Statistical arbitrage: Factor investing approach

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Abstract

We introduce a continuous time model for stock prices in a general factor representation with the noise driven by a geometric Brownian motion process. We derive the theoretical hitting probability distribution for the long-until-barrier strategies and the conditions for statistical arbitrage. We optimize our statistical arbitrage strategies with respect to the expected discounted returns and the Sharpe ratio. Bootstrapping results show that the theoretical hitting probability distribution is a realistic representation of the empirical hitting probabilities. We test the empirical performance of the long-until-barrier strategies using US equities and demonstrate that our trading rules can generate statistical arbitrage profits.

Keywords: Statistical arbitrage; factor models; trading strategies; geometric Brownian motion; Monte Carlo simulation.

JEL: G11, G12, G17

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1 Introduction

Statistical arbitrage strategies emerge from the widely used pairs trading strategy which dates back to 1980s. In pairs trading, it is assumed that there exist two stocks with similar characteristics, which are highly correlated after removing the effects of the common risk factors. Pairs trading exploits market overreaction to new information and whenever the co-integration residual between the two assets is larger than its statistical equilibrium level, the investor shorts the relatively more expensive asset while taking a long position in the cheaper asset (see Elliott et al. (2005) for details on pairs trading). Accordingly, most of the statistical arbitrage trading strategies exploit relative mispricing of assets based on the long-run statistical equilibrium levels or trading signals, and they often require data mining, advanced information technology infrastructure, and trading speed.

In the literature, the existence of statistical arbitrage opportunities refers to a riskless profit opportunity as time or number of trades goes to infinity. Avellaneda and Lee (2010) state that the trading strategies that exploit these opportunities have three common features as follows: (i) trading signals are systematic or rule based, (ii) trading book is market-neutral, and (iii) the mechanism for generating excess returns is statistical. Authors use this definition on US equities to analyse the empirical performance of model driven statistical arbitrage strategies. On the other hand, Hogan et al. (2004) provide a mathematical definition of statistical arbitrage which is used in testing the existence of statistical arbitrage opportunities in US equity markets. Later, Jarrow et al. (2012) revise this definition to relax the variance condition of statistical arbitrage proposed by Hogan et al. (2004).

Amongst others, examples of empirical studies on the performance of statistical arbitrage strategies are given by Elliott et al. (2005), Gatev et al. (2006), Do and Faff (2010), Avellaneda and Lee (2010) Cummins and Bucca (2012), Huck and Afawubo (2015). With the accumulation of big data and availability of necessary infrastructure for the high frequency trading, it also became possible to investigate statistical arbitrage in these environments (Stübinger et al., 2018; Stübinger and Endres, 2018; Stübinger, 2019; Nasekin and Härdle, 2019). Recently, there are also machine learning approaches developed to exploit the statistical arbitrage opportunities in different financial markets (Krauss et al., 2017; Knoll et al., 2019; Huck, 2019). Most of the empirical studies conclude that stock prices appear to contradict the efficient market hypothesis, where excess returns are obtained consistently with the trading signals based on the publicly available information.

In this study, instead of comparing various trading strategies in terms of their excess returns, we show under what conditions statistical arbitrage exists and how it can be exploited in an optimal way. Optimal statistical arbitrage trading for Itô diffusion processes and Ornstein-Uhlenbeck processes are studied by Bertram (2010) and Bertram (2009), respectively. However, these studies do not consider the existence issue with a mathematical definition of statistical arbitrage. One of the few mathematical definitions of statistical arbitrage is given by Bondarenko (2003). In this study, author derives a martingale type no-statistical arbitrage restriction assuming that financial derivatives are traded in the market. Göncü (2015) proves the existence of statistical arbitrage in the Black-Scholes framework via trading strategies that consist of holding an over-performing stock until it hits a deterministic barrier, whereas Göncü and Akyildirim (2017) extend these results for the multi-asset Black-Scholes framework. Focardi et al. (2016) introduce a statistical arbitrage strategy based on dynamic factor models of prices and empirically test it on US equities. Their results show that prices allow for significantly more accurate forecasts than returns in their model and their strategy passes the test for statistical arbitrage.[1]

As it is well known, mathematical programming is a significant part of operations research. Monte Carlo methods have also been proven to be important tools for solving these mathematical optimization problems. Monte Carlo techniques have many different application areas ranging from the traveling salesman problem (Rossier et al. (1986)) to vehicle rooting problems (Jabali et al. (2015)) and from queuing problems (Page (1965)) to stochastic job-shop scheduling (Yoshitomi and Yamaguchi (2003)). In this paper, we also show an application of Monte Carlo methods in finance. In particular, for the first time in the literature, we developed a Quasi-Monte Carlo procedure which significantly reduces the required number of simulations. We also test our theoretical results with the help of Monte Carlo simulations. Our results verify that simulated hitting time probabilities converge to the theoretical hitting time probabilities.

Operations research techniques have been applied to finance problems extensively. These problems have a broad spectrum changing from identifying market imperfections to pricing financial instruments. Among these problems, optimal trading strategies received particular attention from OR researchers because of the stochastic nature of the underlying assets. In this study, we also derive the optimal trading strategies in the framework of a factor model. In particular, we show the optimal trading schedule based on two objective functions: (i) maximizing the discounted expected trading profits or (ii) maximizing a Sharpe ratio type objective function. The statistical arbitrage condition we derive for the factor model allows the trader to optimize her/his trading strategy given her/his beliefs about the drift, volatility, and factor loading of the stock or the portfolio.

Our main contributions to the literature can be summarized as follows. First, we decom-

[1] For other noteworthy studies, see Mayordomo et al. (2014); Jarrow et al. (2019); Lutkebohmert and Sester (2020); Akyildirim et al. (2022).

pose asset returns in a general factor model framework which is similar to the factor model representation given by Avellaneda and Lee (2010). Second, we derive the statistical arbitrage condition in the factor model framework which is written in terms of the stock drift, volatility, and factor betas. Based on this condition, the trader decides the implementation of a long-until-barrier strategy. We show that the statistical arbitrage strategies utilized in this work satisfy the definitions of statistical arbitrage given by Hogan et al. (2004) and Avellaneda and Lee (2010). Third, we derive the theoretical hitting probabilities of the stock prices to the deterministic barrier levels in the factor model (We refer the reader to for instance Helmes et al. (2001), Locatelli (2001), Kraft and Steffensen (2006), Giesecke and Smelov (2013) for the application of hitting times in more general frameworks).Based on the hitting probabilities, we also derive optimal trading rules. Then, we compare the empirical hitting probabilities with the theoretical distribution derived via bootstrapping experiments. Fourth, we propose an efficient Quasi-Monte Carlo simulation method which decreases efficient simulation number significantly. With this methodology, we also verify the convergence of the simulated hitting time probabilities to the theoretical hitting time distribution for the time-dependent barrier levels.[2] We also perform bootstrapping experiments to verify the theoretical and empirical hitting probabilities. In this vein, statistical arbitrage trading strategies are implemented with the stock return data from the US market to check the empirical power of these strategies in terms of out-of-sample performance. Finally, as a robustness, we apply White (2000)'s reality check to test for any potential data snooping bias in the trading strategies considered.

It is important to note that we are not suggesting a method to forecast the future values of these parameters. Each investor might have a different estimate of the physical measure of the stock price returns, and thus, they speculate in the market. As in our empirical examples, the historical estimates obtained from an expanding in-sample estimation window might serve quite well to check the statistical arbitrage condition and generate statistical arbitrage profits. Furthermore, since the existence of our statistical arbitrage condition depends on an inequality, the precision of the parameter estimation does not need to be high as long as the direction of the inequality is estimated correctly.

The motivation for employing the factor model framework is its effectiveness in explaining stock returns, simplicity, and wide-spread use in the financial industry. Analytical tractability of the factor-based model enables us to derive analytical formulas for the hitting time distribution of the stock price for deterministic barriers. Furthermore, factor models are often used for risk management and performance evaluation of portfolio managers. Our statistical

[2]For the other applications of Monte Carlo methods in portfolio optimization, see for instance Christodoulakis (2002); Zhu and Fukushima (2009); Wang and Sloan (2011)

arbitrage methodology can be applied to sector or industry exchange traded funds (ETFs) with a relative value market neutral strategy by a long position in an investor's favourite sector ETF combined with a short position in a less attractive sector ETF. Amongst many choices of factor models in the literature, we employ the well-known Fama and French (1993) three factor model for our empirical examples. However, we are not constrained by the choice of this factor model and our framework applies to any other factor model as well. Another advantage of the factor models is that the estimation of the correlation matrix for thousands of assets is very difficult compared to finding or modelling the correlation between factors that explain the stock returns. Furthermore, when the information explained by the factors are extracted, the idiosyncratic noise in each asset is much closer to normality, which is consistent with the use of Brownian motion for log-returns.

The rest of this article is organized as follows. Section 2 explains the the stock price process in a general factor model. Section 3 presents our statistical arbitrage strategies with the derivation of the statistical arbitrage condition and shows that our statistical arbitrage strategies satisfy the definition by Avellaneda and Lee (2010). It also derives the optimal statistical arbitrage trading rules with respect to two alternative objective functions with possible constraints. Section 5 includes a variety of empirical and simulated experiments verifying the theoretical results. Monte Carlo experiments are conducted to verify the convergence of the simulated hitting probabilities to the derived theoretical hitting distribution of the stock prices for the deterministic barriers, whereas bootstrapping experiments are employed to verify the empirical hitting probabilities and the theoretical probabilities. Furthermore, empirical performance of the statistical arbitrage long-until-barrier strategy is also tested with the US equity market data. Finally, Section 6 concludes the article.

2 Statistical Arbitrage and Stock Price Model

Within our proposed framework, we utilize the following mathematical definition of statistical arbitrage as given by Hogan et al. (2004), but later we also show that our statistical arbitrage strategies satisfy the definition given by Avellaneda and Lee (2010) as well. Given the stochastic process $\{v(t) : t \geq 0\}$ for the discounted cumulative trading profits, which is defined on a probability space $(\Omega, \mathcal{F}, P)$, the statistical arbitrage is defined as follows (Hogan et al., 2004):

Definition 1 *A statistical arbitrage is a zero initial cost, self-financing trading strategy* $\{v(t) : t \geq 0\}$ *with cumulative discounted value* $v(t)$ *such that*

1. $v(0) = 0$

2. $\lim_{t\to\infty} E[v(t)] > 0$,

3. $\lim_{t\to\infty} P(v(t) < 0) = 0$, *and*

4. $\lim_{t\to\infty} var(v(t))/t = 0$ *if* $P(v(t) < 0) > 0, \quad \forall t < \infty$.

It is clear that a standard arbitrage opportunity is a special case of statistical arbitrage. A standard arbitrage strategy V has $V(0) = 0$ (self-financing) with a finite time T such that $P(V(t) > 0) > 0$ and $P(V(t) \geq 0) = 1$ for $t \geq T$ and the proceeds of this profit can be deposited into money market account for the rest of the infinite time horizon.

With regards to our stock price model, let $F_t^j := F_0^j + \sigma_j W_F^j(t)$ for $j = 1, 2, .., p$ be our factor processes where $W_F^1, W_F^2, ..., W_F^p$ are independent Brownian motions and σ_j is the volatility for the factor process F_t^j. We define our factor based stock price process such that it is based on the time averaged value of the integrated factors defined above,

$$S_t = \exp\left(\ln(S_0) + (\alpha - \sigma^2/2)t + \frac{1}{t}\sum_{j=1}^{p}\beta_j \int_0^t F_s^j ds + \sigma W_S(t)\right), \; j = 1, 2, .., p, \tag{1}$$

where W_S and W_F^j are independent Brownian motions from each other for all $j = 1, .., p$. Plugging in the definition of the factor process into the above equation, we obtain the following stock price model with a regular Brownian motion process

$$S_t = \exp\left(\ln(S_0) + (\alpha - \sigma^2/2)t + \sum_{j=1}^{p}\beta_j F_0^j + \frac{1}{t}\sum_{j=1}^{p}\beta_j \int_0^t \sigma_j W_F^j(s) ds + \sigma W_S(t)\right), \; j = 1, 2, .., p, \tag{2}$$

where $\sum_{j=1}^{p}\beta_j \int_0^t \sigma_j W_F^j(s)ds \sim N\left(0, \sum_{j=1}^{p}\frac{\beta_j^2\sigma_j^2 t^3}{3}\right)$. Hence, we obtain

$$S_t = \exp\left(\ln(S_0) + (\alpha - \sigma^2/2)t + \sum_{j=1}^{p}\beta_j F_0^j + \sqrt{\left(\sum_{j=1}^{p}\frac{\beta_j^2\sigma_j^2}{3} + \sigma^2\right)} W(t)\right), \; j = 1, 2, .., p, \tag{3}$$

where $W(t)$ is a convoluted Brownian motion. Moreover, following definitions of the first two moments of S_t will be useful further.

