Extending the Merton Model with Applications to Credit Value Adjustment

Erdinc Akyildirim
e.akyildirim@bradford.ac.uk
School of Management, University of Bradford, Bradford, United Kingdom
Department of Management, Bogazici University, Istanbul, Turkey
Department of Banking and Finance, University of Zurich, Zurich, Switzerland

Alper A. Hekimoglu
European Investment Bank(EIB), Model Validation Unit, Luxembourg
a.hekimoglu@eib.org

Ahmet Sensoy[†]
ahmet.sensoy@bilkent.edu.tr.
Bilkent University, Faculty of Business Administration, Ankara, Turkey
Adnan Kassar School of Business, Lebanese American University, Beirut, Lebanon

Frank J. Fabozzi
EDHEC Business School, 06202 Nice cedex 3, France
frank.fabozzi@edhec.edu

Abstract

Following the global financial crisis, the measurement of counterparty credit risk has become an essential part of the Basel III accord with credit value adjustment being one of the most prominent components of this concept. In this study, we extend the Merton structural credit risk model for counterparty credit risk calculation in the context of calculating the credit value adjustment mainly by estimating the probability of default. We improve the Merton model in a variance-convoluted-gamma environment to include default dependence between counterparties through a linear factor decomposition framework. This allows one to tackle dependence through a systematic common component. Our set-up allows for easier, faster and more accurate fitting for the credit spread. Results confirm that use of the variance-gamma-convolution clearly solves the vanishing credit spread problem for short time-to-maturity or low leverage cases compared to a Brownian motion environment and its modifications.

[*]Ahmet Sensoy gratefully acknowledges support from Turkish Academy of Sciences under its Outstanding Young Scientist Award Programme (TUBA-GEBIP). Frank J. Fabozzi acknowledges the financial support from EDHEC Business School.

[†]Corresponding author. Address: Bilkent University, Faculty of Business Administration, Office: MA118, Cankaya 06800, Ankara, Turkey. E-mail address: ahmet.sensoy@bilkent.edu.tr; Tel.: +90-312-290-2048.

Keywords: Finance, Structural credit risk, Merton model, Variance-gamma process, Credit value adjustment

JEL: C51, C52, G12, G13

1 Introduction

Counterparty risk in the derivatives market arises when a party to an over-the-counter (OTC) derivatives contract may fail to perform on its contractual obligations, causing losses to the other party. Since the failure of Lehman Brothers in September 2008 that almost caused the collapse of the global banking system, counterparty risk has been a major concern in the financial markets all around the world.

During the turmoil of the 2008 financial crisis, total global defaults on debt were $430 billion, up from just $8 billion in 2007. The chain reaction of these defaults was mainly in the financial sector, which makes the counterparty risk a particular problem for banks, insurers, and other financial institutions. Even though some financial institutions have responded to this through improved counterparty risk measurement techniques or risk management departments with higher authorities, the precautions seem not to be enough. As of 2016, global net credit exposure to such risk in the OTC markets was around $3.7 trillion and counterparty credit risk still poses a major threat to the global financial system (BIS, 2016).

In the meantime, the financial regulatory environment has also become stricter in dealing with financial derivatives, highlighting the importance of measuring counterparty credit risk. Obviously, the measurement of this risk is not only a main challenge for financial institutions but also for regulators. Therefore, the new regulatory environment has changed rapidly with the objective of reducing the risk of bank failures and increasing financial stability. Accordingly, this regime shift such as the transition from Basel II (2005) to Basel III (2010) has pointed to the need of an enhanced sensitivity of credit risk measurement and helped in constructing a more robust composition of capital via capital requirements related to counterparty risk exposures.

Equipped with these recent industry and regulatory updates, capital requirements have become linked to more sophisticated measures of counterparty credit risk such as credit value adjustment (CVA), debt value adjustment and their variants. There have been many studies in recent years that have analyzed CVA and its variants. For example, bilateral counterparty risk is closely examined by Hull and White (2012); Brigo et al. (2013) and Fabozzi et al. (2015) in a wrong-way context. In terms of efficient simulation, Bonollo et al. (2015, 2016) lay out a procedure regarding CVA estimation via quantization and Brownian local-time methodologies. The potential volatility of counterparty credit risk has also been captured via value-at-risk of CVA in conjunction with stress testing under extreme market scenarios (Ballotta et al., 2016). More recently, Brigo and Vrins (2018) and Antonelli et al. (2021) consider the problem of CVA computation in the presence of wrong-way risk via change of measures and correlation expansions, respectively.

