:5
        tempTraceSA = [];
        tempTraceSASum = [];
        tempTraceSAMea1 = [];
        tempTraceSAMea2 = [];
        tempTraceSAMea3 = [];
        tempTraceSAMea4 = [];
        tempTraceSAMea5 = [];
        for j = 1:5
            tempTraceSA = [tempTraceSA, trace(rho_SA((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSASum = [tempTraceSASum, trace(rho_SA_Sum((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSAMea1 = [tempTraceSAMea1, trace(rho_SA_Mea1((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSAMea2 = [tempTraceSAMea2, trace(rho_SA_Mea2((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSAMea3 = [tempTraceSAMea3, trace(rho_SA_Mea3((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSAMea4 = [tempTraceSAMea4, trace(rho_SA_Mea4((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSAMea5 = [tempTraceSAMea5, trace(rho_SA_Mea5((5*i-4):(5*i),(5*j-4):(5*j)))];
            if(j==i)
                RDmSA_2 = RDmSA_2 + rho_SA((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSASum_2 = RDmSASum_2 + rho_SA_Sum((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSAMea1_2 = RDmSAMea1_2 + rho_SA_Mea1((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSAMea2_2 = RDmSAMea2_2 + rho_SA_Mea2((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSAMea3_2 = RDmSAMea3_2 + rho_SA_Mea3((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSAMea4_2 = RDmSAMea4_2 + rho_SA_Mea4((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSAMea5_2 = RDmSAMea5_2 + rho_SA_Mea5((5*i-4):(5*i),(5*j-4):(5*j));
            end
        end
    end
    ESA2(counter) = -trace(RDmSA_2*logm(RDmSA_2));
    ESASum2(counter) = -trace(RDmSASum_2*logm(RDmSASum_2));
    ESAMea1_2(counter) = -trace(RDmSAMea1_2*logm(RDmSAMea1_2));
    ESAMea2_2(counter) = -trace(RDmSAMea2_2*logm(RDmSAMea2_2));
    ESAMea3_2(counter) = -trace(RDmSAMea3_2*logm(RDmSAMea3_2));
    ESAMea4_2(counter) = -trace(RDmSAMea4_2*logm(RDmSAMea4_2));
    ESAMea5_2(counter) = -trace(RDmSAMea5_2*logm(RDmSAMea5_2));

    counter=counter+1;
end

% When lambda = 0.3, get the entropy of \rho_2,
% \widetilde{{\rho_{2}}, and \rho_{2,N}^{mea}
% N is equal 0,1,...,4 for Seperate state after symplectic operator SyA.
SA = [ESA2(16); ESASum2(16); ...
    ESAMea1_2(16); ESAMea2_2(16); ESAMea3_2(16); ESAMea4_2(16); ESAMea5_2(16)]
```