Remark 2 *The main reason for choosing an integrated factor construction is final estimation via* (22)*. This construction enabled us to use exact form of factors which are return processes by default. In the regression equation and provides reasonable estimation output. The non-integrated constructions would yield differences in returns due to log-difference of* *stock prices which would cause additional loss of information signal, namely use of white-noise factors in the end. This would possibly deform the significance of regression parameters.*

Remark 3 *In (3) we observe that the $\sum_{j=1}^{p}\beta_j F_0^j$ is independent from time t. Because, after decomposing the factor integral $\frac{1}{t}\sum_{j=1}^{p}\beta_j\int_0^t F_s^j ds$, we can clearly see that we have $\frac{1}{t}\left(\sum_{j=1}^{p}\beta_j F_0^j t\right)+\frac{1}{t}\sum_{j=1}^{p}\beta_j\int_0^t \sigma_j W_F^j(s)ds$ which easily yields $\sum_{j=1}^{p}\beta_j F_0^j+\frac{1}{t}\sum_{j=1}^{p}\beta_j\int_0^t \sigma_j W_F^j(s)ds$.*

$$\mathbb{E}\left(S_t\right) = \exp\left((\alpha+\frac{1}{6}\sum_{j=1}^{p}\beta_j^2\sigma_j^2)T+\sum_{j=1}^{p}\beta_j F_0^j\right) \tag{4}$$

$$\begin{aligned}\mathbb{V}\left(S_t\right) &= \exp\left(2\left(\alpha+\sigma^2+\frac{\sum_{j=1}^{p}\beta_j\sigma_j^2}{3}\right)T+\sum_{j=1}^{p}\beta_j F_0^j\right) \\ &\quad -\exp\left(\left(2\alpha+\frac{\sum_{j=1}^{p}\beta_j\sigma_j^2}{3}\right)T+2\sum_{j=1}^{p}\beta_j F_0^j\right)\end{aligned} \tag{5}$$

Next, we define the following first passage time problem:

$$\tau = \inf\left(t : t \geq 0, S_t \geq S_0(1+k)e^{r_f t}\right) \tag{6}$$

where k is a constant which determines the barrier level together with the risk free rate r_f.

Using the stock price process given in equation (3), we can rewrite the above problem as

$$\hat{\tau} = \inf\left(t : t \geq 0, \left(\alpha-\frac{1}{2}\sigma^2-r_f\right)t+\sum_{j=1}^{p}\beta_j F_0^j+\left(\sqrt{\frac{\sum_{j=1}^{p}\beta_j^2\sigma_j^2}{3}+\sigma^2}\right)W(t) \geq \log(1+k)\right) \tag{7}$$

Next, by using the analytical formulas given by Shreve (2004) for the first passage time density of the Brownian motion, we can write the density for the discounted stock price in the time averaged factor mode. First, we proceed by using the first passage cumulative distribution function of time changed Brownian motion in terms of the variable $\tau(t)$,

$$F(\tau(t)) = \int_0^{\tau(t)} p(\tau(s))d\tau(s) \tag{8}$$

Then first passage time density of $W\left(\tau(t)\right)$ becomes

$$p(t) = \frac{\partial F}{\partial p(\tau(t))}\frac{\partial p(\tau(t))}{\partial t} = p(\tau(t))\frac{\partial\tau(t)}{\partial t} \tag{9}$$

Using (9) we finally arrive at following formula,

$$p(t \in dt) = \frac{\log(1+k) - \left(\sum_{j=1}^{p} \beta_j F_0^j\right)}{\sqrt{2\pi \left(\sum_{j=1}^{p} \frac{\beta_j^2 \sigma_j^2}{3} + \sigma^2\right) t^3}} \exp\left(\frac{-\left(\log(1+k) - \sum_{j=1}^{p} \beta_j F_0^j - \alpha + r_f + \sigma^2/2\right)^2}{2\left(\frac{\sum_{j=1}^{p} \beta_j^2 \sigma_j^2}{3} + \sigma^2\right) t} \right) \quad (10)$$

This gives us the hitting probability of the stock price to the barrier level as follows:

$$\begin{aligned} P(\hat{\tau} \leq t) &= \Phi\left(\frac{(\alpha - r_f - \sigma^2/2)t - M}{\sqrt{Nt}}\right) \\ &\quad + \exp\left(\frac{2(\alpha - r_f - \sigma^2/2)M}{N}\right) \Phi\left(\frac{(-\alpha + r_f + \sigma^2/2)t - M}{\sqrt{Nt}}\right) \end{aligned} \quad (11)$$

where

$$M = \ln(1+k) - \sum_{j=1}^{p} \beta_j F_0^j, \quad (12)$$

$$N = \frac{\sum_{j=1}^{p} \beta_j^2 \sigma_j^2}{3} + \sigma^2, \quad (13)$$

and Φ is the cumulative distribution function for the normal distribution.

3 Statistical Arbitrage Strategy

We first start with a strategy that fails to satisfy the statistical arbitrage in Definition 1. A simple buy-and-hold strategy is shown to fail the conditions of statistical arbitrage due to the exponential growth of the variance of discounted cumulative profits over time whereas long-until-barrier strategies are shown to satisfy the statistical arbitrage conditions. The key in proving the statistical arbitrage conditions is to utilize a barrier level that grows proportional to the risk free rate and to show the finite first passage time to such boundaries. Therefore, by deriving the first passage time distribution under the factor model framework, we are able to design a statistical arbitrage strategy with almost sure finite first passage to the deterministic boundary that guarantees positive profit. The existence of statistical arbitrage is given with respect to certain conditions on the model parameters. Next, we discuss the buy-and-hold strategy.

3.1 Buy-and-Hold Strategy

The variance term derived in equation (5) shows that for $\alpha > r_f$, buy-and-hold strategies fail to yield statistical arbitrage since $\lim_{t\to\infty} var(v(t))/t = \infty$ and there is always a positive probability of loss (see Proposition 2 by Göncü (2015) for details). Buy-and-hold strategy fails to satisfy the Definition 1 since we are not able to control the variance as time increases. To be able to control the variance of the trading profits, we need to reduce the risky asset holding in our portfolio as time increases. Therefore, we introduce a barrier or termination condition to sell the risky asset held and invest all the proceeds in the risk-free account.

3.2 Long-Until-Barrier Strategy

Since the buy-and-hold strategy fails to satisfy the definition of statistical arbitrage, we define a deterministic barrier level $S_0(1+k)e^{r_f t}$ for the risky asset to sell it whenever its price reaches this level. This deterministic barrier grows proportional to the risk-free rate, which guarantees the positivity of the discounted trading profits whenever we close the position in the risky asset. However, in order to show the existence of statistical arbitrage in the sense of Definition 1, we need the finite first passage time of the underlying stochastic process to reach this barrier level.

The discounted cumulative trading profits from this strategy is given by

$$v(t) = \begin{cases} S_0 k & \text{if } \tau_k \in [0,t], \\ S_t e^{-r_f t} - S_0 & \text{else ,} \end{cases} \tag{14}$$

where the first passage time to the barrier is given in equation (6).

As proved in the previous section, for τ defined in equation (6), if the drift term satisfies $\mu = \alpha - \frac{\sigma^2}{2} + \sum_{j=1}^{p} F_0^j > 0$, then the first passage time of the stock price process to the boundary given by $S_0 e^{r_f t}(1+k)$ is guaranteed to be finite and the first passage time distribution is given in equation (11).

Generating statistical arbitrage profits thus boils down to having a good guess about the above condition. The finiteness of the first passage time to the barrier level implies that the variance and the probability of loss in our trading strategy will decay to zero for sufficiently large time t.

3.3 Optimal Barrier (Filter Rule)

The statistical arbitrage condition shows that as long as we have accurate estimation of the model parameters of asset prices, there exists statistical arbitrage opportunities in the economy. In this section, our goal is to find the optimal barrier level or in other words, the optimal limit order level for each time t that maximizes the expected profits or Sharpe ratio in a statistical arbitrage trading. Given the estimates or beliefs about the future alpha (α) and volatility of the stock, we find the optimal level k^* to buy/sell each risky asset in a portfolio. First, we start the with the maximization of the expected profits:

$$\begin{aligned} \max_k E[v(T)] &= \max_k \{E[v(T)|\tau_k < T]P(\tau_k < T) + E[v(T)|\tau_k \geq T]P(\tau_k \geq T)\} \\ &= \max_k \Bigg\{ kS_0(1 - P(\tau_k > T)) + \\ & \quad P(\tau_k > T)S_0 \left(\exp\left(\left(\alpha - r_f + \frac{1}{6}\sum_{j=1}^{p} \beta_j^2 \sigma_j^2 \right) T + \sum_{j=1}^{p} \beta_j F_0^j \right) - 1 \right) \Bigg\} \end{aligned} \tag{15}$$

The first order condition for the optimization is

$$\begin{aligned} \partial E[v(T)]/\partial k &= S_0(P(\tau_k < T)) + \\ & \quad S_0 \frac{\partial P(\tau_k > T)}{\partial k} \left(\exp\left((\alpha - r_f + \frac{1}{6}\sum_{j=1}^{p} \beta_j^2 \sigma_j^2)T + \sum_{j=1}^{p} \beta_j F_0^j \right) - 1 - k \right) = 0, \end{aligned} \tag{16}$$

where the optimal k, denoted by k^*, is the root of the non-linear equation (16). The technical details for the derivation of k^* are given in Appendix A.1.

Our second method to find the optimal k is the optimization of Sharpe ratio which can be defined as

$$\max_k \mathcal{S}_T = \max_k \frac{E[v(T)]}{E[\sqrt{var(v(T))}]} \tag{17}$$

From the above optimization problem we find the optimal k as follows:

$$k^* = \frac{F(k)^2 - F(k)}{F'(k)}, \tag{18}$$

where $F(k)$ refers to equation (11). The details of the derivation can be found in Appendix A.2.

3.4 Monte Carlo Experiments

In this section, we discuss the Monte Carlo experiments utilized to verify our theoretical results. We check whether the hitting probability distribution implied by the proposed model and its parameters are numerically correct. Moreover, we use this exact distribution to generate optimal barrier parameter k^* using different approaches. Therefore, the precision of the theoretical results have importance in this verification procedure.

The simulation of the maximum of Brownian motion is not a trivial process and depends on the simulation of Brownian motion paths. Here we provide a derivation which helps us to simulate this maximum value in an efficient way. Using the conditional distribution for the maximum of Brownian motion together with the given Brownian motion path, we can simulate the maximum value attained over a fixed time period. By using Corollary 3.7.4 (Shreve (2004)) together with Dambis, Dumbins-Schwarz Theorem 1.6 (Revuz and Yor (2004)), we write the conditional distribution of time-changed Brownian motion and its maximum as follows:

$$\begin{aligned} f_{M(\tau(t))|W(\tau(t))}(m|w) &= \frac{f_{M(\tau(t)),W(\tau(t))}(m,w)}{f_{W(\tau(t))}(w)}, \\ &= \frac{2(2m-w)}{\tau(t)\sqrt{2\pi\tau(t)}}\sqrt{2\pi t}e^{-\frac{(2m-w)^2}{2\tau(t)}+\frac{w^2}{2\tau(t)}}, \\ &= \frac{2(2m-w)}{\tau(t)}e^{\frac{-2m(m-w)}{\tau(t)}}. \end{aligned}$$

Using this conditional density, one can derive the maximum process as

$$P(M(\tau(t)) \leq m|W(\tau(t)) = w) = \int_w^m \frac{2(2u-w)}{\tau(t)}e^{\frac{-2u(u-w)}{\tau(t)}}du = 1 - e^{\frac{-2m(m-w)}{\tau(t)}}.$$

Given that any inverse cumulative distribution function is uniformly distributed, we can start by simulating the uniform distribution and continue by inverse transformation method:

$$\begin{aligned} 1 - e^{\frac{-2m(m-w)}{\tau(t)}} &= U, \\ \frac{-2m^2+2mw}{\tau(t)} &= \ln(1-U), \\ 2m^2 - 2mw + \tau(t)(\ln(1-U)) &= 2m^2 - 2mw + C = 0, \end{aligned}$$

where $C = \tau(t)(\ln(1-U))$ and since the above is a standard quadratic equation, it is trivial to find the roots as

$$m_{1,2} = \frac{2w \pm \sqrt{\Delta}}{4} \tag{19}$$

where $\Delta = 4w^2 - 8C$,.

Then by using the positive root from the above formula, we can obtain the maximum of Brownian motion. The exact solution for the roots of maximum process enables us to simulate the maximum of the Brownian motion without using any numerical root finding algorithm. The summary of our methodology can be outlined as follows:

(i) Simulate a Brownian motion path,

(ii) Simulate uniform random variable $U(0,1)$,

(iii) Then use equation (19) to find the roots m_1 & m_2 and select the positive root as the simulated maximum value.

Using the procedure above, we generate the exact simulation of $\max(S_t)$ and obtain more precise and efficient Monte Carlo results without resorting to fully discretized path simulation which can be computationally more expensive. Therefore, at each time point t_k, we simulate the following

$$S_{t_k} = S_0 \exp\left((\alpha - \sigma^2/2)t_k + \sum_{j=1}^{p} \beta_j F_0^j + \sqrt{\left(\sum_{j=1}^{p} \frac{\beta_j^2 \sigma_j^2}{3} + \sigma^2 \right) t_k} M_{t_k} \right) \tag{20}$$

where $M_{t_k} = \max(W_{t_k})|W_{t_k}$ and $W_{t_k} = \sqrt{t_k} Z_k$ where Z_k is a standard normal random variable. The parameters that are used in the simulation exercise are given in Table 3.

Insert Table 3 about here

From Figure 1, we observe that different Monte Carlo procedures perform well, and the error is significantly low even with 10,000 simulations. We obtain the highest precision with 100,000 simulations. In the analysis the exact density and distribution function come from (10) and (11), respectively. The results confirm that the simulated hitting time probabilities converge to the theoretical time probabilities. Moreover, in the analysis, we use Quasi-Monte Carlo simulation methods to achieve variance reduction and observe the minimum efficient number of simulations to converge exact distribution.