Time-changed processes, a special class of Lévy process, have recently been preferred in the literature because they provide a better fit for estimating volatility smile-skew and implied volatility surface (Fabozzi et al., 2014). In the counterparty credit risk context, Ballotta and Fusai (2015) study a CVA estimation procedure within a multivariate structural Lévy process framework. They use a linear factor decomposition for a normal inverse Gaussian process to account for default dependence. In terms of asset pricing, Madan (1998) was the first to study the variance-gamma process for European option pricing (see also Madan and Milne (1991)). Yang and Kanniainen (2017) find that with affine variance specification, variance-gamma return jumps perform best in option pricing in the short-term sample period. In the context of credit derivative pricing, Luciano and Schoutens (2006) use the variance-gamma process in a Black-Cox type model framework. Ballotta et al. (2017) propose a model based on a multivariate construction for Lévy processes for the pricing of FX derivatives, including Quanto products. In a more general setting, Ballotta et al. (2019) adopt a general multivariate model for Lévy processes, which accounts for the impact of dependence between the components of the portfolio, and propose a consistent and computationally efficient estimation procedure that is suitable for portfolio risk measurement. Recently, Gnoatto et al. (2020) propose a deep xVA solver to calculate any type of valuation adjustments (including CVA) for derivative portfolios with a large number of factors. The study uses deep neural networks due to their universal function approximation capability to solve backward stochastic differential equations and later defines a stochastic optimal control problem. This method approximates the portfolio process subject to factors using deep networks. The authors then use this network to calculate the valuation adjustments which are functions of these approximated processes. Our approach differs from that of Gnoatto et al. (2020) in that we derive dependence through factor decompositions and provide analytical expressions using special functions to finally obtain an overall semi-analytical framework to calculate CVA, whereas Gnoatto et al. (2020) use a machine learning approach and cover the entire xVA space.

In this paper, we build upon the earlier studies on counterparty credit risk analysis using a variance-gamma process, a Lévy process with finite moments that differs from many Lévy processes. We start with the framework of the Merton model in the context of a Brownian motion diffusion process, and then extend the same concept for a variance-gamma process where we modify it by a gamma convolution. We use a linear factor decomposition which introduces dependence between asset prices with an underlying variance-gamma convolution process, where we use two models of dependence: (i) a stochastic clock linear factor decomposition, and (ii) a symmetric variance-gamma factor decomposition. These setups lead to the formula for the probability of default (PD) calculation. Finally, after we calibrate the structural PD model to real market credit default swap (CDS) data, we then implement structural model calibration and use these tools to calculate CVA for counterparty credit risk.

Our study differs from earlier ones in two ways. First, we derive and use a gamma convolution density and a multivariate gamma density to arrive at semi-closed form formulas for the purpose of instrument pricing and counterparty credit risk calculations. In order to efficiently implement these, we benefit from different approximation schemes which have successful performances. These pave the way on how factor decompositions could be manipulated to derive useful formulas for option pricing and multivariate asset models. Using the gamma convolution idea, we arrive at a brand-new closed-form formula in a Black-Scholes structure without resorting to characteristic function inversions used in earlier studies to obtain prices of linear Lévy factor models. Moreover, we do not force asset prices to be variance-gamma distributed which creates a fully flexible factor environment. European option pricing (factor based) formula is used both in our option valuation for counterparty credit risk estimation and Merton-type PD estimation. Besides, we obtain the market value of assets, drift and volatility of assets in a Merton model sense. We also use a multivariate gamma model to account for CVA dependence and wrong-way risk by designing different scenarios by changing dependence and sensitivity parameters of the linear factor decomposition.

The rest of this paper is organized as follows. Section 2 extends the Merton structural credit risk model using the variance-gamma convolution process. Section 3 introduces a linear factor model derived by modifying the structural extension. Section 4 provides a European option pricing and a PD formula for linear factor variance-gamma convolution model. Section 5 calibrates the structural PD model to real market data and implements different algorithms to obtain the market value of assets. Section 6 estimates CVA using the extended structural model. Section 7 concludes.

2 Extension of the Merton Model with Lévy Processes

For a brief introduction to the Merton Model, we refer the reader to the Appendix A.0. In the following subsections, we introduce the variance-gamma process and extend the Merton structural credit risk model via the variance-gamma convolution.