```
% Compute Reduce Density matrix for seperate state
% after the Symplecitc operator SyA;
counter = 1;
for lambda = 0:0.02:1
    % Original density matrix.
    rho_SB = lambda*((SyB*a)*(SyB*a)') + (1-lambda)*((SyB*b)*(SyB*b)');
    RDmSB_2 = zeros(5);
    % The density matrix for un-known the outcome;
    rho_SB1 = Proj1*rho_SB*Proj1;
    rho_SB2 = Proj2*rho_SB*Proj2;
    rho_SB3 = Proj3*rho_SB*Proj3;
    rho_SB4 = Proj4*rho_SB*Proj4;
    rho_SB5 = Proj5*rho_SB*Proj5;
    rho_SB_Sum = rho_SB1 + rho_SB2 + rho_SB3 + rho_SB4 + rho_SB5;
    RDmSBSum_2 = zeros(5);
    % Get the measurement density matrix;
    rho_SB_Mea1 = Proj1*rho_SB*Proj1/(trace(rho_SB*Proj1));
    RDmSBMea1_2 = zeros(5);
    rho_SB_Mea2 = Proj2*rho_SB*Proj2/(trace(rho_SB*Proj2));
    RDmSBMea2_2 = zeros(5);
    rho_SB_Mea3 = Proj3*rho_SB*Proj3/(trace(rho_SB*Proj3));
    RDmSBMea3_2 = zeros(5);
    rho_SB_Mea4 = Proj4*rho_SB*Proj4/(trace(rho_SB*Proj4));
    RDmSBMea4_2 = zeros(5);
    rho_SB_Mea5 = Proj5*rho_SB*Proj5/(trace(rho_SB*Proj5));
    RDmSBMea5_2 = zeros(5);
    for i = 1:5
        tempTraceSB = [];
        tempTraceSBSum = [];
        tempTraceSBMea1 = [];
        tempTraceSBMea2 = [];
        tempTraceSBMea3 = [];
        tempTraceSBMea4 = [];
        tempTraceSBMea5 = [];
        for j = 1:5
            tempTraceSB = [tempTraceSB, trace(rho_SB((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBSum = [tempTraceSBSum, trace(rho_SB_Sum((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBMea1 = [tempTraceSBMea1, trace(rho_SB_Mea1((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBMea2 = [tempTraceSBMea2, trace(rho_SB_Mea2((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBMea3 = [tempTraceSBMea3, trace(rho_SB_Mea3((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBMea4 = [tempTraceSBMea4, trace(rho_SB_Mea4((5*i-4):(5*i),(5*j-4):(5*j)))];
            tempTraceSBMea5 = [tempTraceSBMea5, trace(rho_SB_Mea5((5*i-4):(5*i),(5*j-4):(5*j)))];
            if(j==i)
                RDmSB_2 = RDmSB_2 + rho_SB((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBSum_2 = RDmSBSum_2 + rho_SB_Sum((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBMea1_2 = RDmSBMea1_2 + rho_SB_Mea1((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBMea2_2 = RDmSBMea2_2 + rho_SB_Mea2((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBMea3_2 = RDmSBMea3_2 + rho_SB_Mea3((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBMea4_2 = RDmSBMea4_2 + rho_SB_Mea4((5*i-4):(5*i),(5*j-4):(5*j));
                RDmSBMea5_2 = RDmSBMea5_2 + rho_SB_Mea5((5*i-4):(5*i),(5*j-4):(5*j));
            end
        end
    end
    ESB2(counter) = -trace(RDmSB_2*logm(RDmSB_2));
    ESBSum2(counter) = -trace(RDmSBSum_2*logm(RDmSBSum_2));
    ESBMea1_2(counter) = -trace(RDmSBMea1_2*logm(RDmSBMea1_2));
    ESBMea2_2(counter) = -trace(RDmSBMea2_2*logm(RDmSBMea2_2));
    ESBMea3_2(counter) = -trace(RDmSBMea3_2*logm(RDmSBMea3_2));
    ESBMea4_2(counter) = -trace(RDmSBMea4_2*logm(RDmSBMea4_2));
    ESBMea5_2(counter) = -trace(RDmSBMea5_2*logm(RDmSBMea5_2));
```

```
    counter=counter+1;
end

% When lambda = 0.3, get the entropy of \rho_2,
% \widetilde{{\rho_{2}}, and \rho_{2,N}^{mea}
% N is equal 0,1,...,4 for Seperate state after symplectic operator SyB.
SB = [ESB2(16); ESBSum2(16); ...
    ESBMea1_2(16); ESBMea2_2(16); ESBMea3_2(16); ESBMea4_2(16); ESBMea5_2(16)]

end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% MeaTestDriver.m                                          %
% programmed by Lina Wang (email: L.WANG4@bradford.ac.uk)  %
%                                                          %
% Copyright (c) 2008-2009                                  %
% All Rights Reserved                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Initialize Matlab Workspace;
close all;
clear;
clc;