Effectively, Quasi-Monte Carlo simulation methods decrease the efficient simulation number to 10,000 (even 1,000), especially with the Sobol method. However, as it is clear from Table 1, both Halton and Sobol methods achieve better accuracy with less computational burden than the classical Monte-Carlo method. Moreover, we continue our efficiency analysis from a computational cost efficiency perspective. Table 2 shows that our analytical maximum BM simulation method allows us to obtain at least 243 times more computational cost reduction than the classical Monte Carlo.

We also conduct Monte Carlo simulations to test our theoretical results given by equations (4) and (5). In Figure 3, we observe the average trading profits, time-averaged variance, and the probability of loss for the buy-and-hold strategy. As it is consistent with our theoretical results, both mean and time average of variance go to infinity as time increases. However, if we repeat the same experiment for different investment horizons by introducing a barrier in the trading strategy, from Figure 4 it is clear that time average variance decays to zero, which in turn yields statistical arbitrage.

Insert Figures 1 & 3 & 4 about here

4 Estimation

In order to estimate the parameters of our base model given in equation (1), we first express the factors in an equivalent representation using the Ito's lemma. For any factor j, we have the following:

$$\begin{aligned}\beta_j \int_0^t F^j(s)ds &= \beta_j \left(F_0^j t + \sigma_j \int_0^t W_F^j(s)ds \right) \\ &= \beta_j \left(F_0^j t + W_F^j(t)t - \sigma_j \int_0^t s dW_F^j(s) \right)\end{aligned}$$

Then, using the stock price model in equation (1), we obtain:

$$\begin{aligned}\log(S_t) &= \log(S_0) + \left(\alpha - \frac{\sigma^2}{2}\right)t + \sum_{j=1}^p \left(\frac{\beta_j}{t} \int_0^t F^j(s)ds \right) + \sigma W_S(t) \\ &= \log(S_0) + \left(\alpha - \frac{\sigma^2}{2}\right)t + \sum_{j=1}^p \left(\underbrace{\beta_j F_0^j + \beta_j \sigma_j W_F^j(t)}_{\beta_j F^j(t)} \right) \\ &\quad \underbrace{- \sum_{j=1}^p \left(\frac{\beta_j \sigma_j}{t} \int_0^t s dW_F^j(s) \right) + \sigma W_S(t)}_{\eta(t)}\end{aligned}$$

Based on this representation, we obtain logarithmic returns through discretization:

$$\log\left(\frac{S_t}{S_{t-\Delta t}}\right) = \left(\alpha - \frac{\sigma^2}{2}\right)\Delta t + \sum_{j=1}^p \beta_j \Delta F_t^j + \underbrace{\Delta \eta(t)}_{\epsilon_t} \tag{21}$$

where $\epsilon_t \sim N\left(0, \Delta t \left(\frac{\sum_{j=1}^p \beta_j^2 \sigma_j^2}{3} \left(1 - \frac{2\Delta t}{3t}\right) + \sigma^2 \right)\right)$. For large values of t, the variance of the residuals ϵ_t is equal to $\Delta t \left(\frac{\sum_{j=1}^{p} \beta_j^2 \sigma_j^2}{3} + \sigma^2 \right)$.

Then, the regression equation can be written as

$$r_t = a + \sum_{j=1}^{p} \beta_j f_t^j + \epsilon_t \tag{22}$$

where $a = \left(\alpha - \frac{\sigma^2}{2} \right) \Delta t$, $r_t = \log \left(\frac{S_t}{S_{t-\Delta t}} \right)$ and F_t^j is the log of the factor, and thus, $f_t^j := \Delta F_t^j$ becomes the factor log-returns.

From the linear regression in equation (22) and setting the time increments to daily time horizon, we obtain the model parameters. First, the factor coefficients $\hat{\beta}_j$'s are obtained from the regression formula, then we obtain the remaining parameters $\hat{\alpha}$ and $\hat{\sigma}$ as

$$\begin{aligned} \hat{\sigma}_\epsilon^2 &= \frac{1}{T} \sum_{t=1}^{T} \epsilon_t^2 = \sum_{j=1}^{p} \frac{\beta_j^2 \sigma_j^2}{3} \left(1 - \frac{2}{3t} \right) + \hat{\sigma}^2 \\ \implies \hat{\sigma}^2 &= \frac{1}{T} \sum_{t=1}^{T} \epsilon_t^2 - \sum_{j=1}^{p} \frac{\beta_j^2 \sigma_j^2}{3} \left(1 - \frac{2}{3t} \right) \end{aligned} \tag{23}$$

Using this result, we proceed to write α as

$$\hat{\alpha} = \hat{a} + \frac{\hat{\sigma}^2}{2} \tag{24}$$

where $\hat{\alpha}, \hat{a}, \hat{\sigma}$ are sample estimates of α, a, σ. The volatility of each factor is obtained from the sample standard deviation over the considered sample of factors.

5 Empirical Analysis

This section describes the dataset that we use in the empirical testing of our theoretical results together with the estimation methodology explained in the previous section to find the parameters for our factor based stock price model. We also present the results of the bootstrapping experiments (see Section 5.2) comparing the theoretical hitting probabilities with the empirical hitting probabilities of the long-until-barrier type strategies with different investment horizons. Finally, long-until-barrier type strategies are back-tested in terms of out-of-sample performance for the optimal barrier levels that we derived earlier (see Section 5.3).

We conduct the back-testing of long-until-barrier type strategies to verify the theoretical results and find out if such strategies can be utilized as building blocks of more sophisticated trading strategies. Long-until-barrier strategies considered in the back-testing consist of long holding periods of the same portfolio of stocks and thus do not require many transactions. Therefore, transaction costs are assumed to be zero in the out-of-sample back-testing.

5.1 Data

We use the individual stock prices of the companies included in the Russell 3000 Index where the data is obtained from Bloomberg terminal. As factors, we utilize the well-known Fama-French factors which are obtained from Kenneth French's website[3]. In order to have enough number of observations for the empirical investigation, we consider the time period starting from 1 January 1990 up to 30 October 2020. After eliminating the stocks which have missing values more than 10% of the total number of observations, we are left with 480 stocks in the chosen time period. We use 7,770 daily observations of closing prices for stocks and the Fama-French factors to construct 24 equally weighted portfolios such that each of them includes 20 stocks chosen randomly. The monetary value of these 24 different portfolios in time are plotted in Figure 5 with all the initial values scaled to one dollar.

Insert Figure 5 about here

We see that all the portfolios suffer significant drawdown in three periods during the 1990 to 2020 period. The first major drawdown is observed during the 2001 dotcom bubble burst. The second market crash happened during and after the 2008 subprime mortgage crisis, whereas the last one is due to the global Covid-19 pandemic causing extreme downturns in the stock markets. Therefore, the sample period gives us sufficiently rich scenarios for testing the statistical arbitrage strategies and hitting probabilities not only during normal market conditions but also during market crashes.

Insert Table 4 about here

With regards to pricing factors, we employ the Fama-French three factor model where the factors are denoted by SMB (Small Minus Big), HML (High Minus Low), and $R_m - R_f$ (market excess return over the risk-free rate). The descriptive statistics for the log-returns of the equally weighted portfolios are given in Table 4. From this table, we observe that the maximum (minimum) value of 0.00035 (0.00017) for the mean of the daily log-returns is attained by the Portfolio 13 (7) which corresponds to around 8.82% (4.24%) in annual

[3] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

terms. It is also clear from the table that there is not much variation across the portfolios in terms of the daily fluctuations which on average corresponds to about 20% annualized volatility. The daily loss/profit values across the portfolios range from as low as around -16% up to 13% during the time period investigated. As mentioned before, it is important to note that the time period includes both 2001 and 2008 financial crises and the economic crash due to Covid-19 pandemic which are all worse than the great depression in terms of loss in market value. All portfolios have negative skewness, meaning that the left side tail of the log-returns distribution is longer or fatter than the right side of the distribution. Similarly, all the portfolios have kurtosis greater than three which indicates that the distributions have heavier tails than normal distribution because of the extreme events in the sample period.

5.2 Bootstrapping

Theoretical hitting probability distribution derived in equation (11) is tested via bootstrapping experiments. The main idea of bootstrapping follows selecting random subsamples of the complete dataset and comparing the theoretical hitting probability obtained from equation (11) with the actual hitting times of different portfolios of stocks over time. The theoretical hitting probabilities can be calculated for a given barrier level k and time horizon T. The inverse problem can be formulated as the calculation of the barrier level for a given holding period T and probability of hitting the barrier before time T. Therefore, in order to work with fixed probability events over random subsamples with fixed lengths, we vary the barrier level k accordingly across different portfolios of assets and over time. Therefore, we do not need to change the time span of each random subsample and probability, but instead use different implied barrier levels in each bootstrapping subsample.

We briefly summarize the bootstrapping procedure. We start by selecting random block of subsample of observations where the asset is held for T number of days to verify if the barrier is reached or not. By repeating the frequency count of the number of hits relative to the number of subsamples tested, we obtain the empirical probability of hitting the barrier for a given level of fixed probability and corresponding barrier level k. In other words, the bootstrapping procedure repeatedly selects a random starting point in time to check if the barrier is reached for a fixed time period and fixed probability while the corresponding k is calculated each time. For each subsample, the parameters are estimated by equation (22) and solving for the original model parameters. For example, if the first subsample is starting at the trading day 1,000, then we utilize all the past information to estimated model parameters up to day 1,000. Depending on the holding period, we check if the barrier is reached in the next T trading days or not. In the bootstrapping process, the subsampling is repeated for 300 times. By counting the ratio of barrier hits out of 300 trials, we obtain the estimate for the empirical hitting probability at different theoretical probability levels for given holding periods. Note that the barrier level k is calculated for a given time horizon and probability level. Bootstrapping is repeated for 24 different portfolios for different time horizons and different probability levels.

In Figure 6, we consider four different holding periods given as 20, 40, 60 and 80 business days and results are presented for theoretical hitting probabilities of 0.5, 0.6, 0.7, 0.8 and 0.9. Theoretical probabilities are given along the line, whereas the empirical probabilities obtained from the hitting ratios are given for 24 different portfolios. It is often observed that the empirical hitting ratios are relatively more likely than the theoretical probabilities. The normality of the noise term in the stock price model in equation (1) is not a perfect assumption to capture the properties of stock price returns. However, it allows for analytical tractability in the derivation of theoretical hitting density for long-until-barrier type strategies. Therefore, the bootstrapping exercise reveals the fact that asset returns exhibit jumps.

Insert Figure 6 about here

In Figure 6, lower probability levels such as at the 0.5 or 0.6 level, we observe relative larger difference between the empirical hitting probability versus the theoretical probabilities. In Tables 5-8, the empirical and theoretical probabilities are given for each portfolio and the corresponding barrier level k with different holding periods. Each portfolio has different set of parameters for the given sample of observations until the start date of the investment in the stock. Therefore, for each portfolio we calculate the barrier level k that gives us the same probability for the given holding period. From the Tables 5-8, it can be verified that the implied k levels that yield the fixed probability levels are stable across different portfolios with the average implied k values presented in the columns next to the empirical probability obtained from 300 randomly bootstrapped subsamples.

Insert Tables 5 & 6 & 7 & 8 about here

It can be observed that the empirical versus theoretical probabilities show similar behavior with larger differences at the lower probability levels. Note that lower probabilities such as 0.5 or 0.6 correspond to larger barrier level k and thus with the excess kurtosis of stock returns, it is not surprising to observe a higher difference for larger barrier levels. Empirical hitting ratios are higher than the theoretical rates and this shows that the likelihood of obtaining the statistical arbitrage is higher than what the model assumes. Therefore, bootstrapping results verify that the theoretical hitting distribution of the stock prices to the barrier level is a conservative approximation to the empirical hitting probabilities.

5.3 Backtesting the Statistical Arbitrage Strategies

In Section 3, we have given the statistical arbitrage condition for the long-until-barrier strategy and now, we verify the empirical performance of long-until-barrier strategies with the same 24 portfolios of stocks used in the bootstrapping method. For this purpose, we consider the long-until-barrier strategy with the optimal barrier levels calculated from the expected return and Sharpe ratio maximization given in equations (30) and (29), respectively.

In doing so, the first 200 observations are utilized as the initial in sample portion for the empirical backtesting. Parameters of the model are estimated dynamically with all the available past information up to the current trading day. For example, when a long position is opened on day i, all the previous data up to day i is used for the parameter estimation. Based on the estimated parameters, optimal barrier levels k are calculated from the expected return and Sharpe ratio maximization. Similar to the bootstrapping method, out-of-sample backtesting experiments are conducted for 20, 40, 60, and 80 holding days. The investor has the choice of implementing the long-until-barrier strategy at different investment horizons. However, if the investor has a short investment horizon then this strategy should be backtested with a higher frequency dataset instead of the daily price data.

In our framework, reasonable holding periods can be in the order of months. Holding periods that are too short would not be suitable for testing with the daily data, whereas very long holding periods would imply very few instances of the stock price process reaching the barrier levels empirically. Most importantly, the critical parameter is the trend of the stock price process. If the trend is expected to remain bullish for the near future, then the long-until-barrier strategy would perform and achieve its target price and the position is closed. On the other hand, if the investment horizon is much longer, the barrier level should be re-calculated daily with the re-calibrated parameters on a daily basis as well. In our backtesting methodology, an expanding window is utilized and we estimate the model parameters using all the data available until the date we open the long position and do not re-calibrate model parameters during the maximum holding period of 20, 40, 60, and 80 days, respectively. Once the trader opens a long position, we allow for the position to remain open up until three times the investment horizon. If the stock price still does not reach the barrier level by the maximum waiting period then the position is closed at the close price by the end of the maximum waiting period.