2.1 The Variance-Gamma Process

Introduced by Madan and Seneta (1990), the variance-gamma (VG) model is defined as a time-changed Brownian motion with/without drift where time change is a gamma process. Given the probability space $(\Omega, \mathsf{P}, \mathcal{F})$ and for $\sigma, \nu \in \mathsf{R}^+$, $\theta \in \mathsf{R}$, the model can be defined as follows

$$\begin{aligned} B(t,\sigma,\theta) &= \theta t + \sigma W(t), \\ X(t,\sigma,\gamma,\theta) &= B(\gamma(t,1,\nu),\sigma,\theta) = \theta\gamma(t,1,\nu) + \sigma W(\gamma(t,1,\nu)). \end{aligned} \tag{1}$$

The probability density function (pdf) and cumulative distribution function (cdf) of the VG distribution (Hirsa, 2012) are given as

$$f_{X(t)}(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} e^{\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right)} \frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg,$$

$$F_{X(t)}(x) = \mathsf{E}\left(\mathsf{I}_{X(t)\leq x}\right) = \int_0^\infty \mathcal{N}\left(\frac{x-\theta g}{\sigma\sqrt{g}}\right) \frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg,$$

where Γ denotes the Gamma function.

The pdf above can further be written in a semi-closed form (Madan, 1998) as the following

$$f_{X(t)}(x) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{\tau}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{\tau}{\nu})}\left(\frac{(x-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{2-\nu}{4\nu}} K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right), \tag{2}$$

where $\tau = T - t$ and $K_\alpha(y)$ is the modified Bessel function of the second kind. Then the cdf follows

$$F_{X(t)}(x) = \int_{-\infty}^{x} \frac{2e^{\frac{\theta y}{\sigma^2}}}{\nu^{\frac{\tau}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{\tau}{\nu})}\left(\frac{(y-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{2-\nu}{4\nu}} K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(y-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right) dy. \tag{3}$$

Finally, the characteristic function for the VG process can be represented by the following expression

$$\Phi(u,t) = \left(\frac{1}{1 - iu\theta\nu + 0.5u^2\sigma^2\nu}\right)^{\frac{t}{\nu}}, \tag{4}$$

where $i = \sqrt{-1}$.

2.2 The Variance-Gamma Merton Model

By using the formulas given in the previous section, we apply Merton's methodology for the VG process. However, the Merton model requires the estimation of the unobserved market value of the assets, $A(t)$, and its volatility parameter, σ_A. For this purpose, we derive the conditions similar to those used in Merton's structural model. Since the equity value in the structural Merton model satisfies the Black-Scholes (B-S) option pricing model (see Appendix A.0), we can use the following equations to estimate $A(t)$ and σ_A,

$$E(t)\sigma_E dW(t) = \sigma_A \frac{\partial E(t)}{\partial A(t)} dW(t), \tag{5}$$

$$E(t) = A(t)\mathcal{N}(d_1) - De^{-r(T-t)}\mathcal{N}(d_2). \tag{6}$$

The estimation methodology based on the equations above is straightforward for the other Lévy processes with jump components or subordinators. However, due to its pure jump nature, it is not trivial to apply the same reasoning in the VG environment. The system given by equations (5) and (6) alleviates the estimation of $A(t)$ and σ_A considerably. In our VG case, we need to estimate the parameters $A(t), \theta_A, \sigma_A$, and ν_A. In addition to these parameters, θ_E, σ_E, and ν_E are also required since equity is to follow its own VG process to apply Merton's analogy. In order to find these parameters, first we write the SDEs of $A(t)$ and $E(t)$, the asset and equity processes respectively,

$$
\begin{aligned}
dA(t) &= A(t)\left(rdt + dX_A(t)\right), && (7)\\
dE(t) &= E(t)dX_E(t), && (8)
\end{aligned}
$$

where

$$
\begin{aligned}
dX_A(t) &= \theta_A\gamma + \sigma_A W(\gamma), && (9)\\
dX_E(t) &= \theta_E\gamma + \sigma_E dW(\gamma), && (10)
\end{aligned}
$$

and their exponential martingale solutions are given by

$$
\begin{aligned}
A(t) &= A(0)e^{(r-\phi_A(-i))t+X_A(t)}, && (11)\\
E(t) &= E(0)e^{-t\phi_E(-i)+X_E(t)}, && (12)
\end{aligned}
$$

where $\phi_E(-i)$ and $\phi_A(-i)$ are the martingale correction terms, and $X_E(t)$ and $X_A(t)$ are the VG components of these processes, respectively.