% Generate testing data for pure states
% |a> = a1 \otimes a2;
% and
% |b> = b1 \otimes b2;
a1 = [
    0.25 - 0.23i;
    0.05 - 0.07i;
    0.23 - 0.04i;
    0.33 - 0.41i;
    sqrt(1 - abs(0.25 - 0.23i)^2 - abs(0.05 - 0.07i)^2 ...
            - abs(0.23 - 0.04i)^2 - abs(0.33 - 0.41i)^2)
    ];

a2 = [
    0.54 + 0.31i;
    0.11 + 0.21i;
    0.09 + 0.41i;
    0.02 + 0.14i;
    sqrt(1 - abs(0.54 + 0.31i)^2 - abs(0.11 + 0.21i)^2 ...
            - abs(0.09 + 0.41i)^2 - abs(0.02 + 0.14i)^2)
    ];

b1 = [
    0.13 + 0.15i;
    0.25 + 0.32i;
    0.69 + 0.08i;
    0.06 + 0.4i;
    sqrt(1 - abs(0.13 + 0.15i)^2 - abs(0.25 + 0.32i)^2 ...
            - abs(0.69 + 0.08i)^2 - abs(0.06 + 0.4i)^2)
    ];

b2 = [
    0.125 + 0.035i;
    0.25 + 0.061i;
    0.134 + 0.235i;
    0.15 + 0.015i;
    sqrt(1 - abs(0.125 + 0.035i)^2 - abs(0.25 + 0.061i)^2 ...
            - abs(0.134 + 0.235i)^2 - abs(0.15 + 0.015i)^2)
    ];

a = kron(a1,a2);
b = kron(b1,b2);
pi_0 = [
        1, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0
       ];
```

```
pi_1 = [
        0, 0, 0, 0, 0;
        0, 1, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0
       ];
pi_2 = [
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 1, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0
       ];
pi_3 = [
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 1, 0;
        0, 0, 0, 0, 0
       ];
pi_4 = [
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 0;
        0, 0, 0, 0, 1
       ];
II = eye(5);
PA0=kron(pi_0,II);
PA1=kron(pi_1,II);
PA2=kron(pi_2,II);
PA3=kron(pi_3,II);
PA4=kron(pi_4,II);

% Generate parameter 'w';
w = exp(i*2*pi/5);

% Generate Pauli Operators for testing;
X = [
    0, 0, 0, 0, 1;
    1, 0, 0, 0, 0;
    0, 1, 0, 0, 0;
    0, 0, 1, 0, 0;
    0, 0, 0, 1, 0
    ];

Z = [
    w^(-2), 0,      0,      0,      0;
    0,      w^(-1), 0,      0,      0;
    0,      0,      w^(0),  0,      0;
    0,      0,      0,      w^1,    0;
    0,      0,      0,      0,      w^2
    ];

% Generate Symplectic Operator for testing;
X1 = kron(Z^(-1), X^(-2)*Z);
Z1 = kron(X^(2)*Z, X);
X2 = kron(X^(2)*Z, X^(2));
Z2 = kron(X, X^(-1)*Z);
S = Symplectic(X1, Z1, X2, Z2);
```

```
% Generate another Symplectic Operator for testing;
X1c = kron(X*Z, X*Z^(2));
Z1c = kron(X^(-2), X*Z);
X2c = kron(X^(-1), X*Z);
Z2c = kron(X^(-1)*Z, X^(-1)*Z);
Sc = Symplectic(X1c, Z1c, X2c, Z2c);