If the trader has the target holding period of 20 days implementing the long-until-barrier strategy, he opens the position at the closing price of the trading day i and is allowed to wait at most three times the 20 days, i.e. 60 trading days, to reach the barrier to close the position. At the end of the 60 days, the position is closed if the barrier level is not reached yet.

In Table 9, for all portfolios, we present the average returns arising from the out-of-sample backtesting with barrier levels obtained from the expected return maximization algorithm. In Table 10, the out-of-sample backtesting results from the Sharpe ratio maximization algorithm are given. It can be seen that in terms of the average performance across portfolios, barrier levels that optimize expected return and Sharpe ratio to perform indistinguishably similar. It should also be noted that we are using the closing prices in the backtesting methodology and it is possible that such a selection can cause an underestimation of the profitability of the strategy since the highest prices attained intraday are higher than the closing prices with more frequent realizations of the target profit levels. The smoother performance of the equal weighted statistical arbitrage strategies shows that by implementing such strategies across different portfolios or assets would also reduce the probability of loss due to unexpected changes in the trends of assets.

Insert Tables 9 & 10 about here

In Figures 7 and 8, we provide the histograms of the trading profits for the equally weighted statistical arbitrage trading strategies for the out-of-sample period with respect to different investment horizons. In Figure 7, the histogram is given for the statistical arbitrage strategy with the barrier levels derived from the expected return maximization, whereas in Figure 8, the barriers level are obtained from the Sharpe ratio optimization method.

Insert Figures 7 & 8 about here

Overall, the performance of the statistical arbitrage strategies are better at the 80-days investment horizon both with the expected return or Sharpe ratio maximizing barrier levels. For this investment horizon, we reach to an average annualized return of 11% with an annual volatility of 33%. The maximum drawdown in all the strategies are low, showing that such strategies can be implemented over diversified portfolios of assets as long as the trend of the asset remains the same over the short investment horizons.

5.4 Testing for Data Snooping Bias

Statistical arbitrage trading strategies should be tested for the existence of data snooping bias. In particular, trading strategies that utilize the long-until-barrier strategies might suffer from the data snooping effects due to potential biases in the considered time period. Trading scenarios that study the performance of general trading strategies end up with too many combinations or trading rules that produce a wide range of performance results. However, by using expected return and Sharpe ratio maximization methods for the selection of the optimal barrier levels as two main alternatives for the implementation of the trading strategy, we obtain only two types of trading strategies that is applied in the same way for all the wide range of random portfolios constructed. Therefore, our methodology is not expected to suffer from data snooping bias due to its standardized methodology for the implementation of trading strategies. In any case, we control for potential data snooping bias over different sample periods and with new datasets.

Due to the convention in testing trading strategies, White (2000)'s reality check is applied to verify for any potential data snooping bias in our trading strategies. We test the null hypothesis which states that the maximum average return of the trading rules being tested is same as the average return of the benchmark, against the alternative that the maximum average return of the trading rules is greater than the average return of the benchmark. The data mining test is employed on the daily return series obtained from the implementation of the long-until-barrier type trading strategies. Four different bootstrap block sizes are considered, namely 20, 40, 60 and 80 business days of block sub-sampling. As each size of the robustness test, the bootstrapping method is implemented on 500 replications.

Insert Tables 11 about here

In Table 11, the p-values obtained from the White's reality check are given for the expected return and Sharpe ratio maximization strategies, respectively. The findings show that our results do not suffer from data snooping bias since the majority of the portfolios have p-values less than %10. Especially, we observe that the p-values decrease significantly for the Sharpe ratio optimization with increasing holding periods.

6 Conclusion

In this article, we introduce a factor model framework for the statistical arbitrage strategies. Within this framework, we derive the the hitting probability distribution of the stock prices to deterministic barrier levels. Furthermore, Monte Carlo simulations are utilized to verify the theoretical probabilities of the hitting time distribution derived. Derivation of the hitting time distribution of the stock price process to the barrier level is crucial in order to prove the existence of a statistical arbitrage strategy. Although the factor model framework offers us analytical tractability in the derivation of the hitting time distribution of the stock prices to a deterministic barrier level, it has its own limitations. The major limitation is the normality assumption of the noise term of the factor model representation. This assumption reflects itself as the underestimation of the empirical hitting probabilities in comparison to the theoretical hitting time distribution derived. This implies that the likelihood of reaching and obtaining the target profit levels in the long-until-barrier strategies is higher compared to the model assumptions as long as the trend of the stock price process remains positive. Bootstrapping exercises are implemented to verify the behavior of the empirical hitting time probabilities versus the theoretical probabilities from the distribution function derived.

Next, we derive the condition (in terms of the drift, volatility, and factor loadings of the stock price process) that guarantees the existence of the statistical arbitrage. As the second step, we consider alternative methods to obtain best barrier level by optimizing different objective functions. Optimal statistical arbitrage trading strategies are derived based on the probability of hitting a deterministic barrier level to terminate the strategy. We derive the optimal barrier levels to close the long positions for a given investment horizon based on the expected discounted profit, or alternatively, with respect to the Sharpe ratio maximization.

Finally, we apply backtesting to verify the out-of-sample performance of long-until-barrier type strategies across a wide range of portfolios of assets in the US equity market. Results show that although the definition of statistical arbitrage depends on the information on the future values of the model parameters, with relatively short holding periods, most of the time barrier levels are reached and the investor can lock in the excess return. This is simply because during the sample period, we observe overall positive drift and trend in the portfolios with the exception of two major financial turmoil periods. It is also natural that statistical arbitrage strategies introduced in our theoretical framework are not perfect and they also show higher drawdown during unexpected market crashes. However, with the use of intraday data, such strategies can be modified to control for drawdown at intraday holding periods and accompanying stop-loss levels can be introduced for designing trading strategies that can handle the market risk better.

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Figure 1: Monte Carlo and Quasi Monte Carlo Convergence of empirical hitting probabilities to the theoretical hitting probabilities derived in the hitting time distribution.

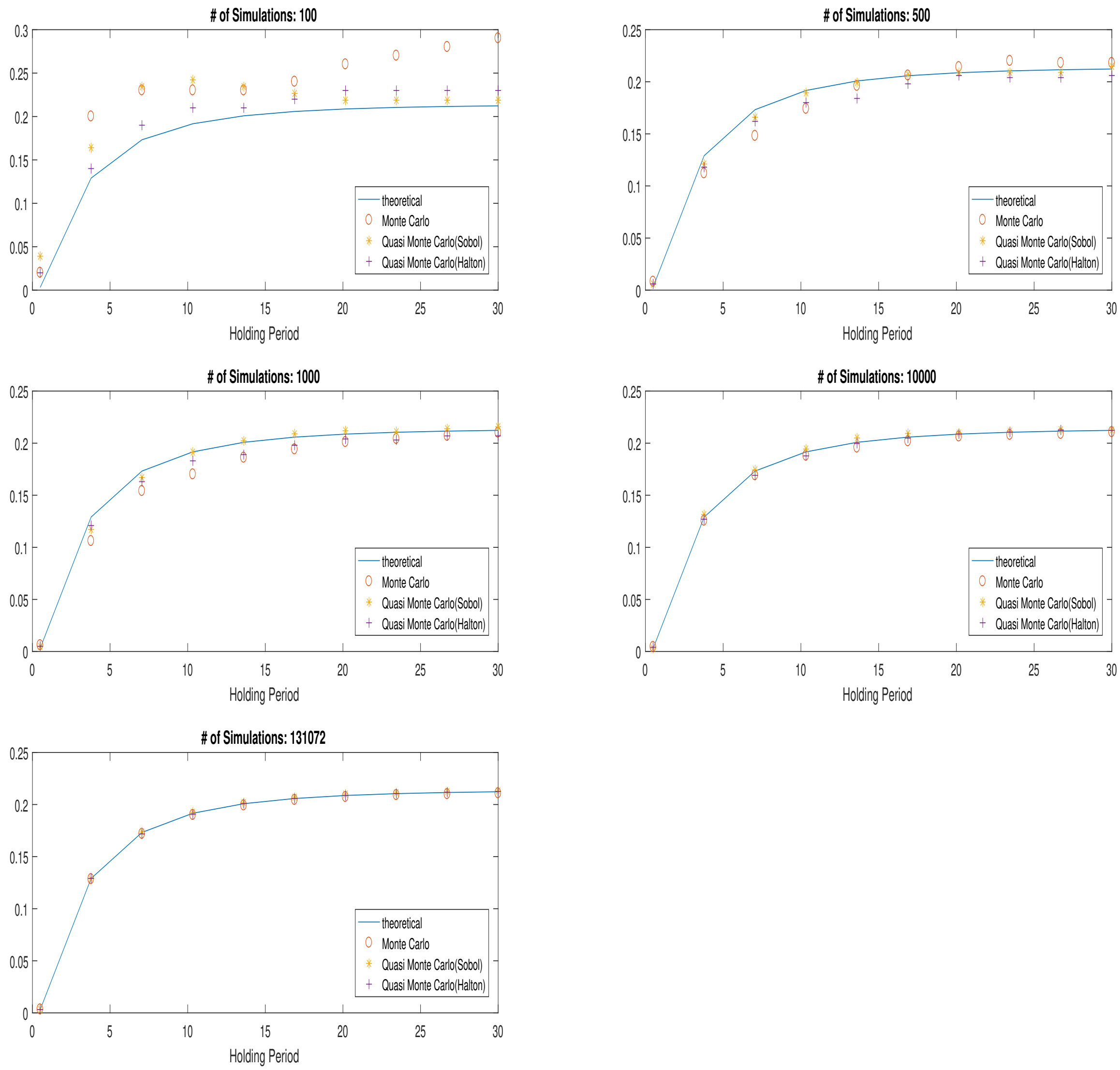

Figure 2: Concavity of Objective Functions

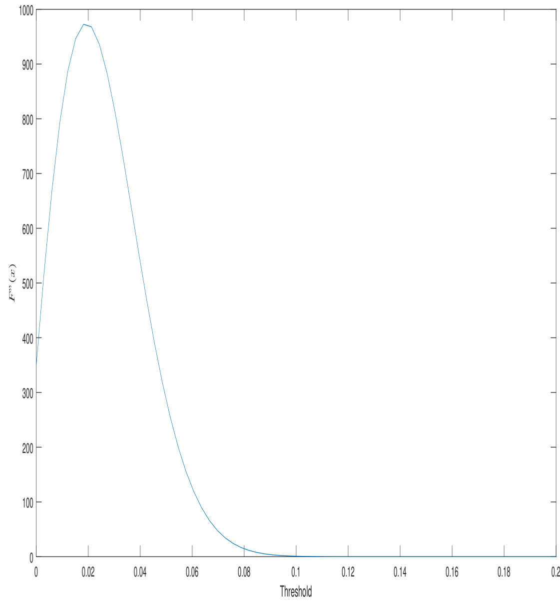

Table 1: Monte Carlo and Quasi Monte Carlo Convergence and Error Analyses.

# of Simulations	MC Sim error	QMC Sim error(Halton)	QMC Sim error(Sobol)
100	0.050328	0.016328	0.02689
500	0.009675	0.008443	0.0029812
1000	0.011443	0.007043	0.0034108
10000	0.003133	0.001416	0.001756
100000	0.001351	0.000313	0.00073465

Table 2: Analytical Max BM Efficiency vs Classical Max BM Simulation Efficiency.

Bound.	**# of Sim**	Classical MC $\mathbb{P}(M(t) > B)$	Max BM $\mathbb{P}(M(t) > B)$	Max BM **Comp. (Seconds)**	Classical BM simulation **Comp. (Seconds)**	**Comp. Cost Reduction**
0.0	1000	0.9934	1.0000	0.210195	0.000864	243
0.125	5000	0.5300	0.5326	0.987166	0.001723	572
0.25	0000	0.2093	0.2097	1.885606	0.001456	1295
0.375	15000	0.0562	0.0596	2.924059	0.003374	866
0.5	20000	0.0124	0.0125	3.839442	0.004928	779

Figure 3: Evolution of mean, time averaged variance and probability of loss for the buy-and-hold strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years.