The equity $E(t)$ of the company, which is the option price in Merton's context, is a function of $A(t)$. Using Ito's lemma for jump processes (Cont and Tankov, 2004), we can write the following partial integro differential equation (PIDE) for the equity process $E(t)$[1],

$$
\begin{aligned}
dE(t) = \Bigg(\int_{-\infty}^{\infty}\Big[E(A(t-)e^{X_A(t)},t) - E(A(t-),t) - \frac{\partial E(A(t),t)}{\partial A(t)}(e^{X_A(t)}-1)\Big]\nu(dx) \qquad (13)\\
- rE(t) + \frac{\partial E(t)}{\partial t} + r\frac{\partial E(t)}{\partial A(t)}A(t)\Bigg)dt + \frac{\partial E(t)}{\partial A(t)}dX_A(t)A(t).
\end{aligned}
$$

Similar to the Brownian motion case, the VG implied call option price can be found by setting the dt part of the VG PIDE equal to zero. Then by using the equations (13) and (8), we write the following

$$
\frac{\partial E(t)}{\partial A(t)}dX_A(t)A(t) = E(t)dX_E(t). \qquad (14)
$$

Using equation (A.9) in the Appendix A.1, we can write $\frac{\partial E(t)}{\partial A(t)} = F_{VG}(d_1)$ where F_{VG} is the cumulative distribution function under the VG process. From here, we obtain the following system

[1] In equation (13), the notations $E(A(t-),t)$, $E(A(t),t)$, and $E(A(t-)e^{X_A(t)},t)$ denote that the equity value at time t depends on $A(t-)$, $A(t)$, and $A(t-)e^{X_A(t)}$, respectively. $A(t-)$ and $A(t-)e^{X_A(t)}$ refer to the asset value before and after the jump, respectively.

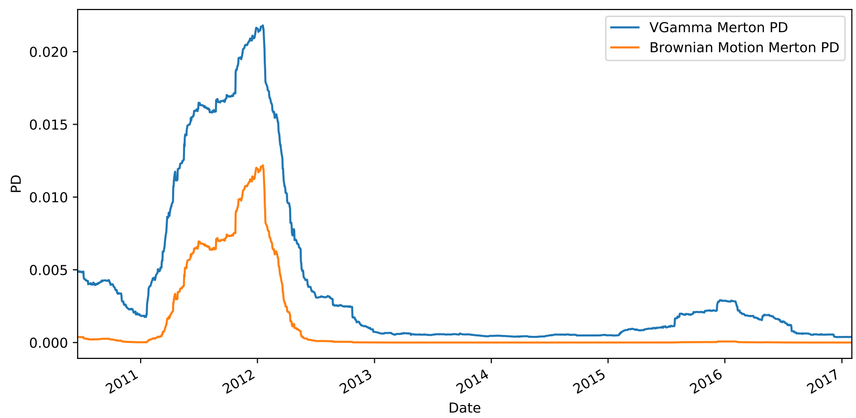


Figure 1: VG Merton model PD and Brownian motion Merton model PD for JP Morgan between the years 2010 and 2017.

of equations

$$\begin{aligned} E(t)\theta_E &= F_{VG}(d_1)\theta_A A(t), \\ E(t)\sigma_E &= F_{VG}(d_1)\sigma_A A(t), \\ \nu_A &= \nu_E, \end{aligned} \tag{15}$$

where ν_A and ν_E are the gamma scale parameters; (θ_A, σ_A) and (θ_E, σ_E) are the drift-volatility parameters sets for the VG processes of $A(t)$ and $E(t)$, respectively. We set the gamma scale parameters to be the same since we assume $\sigma-$algebra, $\mathcal{G}$, generated by the subordinator in both VG processes, are the same as they are for the same company. Similar to equation (6), we write the call option price under the VG process as follows

$$E(t) = A(t)F_{VG}(d_1) - De^{-r(T-t)}F_{VG}(d_2). \tag{16}$$

From the four equations given by (15) and (16), we estimate the parameters $A(t), \theta_A, \sigma_A$, and ν_A by using the simultaneous equations method of Ronn and Verma (1986). Then we calculate the risk-neutral PD from $F_{VG}(d_2)$.