[Sp, En, SA, SB] = Mea(a,b, PA0,PA1,PA2,PA3,PA4,S,Sc)
```

Bibliography

[1] H. Weyl, *Theory of groups and quantum mechanics*, Dover, New York, 1950.

[2] J. Schwinger, *Unitary operator bases*, Proc.Natl.Acad. Sci. U.S.A. 46, 570, 1960.

[3] J. Schwinger, *Quantum kinematics and dynamics*, Benjamin, New York, 1970.

[4] A.Vourdas, *Symplectically entangled states and their applications to coding*, J. Phys. A 37, 3305, 2004.

[5] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Quantum entanglement*, quant-ph/0702225, 2007.

[6] H. J. Briegel and R. Raussendorf, *Persistent entanglement in arrays of interacting particles*, Phys. Rev. Lett. 86, 910, 2001.

[7] A. hutton and S. Bose, *Mediated entanglement and correlations in a star network of interacting spins*, Phys. Rev. A69, 042312, 2004.

[8] P. Zanardi, *Entanglement of quantum evolutions*, Phys. Rev. A63, 040304, 2001.

[9] P. Stelmachovic and V. Buzek, *Quantum-information approach to the Ising model: Entanglement in chains of qubits*, Phys. Rev. A70, 032313, 2004.

[10] V. Vedral, *The role of relative entropy in quantum information theory*, Rev. Mod. Phys. 74, 197, 2002.

[11] W. Dur, G. Vidal, J. I. Cirac, n. *Linden and S. Popescu, entanglement capabilities of nonlocal hamiltonians*, Phys. Rev. Lett. 87, 137901, 2001.

[12] W. Dur and J. I. Cirac, *Nonlocal operations: Purification, storage, compression, tomography, and probabilistic implementation*, Phys. Rev. A64, 012317, 2001.

[13] J. I. Cirac, W. Dur, B. Kraus and M. Lewenstein, *Entangling operations and their implementation using a small amount of entanglement*, Phys. Rev. Lett. 86, 544, 2001.

[14] Courant, Richard and Hilbert, David , *Methods of mathematical physics*, Vol. I, Interscience, 1953.

[15] C. W. Gardiner, *Quantum noise*, Springer, Berlin, 1991.

[16] J.S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics 1, 195-200,1964.

[17] A. Einstein, B. Podolsky, N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?* Phys. Rev. 47, 777, 1935.

[18] E. Schrödinger, *Die Gegenwärtige Situation in der Quantenmechanik*, Naturwiss, 23. 807. 823 and 844, 1935.

[19] I. Percival, *Quantum state diffusion*, Queen mary and westfield college, London, england, 1991.

[20] J. F. Clauser and A. Shimony. *Bell's theorem: experimental tests and implications*, Rep. Prog. Phys., 41: 1881, 1978.

[21] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. H. Shih, *New high-intensity source of polarizatiion-ectangled photon pairs*, Phys. Rev. Lett, 75:4337, 1995.

[22] J. Von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1932.

[23] N. J. Cerf, C. Adami, *Negative entropy and information in quantum mechanics*, Phys. Rev. Lett. 79, 5194, 1997.

[24] N. J. Cerf, C. Adami, *Negative entropy in quantum information theory*, arXiv:quant-ph/9610005v1, 1996.

[25] M. Ferrero, A. van der Merwe (Eds.), *New developments on fundamental problems in quantum physics, fundamental theories of physics*, vol. 81, Kluwer Academic, Dordrecht, pp. 77-84, 1997.

[26] N. J. Cerf, C. Adami, *Information theory of quantum entanglement and measurement*, Phys. D, 120, 62-81, 1998.

[27] G. Vidal, R. F. Werner, *Computable measure of entanglement*, Phys. Rev. A, 65, 032314, 2002.

[28] G. Vidal and J. Mod, *Entanglement monotones*, Opt. 47, 355, 2000.

[29] A.Peres, Quantum Theory: *Concepts and methods*, Kluwer, Dordrecht, 1993.

[30] G. Jaeger, *Quantum information an overview*, Springer Science+Business Media, LLC, 2007.

[31] Nielsen, M. A., and I. L. Chuang, *Quantum computing and quantum information*, Cambridge University Press; Cambridge, 2000.

[32] K. Tadaki, *Upper bound by Kolmogorov complexity for the probability in computable POVM measurement*, arXiv:quant-ph/0212071v1, 2002.

[33] T. Decker, D. Janzing, M. Roetteler, *Implementation of group-covariant POVMs by orthogonal measurements*, arXiv:quant-ph/0407054v1, 2004.