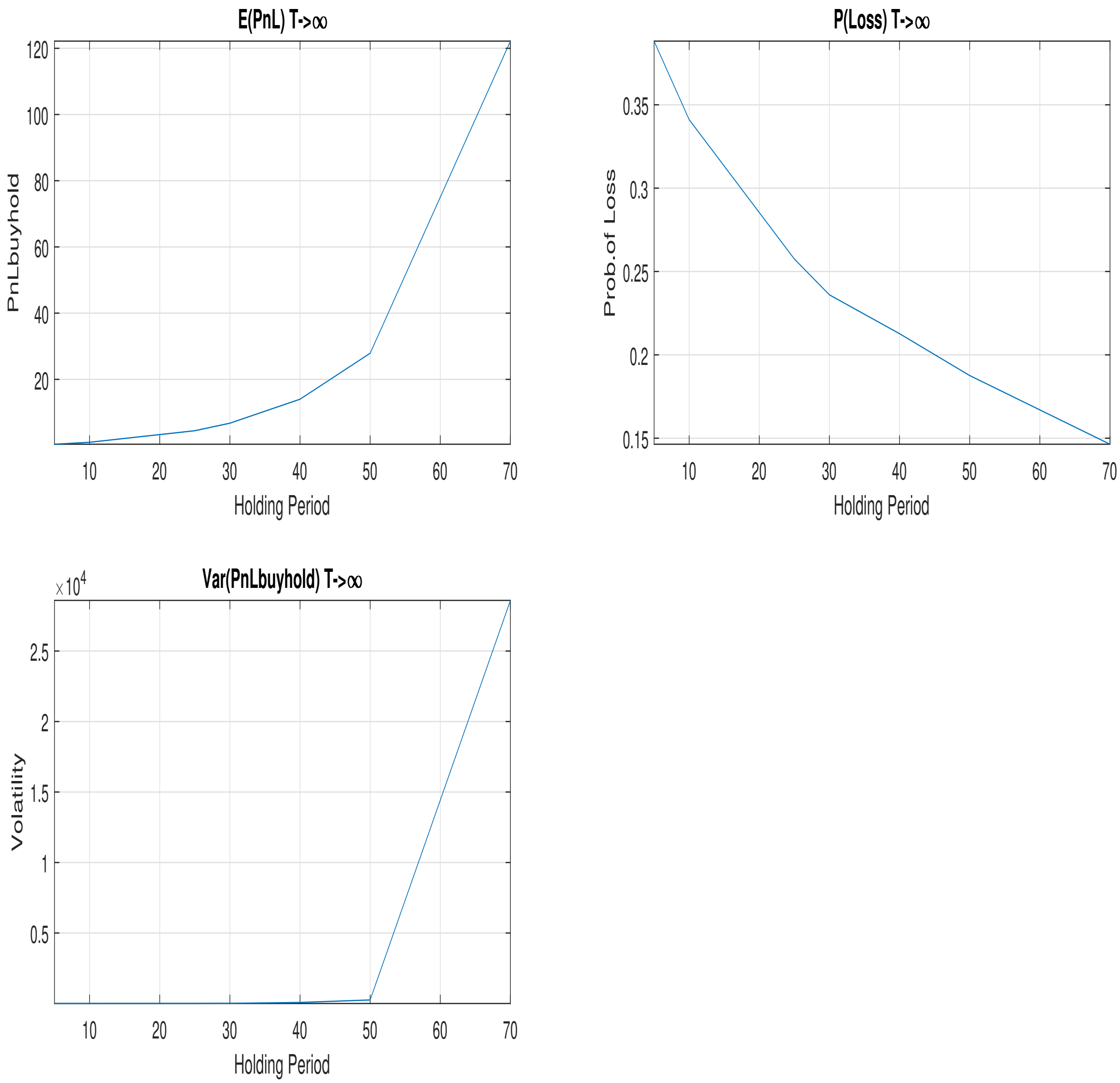

Figure 4: Evolution of mean, time averaged variance and probability of loss for the long-until-barrier strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years.

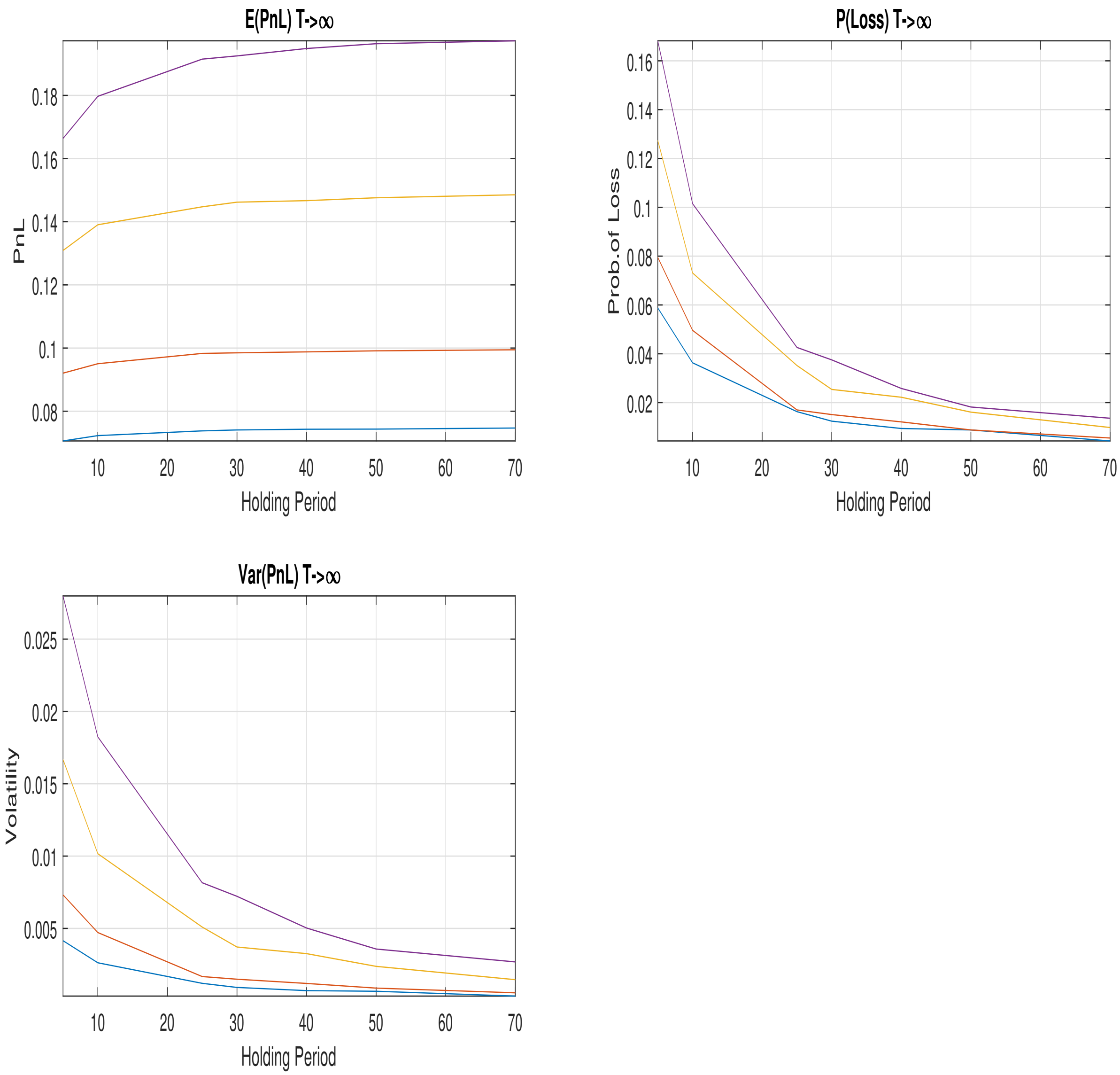

Table 3: Parameters considered throughout the Monte Carlo experiments.

Stock parameters			
σ	0.0002		
α	0.0774		
Factor parameters			
	$R_m - R_f$	SMB	HML
σ_j	0.1254	0.0725	0.0644
β_j	1.0960	0.4270	0.2300
F_0	-0.0016	-0.0126	0.0210

Table 4: Descriptive statistics for the daily-log returns of the equally weighted portfolios of Russell 3000 stocks for the period 01 Jan 1990 to 30 Oct 2020. P-i refers to i-th portfolio.

	mean	**median**	**std**	**min**	**max**	**skewness**	**kurtosis**
P-1	0.00027	0.00059	0.013	-0.150	0.130	-0.592	16.333
P-2	0.00021	0.00058	0.012	-0.145	0.117	-0.589	15.309
P-3	0.00021	0.00059	0.013	-0.146	0.096	-0.799	14.284
P-4	0.00026	0.00053	0.011	-0.136	0.106	-0.517	18.605
P-5	0.00031	0.00045	0.012	-0.136	0.104	-0.694	16.758
P-6	0.00019	0.00050	0.012	-0.143	0.086	-0.690	14.074
P-7	0.00017	0.00048	0.013	-0.140	0.089	-0.599	11.902
P-8	0.00025	0.00059	0.014	-0.126	0.113	-0.508	13.754
P-9	0.00029	0.00059	0.013	-0.138	0.100	-0.630	14.073
P-10	0.00022	0.00053	0.013	-0.124	0.088	-0.690	13.256
P-11	0.00034	0.00064	0.012	-0.120	0.092	-0.414	13.140
P-12	0.00032	0.00069	0.013	-0.108	0.097	-0.494	12.091
P-13	0.00035	0.00073	0.013	-0.168	0.111	-0.730	15.627
P-14	0.00021	0.00052	0.012	-0.120	0.103	-0.613	16.095
P-15	0.00028	0.00048	0.013	-0.155	0.092	-0.546	14.144
P-16	0.00025	0.00065	0.012	-0.135	0.103	-0.718	15.510
P-17	0.00020	0.00060	0.014	-0.151	0.097	-0.639	13.624
P-18	0.00027	0.00075	0.013	-0.136	0.097	-0.510	12.739
P-19	0.00031	0.00053	0.012	-0.128	0.123	-0.396	15.769
P-20	0.00025	0.00076	0.013	-0.137	0.112	-0.636	15.083
P-21	0.00025	0.00047	0.012	-0.119	0.106	-0.454	13.021
P-22	0.00030	0.00065	0.012	-0.117	0.122	-0.368	16.631
P-23	0.00030	0.00056	0.012	-0.134	0.110	-0.757	15.801
P-24	0.00026	0.00059	0.014	-0.131	0.111	-0.405	11.203
average	0.00026	0.00058	0.013	-0.135	0.104	-0.583	14.534

Figure 5: The price series of twenty four stock portfolios in the time period from 1/1/1990 to 30/10/2020 with all the initial prices scaled to one dollar.

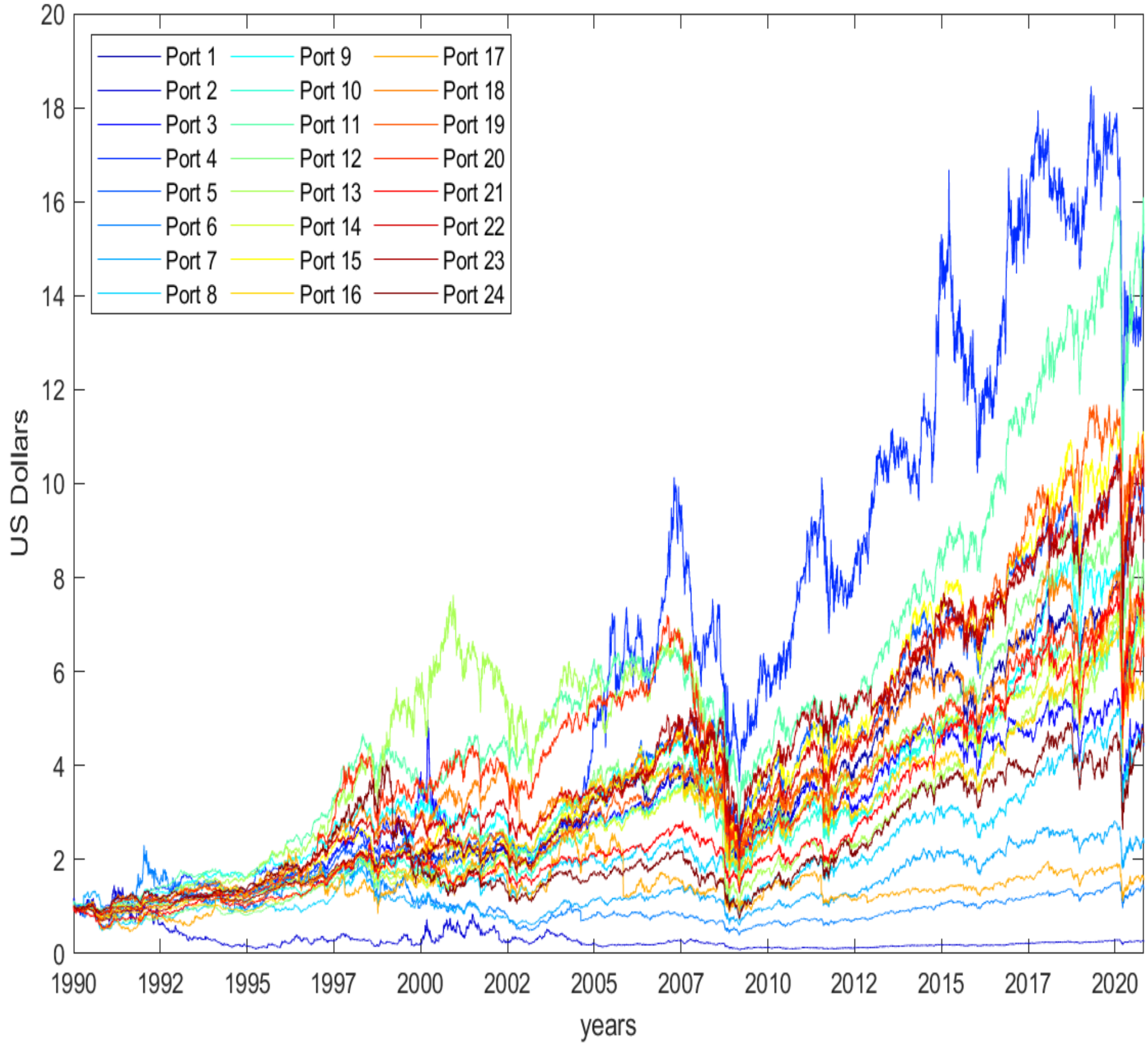

Figure 6: Empirical hitting probabilities of 24 portfolios to the barrier level for a given holding period of 20-, 40-, 60-, 80-days compared to theoretical probabilities.