For an empirical application regarding these findings, we use JP Morgan's balance sheet liabilities and its market value of equity data for the years between 2010 and 2017. We compare the VG Merton model and Brownian motion Merton model where the latter is a conventional model used for structural PD estimation. We see in Figure 1 that the VG model clearly provides more realistic results compared to the Brownian motion model due to its better fit properties for financial data (Geman and Ane, 1996). This is also important in case the PD from the model is used as an input (e.g., for the CDS spread calculation). Moreover, as stated by Gemmil and Marra (2019), the Merton model needs a fat tail correction due to its unrealistic assumption of a log-normal asset value distribution. Without any modifications, the Brownian motion Merton model is difficult to calibrate to a non-vanishing PD in a short-term horizon without assuming particularly high short-term volatility (Brigo et al., 2013). Accordingly, we see in Figure 1 that the Brownian motion model shows a vanishing PD for the low risk area in the late periods of our sample which is not realistic due to the fat-tail property. This comes from the fact that unless there is an increase in short-term volatility, the classical Merton model estimates extremely low PDs, which becomes a non-robust benchmark for the market data. However, the VG model shows a reasonable PD given the liability level due to its ability to account for the fat-tail effect.

3 Variance Gamma Convolution Factor Model

Let $X(t)$ be an R-valued VG process with probability space $(\Omega, \mathcal{F}, P)$ and let $\gamma_X(t)$ be the related R-valued gamma process. We further assume that $I(t)$ is another R-valued VG process and $\gamma_I(t)$ is an R-valued gamma process. Likewise, let $\gamma_Z(t)$ be the related univariate gamma process which is independent from $\gamma_I(t)$ and we further conjecture $\gamma_X(t) = \gamma_I(t) + c^2\gamma_Z(t)$. Then the structure of the linear VG convolution (VGC) factor model can be written as,

$$X(t) = \theta_X \gamma_X(t) + \sigma_X(I(t) + cZ(t)), \tag{17}$$

where $c \in \mathsf{R}$ shows the sensitivity of $X(t)$ to the systematic component $Z(t)$, θ_X is the drift parameter, and σ_X is the diffusion parameter. Futhermore, by definition, we can write $I(t)$ and $Z(t)$ as

$$\begin{aligned} I(t) = W_I(\gamma_I(t)) = \sqrt{\gamma_I} W_I(t), \\ Z(t) = W_Z(\gamma_Z(t)) = \sqrt{\gamma_Z} W_Z(t). \end{aligned} \tag{18}$$

Using the fact that sum of two normal random variables is again a normal random variable and conditioning on γ_I and γ_Z, we obtain the following

$$\sqrt{\gamma_I} W_I(t) + c\sqrt{\gamma_Z} W_Z(t) \overset{d}{=} \sqrt{(\gamma_I + c^2\gamma_Z)} W(t) \sim \mathcal{N}\left(0, t(\gamma_I + c^2\gamma_Z)|\gamma_I, \gamma_Z\right),$$

where $\gamma_I \sim Ga(\frac{1}{\nu_I}, \nu_I)$, $\gamma_Z \sim Ga(\frac{1}{\nu_Z}, \nu_Z)$, and $W(t)$ is another Brownian motion.

Then using these we conclude that

$$X(t) \overset{d}{=} \theta_X \gamma_X(t) + \sigma_X W\left(\gamma_X(t)\right),$$

where $\gamma_X(t) = \gamma_I(t) + c^2\gamma_Z(t)$ is our new subordinator which is a linear sum of two weighted gamma random variables.

We can also write $X(t)$ in a more compact form as

$$X(t) = \theta_X \gamma_X(t) + \sigma_X W(\gamma_X(t)), \tag{19}$$

and for the stochastic clock factor decomposition case, we have the structure of the VG process

$$X(t) = \theta_X \left(\gamma_I(t) + a\gamma_Z(t)\right) + \sigma_X W\left(\gamma_I(t) + a\gamma_Z(t)\right), \tag{20}$$

where the coefficient $a \in \mathsf{R}^+$ shows the sensitivity to systematic stochastic clock component. We now write two VGC random variables having this structure in an explicit time-changed form to see what drives the co-movement,

$$\begin{aligned} X_1(t) &= \theta_{1X}\left(\gamma_{1I}(t) + a_1\gamma_Z(t)\right) + \sigma_{1X} W_1(t)\sqrt{(\gamma_{1I} + a_1\gamma_Z)}, \\ X_2(t) &= \theta_{2X}\left(\gamma_{2I}(t) + a_2\gamma_Z(t)\right) + \sigma_{2X} W_2(t)\sqrt{(\gamma_{2I} + a_2\gamma_Z)}. \end{aligned} \tag{21}$$

From the above equations, we observe that the main factor driving co-movement is the γ_Z, the systematic term in the decomposed time-change forms $\sqrt{(\gamma_{1I} + a_1\gamma_Z)}$ and $\sqrt{(\gamma_{2I} + a_2\gamma_Z)}$, whereas $W_1(t)$ and $W_2(t)$ are uncorrelated.