[34] B.Reznik, *Remote generalized measurements (POVMs) require non-maximal entanglement*, arXiv:quant-ph/0203055v1, 2002.

[35] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge university press, 2000.

[36] A.Vourdas, *Quantum systems with finite Hilbert space*, Rep. Prog. Phys. 67, 267, 2004.

[37] A. Vourdas, *Transformations in the angle-angular-momentum phase space*, Phys. Rev. A, 43. 1564, 1991.

[38] A. Vourdas. *The angle-angular momentum quantum phase space*, J. Phys. A, 29. 4275, 1996.

[39] G. G. Athanasiu, E. G. Floratos and S. Nicolis, *Holomorphic quantization on the torus and finite quantum mechanics*, J. Phys. A, 29. 6737, 1996.

[40] D. Judge, *On the uncertainty relation for* ℓ_Z *and* φ, Phys. Lett., 5. 189, 1963.

[41] L. Auslander and R. Tolimieri, *Is computing with the finite fourier transform pure or applied mathematics?* Bull. Am. Math. Soc, 1 847, 1979.

[42] L. D. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Mathematics Subject Classification 47A60 - 47A67 - 22D25, 1991.

[43] J. R. Klauder, E. C. G. Sudarshan, *Fundamentals of quantum optics*, New York: Benjiamin, 1968.

[44] J. R. Klauder, B. S. Skagerstam, *Conherent states*, Spingpore: World Scientific, 1985.

[45] R. Loudon, *The quantum theory of light*, Oxford: Oxford University Press, 2000;

[46] A. Perelomov, *Generalized coherent states and their applications*, Berlin: Springer, 1986

[47] D. F. Walls, G. Milburn, *Quantum optics*, Berlin: Springer, 1994.

[48] Y. S. Kim, M. E. Noz, *Theory and applications of the poincare group*, Amsterdam: Reidel, 1986.

[49] Y. S. Kim, M. E. Noz, *Phase space picture of quantum mechanics*, Singapore: World Scientific, 1991.

[50] F. Strocchi, *An introduction to the mathematical structure of quantum mechanics: a short course for mathematicians*, World Scientific, 2005.

[51] Ronald Gautreau, William Savin, *Schaum's Outline of Theory and Problems of Modern Physics: modern physics*, McGraw-Hill Professional, 1999.

[52] E. U. Condon and H. Odabasi, *Atomic structure*, CUP Archive, 1980.

[53] D.Deutsch, *Uncertainty in quantum measurements*, Phys. Rev. Lett, 50 631, 1983.

[54] M. H. Partovi, *Entropic formulation of uncertainty for quantum measurements*, Phys. Rev. Lett, 50 1883, 1983.

[55] H. Maassen and J. B. M. Uffink, *Generalized entropic uncertainty relations*, Phys. Rev. Lett, 60 1103, 1988.

[56] K.Kraus, *Complementary observables and uncertainty relations*, Phys. Review D, 35 3070, 1987.

[57] V. Giovannetti, *Separability conditions form entropic uncertainty relations*, Phys. Review A, 70 012102, 2004

[58] Y. S. Kim and M. E. Noz, *Theory and aplications of the poincare group*, Amsterdam: Reidel, 1986.

[59] Y. S. Kim and M. E. Noz, *Phase space picture of quantum mechanics*, Singapore: World Scientific, 1991.

[60] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge: Cambridge University Press, 1984.

[61] A. H. Kastrup, *Quantization of the optical phase space* $S^2 = \{\phi mod 2\pi, I > 0\}$ *in terms of the group* $SO^{\uparrow}(1,2)$, 51 975-1134, 2003.

[62] A. Vourdas, *Quantum systems with finite hilbert space: Galois fields in quantum mechanics*, J. Phys. A: Math. Theor. 40. R285-R331, 2007.

[63] A. Vourdas, *Quantum systems with finite hilbert space*, Rep. Prog. Phys. 67. 267-320, 2004.

[64] D. D'Alessandro, *Introduction to quantum control and dynamics*, CRC Press, 2008.

[65] D.W. Robinson, D. Ruelle, *Mean entropy of states in classical statistical mechanics*, Commun. Math. Phys. 5, 288, 1967.

[66] O. E. Lanford and D.W. Robinson, *Mean entropy of states in quantum-statistical systems*, J. Math. Phys. 9, 1120, 1968.

[67] E.H. Lieb, M.B. Ruskai, *A fundamental property of quantum-mechanical entropy*, Phys. Rev. Lett. 30, 434, 1973.

[68] P. Hayden, R. Jozsa, D. Petz, A. Winter, *Structure of states which satisfy strong subadditivity of quantum entropy with equality*, Commun. Math. Phys. 246, 359, 2004.