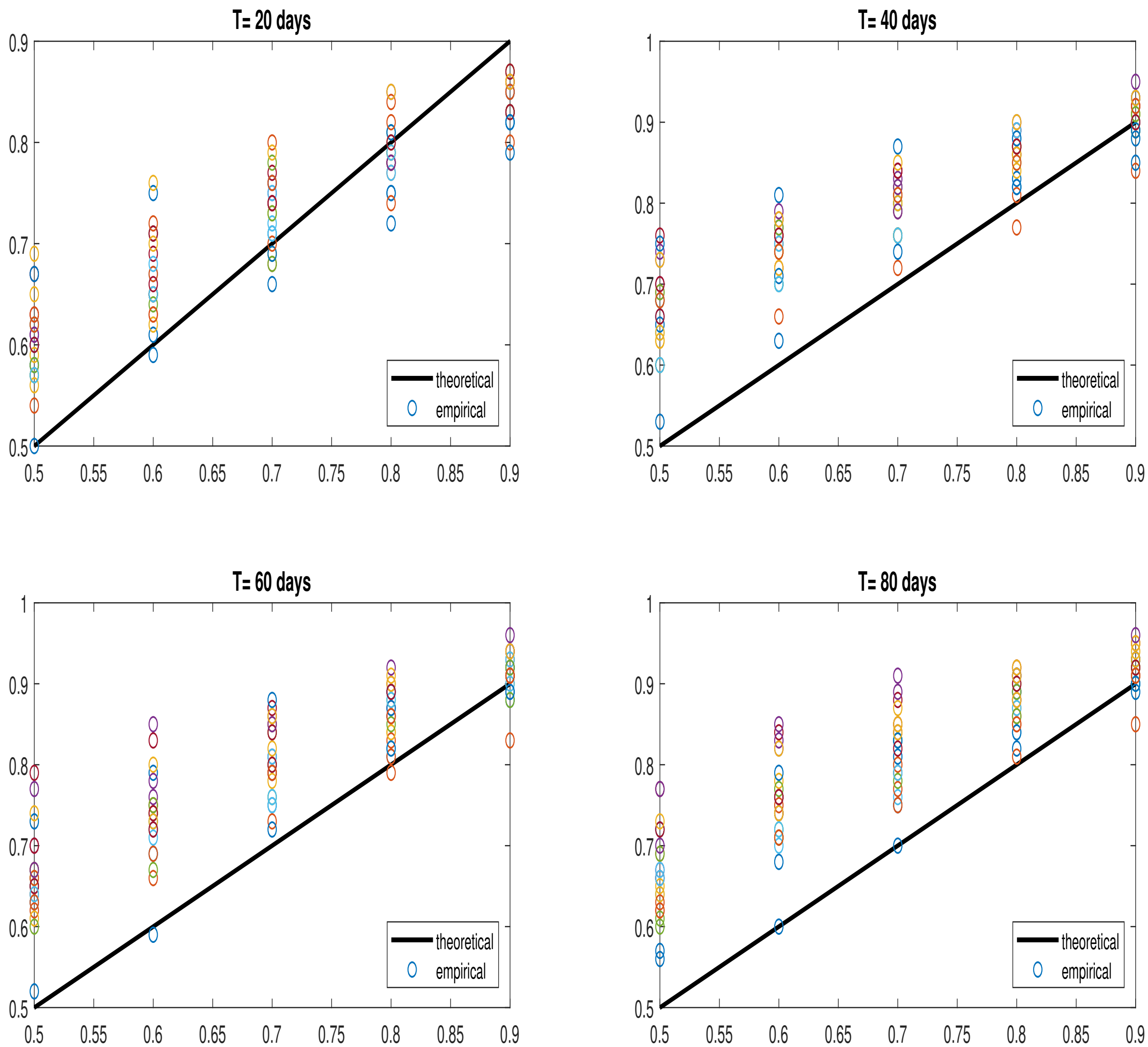

Figure 7: Out-of-sample performance of the long-until-barrier strategies using the optimal barrier levels from expected return maximization where each position is held for different investment horizons of 20-, 40-, 60-, and 80-days, respectively. The out-of-sample test period is divided into time periods with given investment horizons.

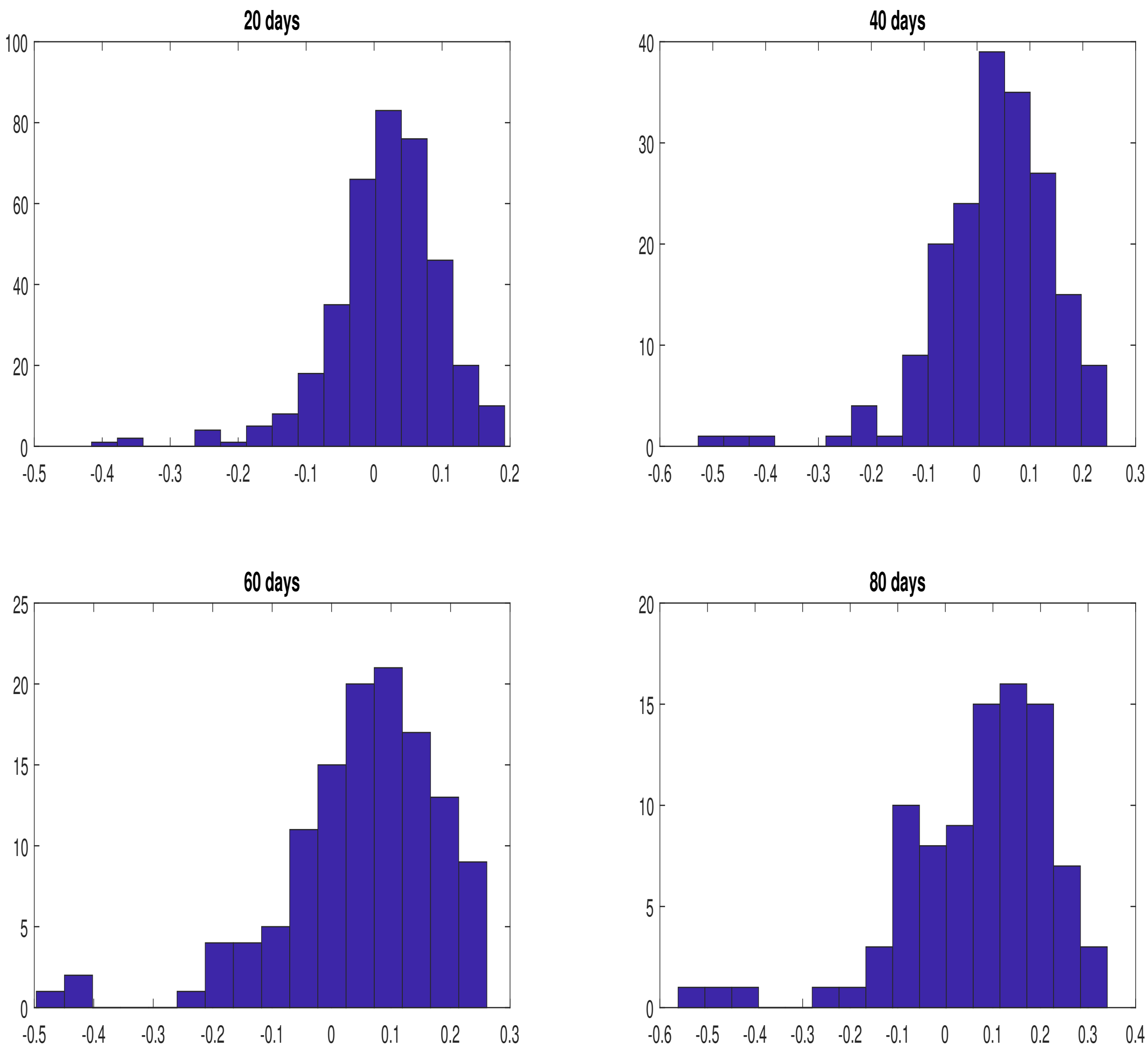

Figure 8: Out-of-sample performance of the long-until-barrier strategies using the optimal barrier levels from Sharpe ratio maximization where each position is held for different investment horizons of 20-, 40-, 60-, and 80-days, respectively. The out-of-sample test period is divided into time periods with given investment horizons.

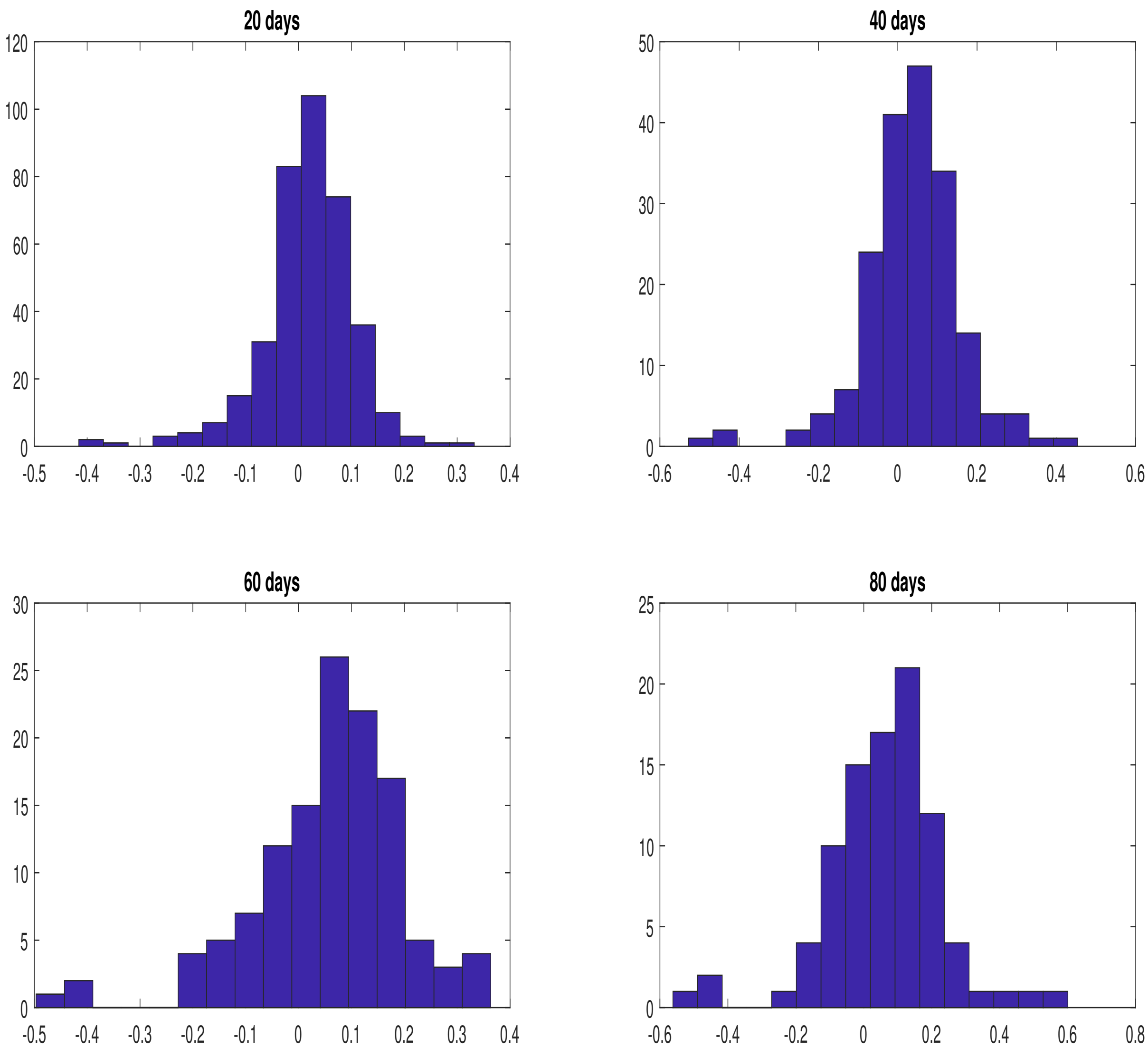

Table 5: Empirical hitting probabilities of 24 portfolios together with the corresponding barrier levels k for a holding period of T= 20 days. theoretical probability (tp), empirical probability (ep), implied k (impK). P-i refers to i-th portfolio.