Although these two decompositions given by equations (17) and (20) look similar, the main difference is the components of dependence. In the VGC factor form, the components are themselves symmetric VG variables and by the gamma convolution we have a final time change similar to the stochastic clock form.[2] Therefore, in VGC factor form, due to the common $Z(t)$ term, the dependence contains both the effect of the normally distributed variable and its gamma time change. However, in the stochastic clock decomposition, the dependence comes only from the time change component. Moreover, we decompose the time change as idiosyncratic and systematic, and systematic stochastic clock component drives the co-movement.

It is important to note that in the construction of equation (17), the final result will involve a multivariate normal distribution function conditioned on the gamma time change. This is computationally more expensive compared to the univariate normal distribution function which is the case for the stochastic clock factor model in the construction of equation (20) where normally distributed components between two VGC processes are uncorrelated (here uncorrelated variables refer to $W_1(t)$ and $W_2(t)$ in equation (21)).

Abovementioned methods are useful for cases that require dependence between counterparties and the derivative's underlying asset. The dependent VG process in the context of credit risk has

[2] We call the construction in equation (17) a linear VGC factor framework because the final variable $X(t)$ is no longer a VG random variable. This results from the fact that the time change is a convolution of two gamma variables with completely different parameters which in turn is not a gamma variable anymore.

been studied by Moosbrucker (2006) and in a general multivariate setting by Luciano and Schoutens (2006) and Ballotta and Bonfiglioli (2016). The use of such models has some advantages, particularly for calibration associated with dependence structure. In this regard, Fiorani et al. (2010) mention the difficulty of calibration in copula-based models. However, Fabozzi et al. (2015) use a trivariate copula model to integrate the wrong-way risk into the counterparty credit risk calculation and show the efficiency of their model. Still, we use a linear factor structure with a common component because of its flexible structure and simplicity, and we show that this approach solves the problem of both dependence and calibration. In particular, although our linear VGC factor process is similar to the one used by Ballotta and Fusai (2015), Ballotta and Bonfiglioli (2016), and the stochastic clock factor process of Luciano and Schoutens (2006), there are significant differences between our approach and theirs. Our first decomposition structure involves a symmetric variance gamma process for which the tractability and closed-form solutions are possible while preserving full flexibility and closed-form representations (e.g., our process allows the sum of linear factors to be no longer variance gamma together with closed-form solutions). In our second decomposition that is the factor stochastic clock model, different from Luciano and Schoutens (2006) we decompose the time change into the systematic and idiosyncratic factors and then we use the gamma convolution result and a multivariate gamma structure to model dependence. Then we present appropriate closed-form solutions for pricing and credit risk purposes. Finally, we use an efficient approximation for this gamma convolution which enables us to derive closed/semi-closed form solutions in the context of CVA.

4 Gamma Convolution and Multivariate Gamma Model

In this section, we present some useful results which are utilized for price and dependence modelling. The first result is related to the exact expression of the convolution of two gamma random variables (see Di Salvo (2008)). Di Salvo (2008) derives an exact formula for the sum of weighted gamma variables with different shapes but the same scale parameters involving multiple hypergeometric functions. We now present a further result again for the sum of two weighted gamma random variables but with different shape and scale parameters in the context of our parametrization, which we present a detailed proof in the Appendix A.2.

Moreover, we present a bivariate gamma distribution result where we make use of dependence between default and price processes. Although we follow a similar approach to Mathai and Moschopoulos (1991) when we define dependence in our setting, our parametrization is different from theirs. What is more, in the two variable case, we derive a special form different than their representation which turns out to be more efficient for our purposes. Our derivations which involve Humbert's hypergeometric function of two variables are given in the Appendix A.4.