	tp = 0.5		tp = 0.6		tp = 0.7		tp = 0.8		tp = 0.9	
	ep	impK	ep	impK	ep	impK	ep	impK	ep	impK
P-1	0.58	0.017	0.61	0.013	0.71	0.010	0.75	0.007	0.83	0.004
P-2	0.57	0.019	0.63	0.015	0.68	0.011	0.77	0.008	0.86	0.005
P-3	0.56	0.016	0.62	0.013	0.69	0.009	0.78	0.006	0.85	0.004
P-4	0.58	0.020	0.65	0.015	0.75	0.011	0.78	0.008	0.85	0.004
P-5	0.57	0.021	0.64	0.016	0.68	0.012	0.78	0.008	0.83	0.005
P-6	0.58	0.022	0.67	0.017	0.72	0.012	0.77	0.008	0.82	0.005
P-7	0.63	0.021	0.69	0.016	0.77	0.012	0.79	0.008	0.83	0.005
P-8	0.60	0.020	0.65	0.015	0.69	0.011	0.75	0.008	0.82	0.005
P-9	0.63	0.018	0.67	0.014	0.80	0.010	0.84	0.007	0.86	0.004
P-10	0.59	0.020	0.65	0.015	0.73	0.011	0.81	0.008	0.85	0.004
P-11	0.61	0.019	0.71	0.014	0.76	0.011	0.85	0.007	0.86	0.005
P-12	0.62	0.019	0.71	0.015	0.78	0.011	0.79	0.008	0.83	0.005
P-13	0.62	0.017	0.68	0.013	0.75	0.010	0.81	0.007	0.85	0.004
P-14	0.67	0.020	0.71	0.015	0.74	0.011	0.78	0.008	0.83	0.005
P-15	0.67	0.018	0.75	0.014	0.76	0.010	0.81	0.007	0.82	0.004
P-16	0.54	0.021	0.63	0.016	0.70	0.012	0.74	0.008	0.80	0.005
P-17	0.65	0.019	0.70	0.014	0.74	0.011	0.79	0.007	0.87	0.005
P-18	0.61	0.018	0.65	0.014	0.74	0.010	0.78	0.007	0.85	0.004
P-19	0.58	0.021	0.65	0.017	0.73	0.012	0.80	0.008	0.87	0.005
P-20	0.57	0.017	0.65	0.013	0.71	0.010	0.79	0.007	0.85	0.004
P-21	0.60	0.017	0.66	0.013	0.74	0.010	0.80	0.007	0.87	0.004
P-22	0.50	0.021	0.59	0.016	0.66	0.012	0.72	0.008	0.79	0.005
P-23	0.62	0.019	0.72	0.015	0.76	0.011	0.82	0.008	0.85	0.005
P-24	0.69	0.018	0.76	0.014	0.79	0.011	0.85	0.007	0.86	0.005
average	0.60	0.019	0.67	0.015	0.73	0.011	0.79	0.007	0.84	0.004

Table 6: Empirical hitting probabilities of 24 portfolios together with the corresponding barrier levels k for a holding period of T= 40 days. theoretical probability (tp), empirical probability (ep), implied k (impK). P-i refers to i-th portfolio.

	tp = 0.5		tp = 0.6		tp = 0.7		tp = 0.8		tp = 0.9	
	ep	impK	ep	impK	ep	impK	ep	impK	ep	impK
P-1	0.66	0.025	0.70	0.019	0.76	0.014	0.83	0.009	0.88	0.005
P-2	0.68	0.027	0.72	0.021	0.76	0.015	0.81	0.010	0.92	0.006
P-3	0.63	0.023	0.72	0.018	0.76	0.013	0.84	0.009	0.89	0.005
P-4	0.70	0.028	0.74	0.022	0.83	0.016	0.89	0.011	0.90	0.006
P-5	0.68	0.029	0.74	0.023	0.80	0.017	0.86	0.011	0.91	0.006
P-6	0.64	0.032	0.70	0.024	0.79	0.018	0.88	0.012	0.93	0.006
P-7	0.76	0.030	0.78	0.023	0.84	0.017	0.89	0.011	0.93	0.006
P-8	0.65	0.028	0.71	0.022	0.80	0.016	0.82	0.010	0.89	0.006
P-9	0.69	0.026	0.75	0.020	0.79	0.014	0.87	0.010	0.90	0.006
P-10	0.64	0.028	0.76	0.022	0.80	0.016	0.87	0.010	0.91	0.006
P-11	0.74	0.026	0.77	0.020	0.79	0.015	0.87	0.010	0.95	0.006
P-12	0.69	0.027	0.77	0.021	0.82	0.015	0.87	0.010	0.89	0.006
P-13	0.68	0.024	0.75	0.019	0.81	0.014	0.89	0.009	0.93	0.005
P-14	0.70	0.028	0.76	0.022	0.84	0.016	0.86	0.011	0.92	0.006
P-15	0.75	0.026	0.81	0.020	0.87	0.015	0.88	0.010	0.89	0.005
P-16	0.60	0.031	0.66	0.024	0.72	0.017	0.77	0.011	0.84	0.006
P-17	0.63	0.027	0.74	0.021	0.81	0.015	0.86	0.010	0.90	0.006
P-18	0.73	0.026	0.79	0.020	0.82	0.015	0.90	0.010	0.93	0.005
P-19	0.60	0.030	0.70	0.023	0.76	0.017	0.85	0.011	0.91	0.006
P-20	0.60	0.024	0.70	0.019	0.76	0.014	0.82	0.009	0.90	0.005
P-21	0.66	0.024	0.76	0.018	0.84	0.013	0.87	0.009	0.90	0.005
P-22	0.53	0.030	0.63	0.023	0.74	0.017	0.82	0.011	0.85	0.006
P-23	0.68	0.027	0.74	0.021	0.81	0.015	0.85	0.010	0.92	0.006
P-24	0.73	0.026	0.78	0.020	0.85	0.015	0.90	0.010	0.93	0.006
average	0.67	0.027	0.74	0.021	0.80	0.015	0.86	0.010	0.91	0.006

Table 7: Empirical hitting probabilities of 24 portfolios together with the corresponding barrier levels k for a holding period of T= 60 days. theoretical probability (tp), empirical probability (ep), implied k (impK). P-i refers to i-th portfolio.

	tp = 0.5		tp = 0.6		tp = 0.7		tp = 0.8		tp = 0.9	
	ep	impK	ep	impK	ep	impK	ep	impK	ep	impK
P-1	0.63	0.031	0.69	0.024	0.75	0.017	0.81	0.011	0.88	0.006
P-2	0.62	0.034	0.75	0.026	0.79	0.019	0.81	0.012	0.93	0.007
P-3	0.65	0.029	0.72	0.022	0.78	0.016	0.84	0.011	0.90	0.006
P-4	0.70	0.035	0.76	0.027	0.81	0.020	0.87	0.013	0.91	0.007
P-5	0.66	0.036	0.74	0.028	0.84	0.020	0.89	0.013	0.91	0.007
P-6	0.63	0.039	0.69	0.030	0.75	0.022	0.86	0.014	0.94	0.007
P-7	0.79	0.037	0.83	0.029	0.87	0.021	0.90	0.014	0.94	0.007
P-8	0.65	0.034	0.73	0.027	0.79	0.019	0.87	0.013	0.89	0.007
P-9	0.66	0.032	0.75	0.024	0.79	0.018	0.83	0.012	0.91	0.006
P-10	0.61	0.035	0.74	0.027	0.85	0.019	0.90	0.013	0.93	0.007
P-11	0.77	0.033	0.85	0.025	0.86	0.018	0.89	0.012	0.94	0.007
P-12	0.67	0.034	0.75	0.026	0.80	0.019	0.88	0.012	0.88	0.007
P-13	0.67	0.030	0.72	0.023	0.81	0.017	0.88	0.011	0.93	0.006
P-14	0.65	0.035	0.74	0.027	0.84	0.020	0.86	0.013	0.92	0.007
P-15	0.73	0.032	0.79	0.024	0.88	0.018	0.89	0.012	0.92	0.006
P-16	0.64	0.038	0.66	0.029	0.73	0.021	0.79	0.014	0.83	0.007
P-17	0.67	0.033	0.73	0.025	0.82	0.018	0.84	0.012	0.91	0.007
P-18	0.67	0.032	0.78	0.025	0.85	0.018	0.92	0.012	0.96	0.006
P-19	0.60	0.038	0.67	0.029	0.76	0.021	0.85	0.014	0.92	0.007
P-20	0.64	0.030	0.71	0.023	0.76	0.017	0.82	0.011	0.90	0.006
P-21	0.70	0.029	0.72	0.022	0.80	0.016	0.89	0.011	0.91	0.006
P-22	0.52	0.037	0.59	0.029	0.72	0.021	0.82	0.014	0.89	0.007
P-23	0.63	0.034	0.69	0.026	0.79	0.019	0.86	0.012	0.91	0.007
P-24	0.74	0.032	0.80	0.025	0.86	0.018	0.91	0.012	0.94	0.007
average	0.66	0.034	0.73	0.026	0.80	0.019	0.86	0.012	0.91	0.007

Table 8: Empirical hitting probabilities of 24 portfolios together with the corresponding barrier levels k for a holding period of T= 80 days. theoretical probability (tp), empirical probability (ep), implied k (impK). P-i refers to i-th portfolio.

	tp = 0.5		tp = 0.6		tp = 0.7		tp = 0.8		tp = 0.9	
	ep	impK	ep	impK	ep	impK	ep	impK	ep	impK
P-1	0.57	0.036	0.68	0.027	0.75	0.020	0.82	0.013	0.89	0.007
P-2	0.62	0.039	0.75	0.030	0.79	0.022	0.85	0.014	0.91	0.007
P-3	0.72	0.033	0.78	0.026	0.83	0.019	0.87	0.012	0.92	0.006
P-4	0.67	0.040	0.82	0.031	0.84	0.023	0.89	0.015	0.93	0.008
P-5	0.61	0.042	0.72	0.032	0.83	0.024	0.90	0.015	0.92	0.008
P-6	0.64	0.045	0.70	0.035	0.76	0.025	0.87	0.016	0.91	0.008
P-7	0.77	0.043	0.84	0.033	0.88	0.024	0.92	0.016	0.95	0.008
P-8	0.66	0.040	0.76	0.031	0.81	0.022	0.89	0.015	0.90	0.008
P-9	0.69	0.037	0.76	0.028	0.80	0.021	0.86	0.014	0.91	0.007
P-10	0.64	0.040	0.74	0.031	0.87	0.023	0.92	0.015	0.95	0.008
P-11	0.77	0.038	0.85	0.029	0.89	0.021	0.91	0.014	0.93	0.007
P-12	0.69	0.039	0.77	0.030	0.83	0.022	0.89	0.014	0.90	0.008
P-13	0.67	0.035	0.74	0.027	0.83	0.020	0.88	0.013	0.93	0.007
P-14	0.66	0.041	0.74	0.031	0.85	0.023	0.88	0.015	0.93	0.008
P-15	0.72	0.037	0.79	0.028	0.83	0.021	0.91	0.014	0.93	0.007
P-16	0.63	0.044	0.71	0.034	0.77	0.025	0.81	0.016	0.85	0.008
P-17	0.65	0.038	0.74	0.029	0.84	0.021	0.88	0.014	0.93	0.007
P-18	0.70	0.037	0.83	0.029	0.91	0.021	0.91	0.014	0.96	0.007
P-19	0.60	0.044	0.71	0.034	0.78	0.024	0.86	0.016	0.92	0.008
P-20	0.66	0.035	0.72	0.027	0.79	0.020	0.84	0.013	0.90	0.007
P-21	0.72	0.034	0.76	0.026	0.82	0.019	0.90	0.012	0.92	0.006
P-22	0.56	0.043	0.60	0.033	0.70	0.024	0.84	0.016	0.90	0.008
P-23	0.62	0.039	0.71	0.030	0.75	0.022	0.85	0.014	0.91	0.007
P-24	0.73	0.037	0.82	0.029	0.85	0.021	0.91	0.014	0.94	0.007
average	0.67	0.039	0.75	0.030	0.82	0.022	0.88	0.014	0.92	0.007

Table 9: The out-of-sample backtesting results from the expected return maximization algorithm for the target holding period of 20-, 40-, 60-, 80- days. P-i refers to i-th portfolio. Rets (Vol) stands for the annualized returns (volatility). MaxDD is the maximum drawdown.

	20-days				40-days				60-days				80-days			
	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD
P-1	0.11	0.25	0.18	0.25	0.09	0.24	0.24	0.25	0.10	0.21	0.34	0.25	0.13	0.37	0.35	0.23
P-2	0.07	0.22	0.14	0.23	0.07	0.22	0.19	0.23	0.08	0.22	0.26	0.23	0.10	0.28	0.29	0.23
P-3	0.09	0.26	0.15	0.22	0.08	0.24	0.20	0.22	0.09	0.23	0.27	0.22	0.14	0.47	0.27	0.21
P-4	0.09	0.17	0.23	0.22	0.09	0.17	0.30	0.22	0.09	0.17	0.40	0.22	0.09	0.17	0.45	0.22
P-5	0.13	0.23	0.24	0.22	0.11	0.21	0.33	0.22	0.12	0.19	0.46	0.22	0.15	0.32	0.48	0.22
P-6	0.08	0.23	0.15	0.21	0.07	0.21	0.20	0.21	0.07	0.27	0.27	0.21	0.14	0.41	0.25	0.21
P-7	0.08	0.27	0.12	0.21	0.06	0.25	0.15	0.21	0.07	0.25	0.20	0.21	0.16	0.62	0.20	0.18
P-8	0.13	0.31	0.18	0.21	0.09	0.27	0.21	0.21	0.10	0.29	0.31	0.21	0.18	0.52	0.30	0.20
P-9	0.12	0.29	0.18	0.22	0.10	0.27	0.23	0.22	0.10	0.25	0.30	0.22	0.15	0.46	0.31	0.21
P-10	0.09	0.26	0.15	0.20	0.08	0.25	0.18	0.20	0.09	0.23	0.26	0.20	0.10	0.38	0.26	0.19
P-11	0.11	0.18	0.29	0.19	0.13	0.18	0.40	0.19	0.13	0.18	0.54	0.19	0.20	0.31	0.50	0.19
P-12	0.12	0.23	0.23	0.19	0.12	0.23	0.31	0.19	0.13	0.24	0.42	0.19	0.17	0.38	0.43	0.19
P-13	0.15	0.28	0.23	0.25	0.13	0.26	0.30	0.25	0.14	0.25	0.41	0.25	0.20	0.46	0.41	0.25
P-14	0.10	0.22	0.18	0.20	0.08	0.20	0.24	0.20	0.09	0.19	0.33	0.20	0.09	0.21	0.39	0.20
P-15	0.13	0.28	0.19	0.23	0.10	0.23	0.27	0.23	0.12	0.23	0.37	0.23	0.19	0.42	0.36	0.23
P-16	0.10	0.23	0.19	0.22	0.09	0.22	0.24	0.22	0.10	0.22	0.34	0.22	0.13	0.31	0.38	0.21
P-17	0.10	0.28	0.14	0.23	0.08	0.26	0.18	0.23	0.09	0.32	0.27	0.23	0.13	0.48	0.23	0.22
P-18	0.10	0.25	0.18	0.21	0.10	0.23	0.25	0.21	0.10	0.23	0.32	0.21	0.14	0.38	0.35	0.21
P-19	0.11	0.21	0.23	0.22	0.12	0.23	0.30	0.22	0.12	0.21	0.43	0.22	0.15	0.27	0.44	0.20
P-20	0.10	0.24	0.18	0.22	0.10	0.24	0.25	0.22	0.11	0.31	0.33	0.22	0.16	0.45	0.31	0.21
P-21	0.10	0.22	0.20	0.20	0.10	0.22	0.28	0.20	0.09	0.20	0.34	0.20	0.10	0.22	0.39	0.19
P-22	0.12	0.21	0.24	0.21	0.12	0.22	0.32	0.21	0.11	0.18	0.46	0.21	0.12	0.21	0.51	0.21
P-23	0.12	0.20	0.25	0.22	0.11	0.21	0.34	0.22	0.11	0.19	0.45	0.22	0.23	0.40	0.45	0.20
P-24	0.11	0.29	0.15	0.22	0.10	0.28	0.20	0.22	0.10	0.28	0.28	0.22	0.13	0.46	0.27	0.19

Table 10: The out-of-sample backtesting results from the Sharpe ratio maximization algorithm for the target holding period of 20-, 40-, 60-, 80- days. P-i refers to i-th portfolio. Rets (Vol) stands for the annualized returns (volatility). MaxDD is the maximum drawdown.

	20-days				40-days				60-days				80-days			
	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD	Rets	Vol	Sharpe	MaxDD
P-1	0.11	0.25	0.18	0.25	0.08	0.23	0.22	0.25	0.11	0.23	0.34	0.23	0.13	0.37	0.44	0.23
P-2	0.07	0.22	0.13	0.23	0.07	0.23	0.16	0.23	0.08	0.21	0.27	0.23	0.09	0.35	0.26	0.23
P-3	0.09	0.26	0.15	0.22	0.07	0.25	0.16	0.22	0.08	0.25	0.21	0.22	0.10	0.49	0.24	0.21
P-4	0.09	0.18	0.21	0.22	0.09	0.19	0.28	0.22	0.09	0.17	0.38	0.22	0.09	0.19	0.39	0.22
P-5	0.11	0.22	0.23	0.22	0.11	0.22	0.30	0.22	0.12	0.20	0.42	0.22	0.13	0.32	0.44	0.22
P-6	0.08	0.23	0.15	0.21	0.06	0.22	0.17	0.21	0.09	0.25	0.29	0.21	0.09	0.48	0.27	0.21
P-7	0.05	0.27	0.08	0.21	0.05	0.27	0.10	0.21	0.06	0.26	0.15	0.21	0.09	0.69	0.21	0.21
P-8	0.11	0.28	0.17	0.21	0.09	0.27	0.19	0.21	0.12	0.32	0.29	0.21	0.15	0.67	0.32	0.21
P-9	0.10	0.28	0.16	0.22	0.10	0.30	0.19	0.22	0.10	0.26	0.24	0.22	0.14	0.54	0.29	0.22
P-10	0.09	0.27	0.13	0.20	0.08	0.26	0.17	0.20	0.08	0.24	0.24	0.20	0.08	0.41	0.23	0.19
P-11	0.13	0.19	0.29	0.19	0.13	0.19	0.39	0.19	0.13	0.18	0.50	0.19	0.15	0.29	0.55	0.17
P-12	0.12	0.24	0.21	0.19	0.12	0.23	0.29	0.19	0.11	0.22	0.36	0.19	0.13	0.40	0.40	0.19
P-13	0.13	0.27	0.20	0.25	0.12	0.26	0.25	0.25	0.12	0.24	0.35	0.25	0.18	0.45	0.46	0.25
P-14	0.09	0.23	0.17	0.20	0.07	0.22	0.19	0.20	0.08	0.21	0.27	0.20	0.09	0.23	0.33	0.20
P-15	0.12	0.27	0.18	0.23	0.10	0.25	0.22	0.23	0.11	0.24	0.31	0.23	0.14	0.46	0.41	0.23
P-16	0.10	0.24	0.18	0.22	0.09	0.24	0.22	0.22	0.10	0.22	0.29	0.22	0.12	0.35	0.36	0.21
P-17	0.07	0.26	0.12	0.23	0.07	0.26	0.16	0.23	0.11	0.31	0.27	0.23	0.16	0.77	0.28	0.22
P-18	0.10	0.26	0.17	0.21	0.09	0.25	0.21	0.21	0.11	0.24	0.29	0.21	0.13	0.39	0.35	0.21
P-19	0.11	0.21	0.23	0.22	0.10	0.21	0.27	0.22	0.10	0.20	0.35	0.22	0.11	0.33	0.37	0.20
P-20	0.09	0.25	0.16	0.22	0.09	0.25	0.23	0.22	0.10	0.25	0.31	0.22	0.14	0.52	0.39	0.22
P-21	0.10	0.23	0.19	0.20	0.08	0.21	0.23	0.20	0.10	0.21	0.31	0.20	0.10	0.24	0.34	0.19
P-22	0.12	0.22	0.24	0.21	0.11	0.21	0.30	0.21	0.12	0.18	0.47	0.21	0.13	0.23	0.51	0.21
P-23	0.13	0.22	0.25	0.22	0.11	0.21	0.29	0.22	0.12	0.20	0.43	0.20	0.13	0.29	0.49	0.20
P-24	0.09	0.29	0.13	0.22	0.08	0.28	0.18	0.22	0.10	0.29	0.26	0.20	0.13	0.66	0.28	0.20

Table 11: Probability values coming from the White's reality check for the expected return and Sharpe ratio maximization algorithms are given for the 20-, 40-, 60-, and 80- days holding periods, respectively. P-i refers to i-th portfolio.

	Expected Return				Sharpe Ratio			
	20-days	40-days	60-days	80-days	20-days	40-days	60-days	80-days
P-1	0.044	0.014	0.024	0.028	0.028	0.026	0.038	0.036
P-2	0.122	0.078	0.092	0.122	0.144	0.092	0.048	0.066
P-3	0.110	0.078	0.094	0.124	0.118	0.092	0.110	0.080
P-4	0.012	0.008	0.012	0.014	0.018	0.002	0.008	0.006
P-5	0.006	0.006	0.010	0.002	0.012	0.002	0.008	0.004
P-6	0.102	0.082	0.104	0.124	0.116	0.068	0.046	0.036
P-7	0.270	0.238	0.270	0.308	0.422	0.342	0.340	0.300
P-8	0.064	0.048	0.080	0.076	0.084	0.046	0.054	0.102
P-9	0.092	0.088	0.072	0.066	0.110	0.062	0.064	0.058
P-10	0.148	0.138	0.102	0.154	0.192	0.160	0.124	0.144
P-11	0.000	0.000	0.004	0.002	0.002	0.000	0.000	0.000
P-12	0.004	0.008	0.010	0.010	0.022	0.010	0.014	0.006
P-13	0.018	0.008	0.012	0.008	0.042	0.024	0.028	0.002
P-14	0.024	0.016	0.024	0.032	0.046	0.034	0.012	0.028
P-15	0.038	0.044	0.028	0.038	0.048	0.036	0.060	0.042
P-16	0.026	0.022	0.014	0.010	0.038	0.022	0.032	0.018
P-17	0.110	0.144	0.134	0.128	0.162	0.118	0.052	0.020
P-18	0.056	0.036	0.048	0.028	0.066	0.066	0.092	0.044
P-19	0.008	0.002	0.008	0.008	0.006	0.012	0.008	0.000
P-20	0.084	0.074	0.050	0.084	0.182	0.102	0.102	0.028
P-21	0.058	0.036	0.040	0.026	0.046	0.036	0.020	0.024
P-22	0.000	0.002	0.004	0.000	0.004	0.002	0.000	0.000
P-23	0.002	0.004	0.002	0.008	0.000	0.000	0.000	0.002
P-24	0.116	0.146	0.144	0.142	0.196	0.182	0.102	0.052

APPENDIX

A.1. Derivation of the optimal k

We can further rewrite equation (16) in a more compact form,

$$\begin{aligned}\partial E[v(T)]/\partial k &= \frac{\partial\left((k-C)F(k)+C\right)}{\partial k}\\ &= (k-C)\,\frac{\partial F(k)}{\partial k}+F(k)\\ &= k+\frac{F(k)}{F'(k)}-C=0\end{aligned} \tag{25}$$

where $F(k)$ refers to equation (11) and

$$C=\exp\left(\left(\alpha-r_f+\frac{1}{6}\sum_{j=1}^{p}\beta_j^2\sigma_j^2\right)T+\sum_{j=1}^{p}\beta_j F_0^j\right)-1$$

Here it's important to note that (15) is strictly concave for k $\in[0,\infty]$. We can see that from (16)

$$\partial^2 E[v(T)]/\partial k^2 = 1-\frac{F''(k)F(k)}{F'(k)^2}.$$

and the second derivative of $F(k)$ which is the second derivative of $P(\tau_k<T)$ is positive in that interval. Therefore, the second derivative of objective function (15) becomes negative, consequently strict concavity and a unique maximum is achieved.

A.2. Derivation of the optimal k from the Sharpe Ratio

To compute the ratio given by 29, we need the variance of the cumulative profits $v(t)$. Hence,

$$E[var(v(T))]=var(v(T)|\tau_k>T)P(\tau_k>T)+0P(\tau_k<T)\ ,$$

and by using our stock model definition in equation (3), we obtain

$$E[\sqrt{var(v(T))}]=S_0\sqrt{\left(e^{2\left(\alpha-r_f+\sigma^2+\frac{\sum_{j=1}^{p}\beta_j\sigma_j^2}{3}\right)T+\sum_{j=1}^{p}\beta_jF_0^j}-1\right)P(\tau_k>T)+0P(\tau_k<T)}.$$

This finally leads to

$$\max_k \mathcal{S}_T = \max_k \frac{S_0\left(k-C\right)F(k)+C}{S_0\sqrt{\left(e^{2\left(\alpha-r_f+\sigma^2+\frac{\sum_{j=1}^{p}\beta_j\sigma_j^2}{3}\right)T+\sum_{j=1}^{p}\beta_j F_0^j}-1\right)\left(1-F(k)\right)}}, \tag{26}$$

and the first order condition is derived as

$$\frac{1}{\hat{C}}\left(\frac{F'(k)\left(F(k)\left(k-C\right)+C\right)+F'(k)\left(1-F(k)\right)\left(k-C\right)+F(k)\left(1-F(k)\right)}{\left(1-F(k)\right)^{3/2}}\right)=0 \tag{27}$$

where $\hat{C}=\sqrt{\left(e^{2\left(\alpha-r_f+\sigma^2+\frac{\sum_{j=1}^{p}\beta_j\sigma_j^2}{3}\right)T+\sum_{j=1}^{p}\beta_j F_0^j}-1\right)}$

Then this leads to the equation,

$$\begin{aligned} kF(k)F'(k)-CF(k)F'(k)+CF'(k)+ \\ kF'(k)-kF'(k)F(k)-kF'(k)+kF'(k)F(k)+F(k)-F(k)^2=0 \\ =kF'(k)+F(k)-F(k)^2=0 \end{aligned} \tag{28}$$

We finally obtain:

$$kF'(k)+F(k)-F(k)^2=0 \implies k^*=\frac{F(k)^2-F(k)}{F'(k)} \tag{29}$$

Insert Figure 2 about here

Clearly, we can find the optimal k^* by finding the roots of the above equation. By using the cumulative distribution function for the first passage time density, equation (25) can be written as

$$k+\frac{\Phi\left(A\right)+e^{\gamma\left(\ln(1+k)-\sum_{j=1}^{p}\beta_j F_0^j\right)}\Phi\left(B\right)}{-\frac{1}{(k+1)}\left[\frac{\phi(A)}{\sigma}+e^{\gamma\left(\ln(1+k)-\sum_{j=1}^{p}\beta_j F_0^j\right)}\left(\gamma\Phi\left(B\right)-\frac{1}{\sigma\sqrt{t}}\phi\left(B\right)\right)\right]}+C=0$$

where

$$\begin{aligned} A &= \frac{(\alpha - r_f - \frac{1}{2}\sigma^2)t - \ln(1+k) + \sum_{j=1}^{p} \beta_j F_0^j}{\sigma\sqrt{t}}, \\ B &= \frac{(-\alpha + r_f + \frac{1}{2}\sigma^2)t - \ln(1+k) + \sum_{j=1}^{p} \beta_j F_0^j}{\sigma\sqrt{t}}, \\ \gamma &= 2\left(\alpha - r_f - \frac{1}{2}\sigma^2\right), \\ \sigma_s &= \sqrt{\left(\frac{\sum_{j=1}^{p} \beta_j^2 \sigma_j^2}{3} + \sigma^2\right)} \quad . \end{aligned} \tag{30}$$