The gamma sum in our case which is formulated as $\gamma_X(t) = \gamma_I(t) + c^2\gamma_Z(t)$ is a convolution of two gamma variables with weight/sensitivity components and is equipped with completely different shape and scale parameters. The following theorem presents the density of this process:

Theorem 1. Let $\gamma(\nu_1, \beta_1)$ and $\gamma(\nu_2, \beta_2)$ be two independent gamma random variables with the density given by

$$f(\gamma_i, \nu_i, \beta_i) = \frac{\gamma_i^{\nu_i - 1} e^{\frac{-\gamma_i}{\beta_i}}}{\Gamma(\nu_i)} \beta_i^{-\nu_i} \qquad for \quad i = 1, 2. \tag{22}$$

Then the new variable $\zeta\left(\nu_1, \nu_2, \beta_1, \beta_2\right) = c_1 \gamma_1 \left(\nu_1, \beta_1\right) + c_2 \gamma_2 \left(\nu_2, \beta_2\right)$ will have the density,

$$f\left(\zeta, \nu_1, \beta_1, \nu_2, \beta_2\right) = \zeta^{(\nu_1 + \nu_2 - 1)} e^{-\frac{\zeta}{c_2 \beta_2}} (c_1 \beta_1)^{-\nu_1} (c_2 \beta_2)^{-\nu_2} \frac{{}_1F_1\left(\nu_1, \nu_1 + \nu_2, \left(\frac{1}{c_1 \beta_1} - \frac{1}{c_2 \beta_2}\right) \zeta\right)}{\Gamma(\nu_1 + \nu_2)}, \tag{23}$$

and the corresponding characteristic function is,

$$\Phi_{\zeta_2}(u, \nu_1, \nu_2, \beta_1, \beta_2, c_1, c_2)^{3} = C \left(\frac{2}{c_2 \beta_2} - \frac{1}{c_1 \beta_1} - iu\right)^{-\nu_1 - \nu_2} \times F \left(\nu_2, \nu_1 + \nu_2, \nu_1 + \nu_2, \frac{k}{k - \left(\frac{1}{c_2 \beta_2} - iu\right)}\right) \tag{24}$$

where $C = (c_2 \beta_2)^{-\nu_2} (c_1 \beta_1)^{-\nu_1}$, ${}_1F_1$ is confluent hypergeometric function of the first kind and F is Gaussian hypergeometric function.

Proof. See Appendix A.2 and equation (A.13). □

We then present Theorem 2 to model dependence between processes used in the pricing and credit risk modelling in our setting,

Theorem 2. Let $Z_1\left(\nu_1, \beta_1\right) = X_1 + c_1 Y$ and $Z_2\left(\nu_2, \beta_2\right) = X_2 + c_2 Y$ be correlated random variables then their joint density can be defined, when $Z_1 \leq Z_2$,

$$\begin{aligned} f(z_1, z_2, \nu_1, \nu_2, \nu_3, \beta_1, \beta_2, \beta_3) = \quad & \frac{z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} e^{\frac{-z1}{\beta_1}} e^{\frac{-z2}{\beta_2}}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} B(\nu_1, \nu_2) \\ & \times \Phi_1 \left(\nu_1, 1 - \nu_3, \nu_1 + \nu_2; \frac{z_1}{z_2}, z_1 \left(\frac{1}{\beta_2} + \frac{1}{\beta_3} - \frac{1}{c_1 \beta_1}\right)\right), \end{aligned} \tag{25}$$

and when $Z_1 \geq Z_2$ we have,

$$\begin{aligned} f(z_1, z_2, \nu_1, \nu_2, \nu_3, \beta_1, \beta_2, \beta_3) = \quad & \frac{z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} e^{\frac{-z1}{\beta_1}} e^{\frac{-z2}{\beta_2}}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} B(\nu_1, \nu_3) \\ & \times \Phi_1 \left(\nu_1, 1 - \nu_2, \nu_1 + \nu_3; \frac{z_2}{z_1}, z_2 \left(\frac{1}{\beta_3} + \frac{1}{\beta_2} - \frac{1}{c_2 \beta_1}\right)\right), \end{aligned} \tag{26}$$

where Φ_1 is a confluent hypergeometric function of two variables (Humbert (1922)), $B(x, y)$ is beta function, and X_1, X_2, Y are distributed with $Ga(\nu_1, \beta_1), Ga(\nu_2, \beta_2), Ga(\nu_3, \beta_3)$, respectively.

[3] Setting $c_2 \beta_2 = c_1 \beta_1$ one can easily recover a gamma random variable characteristic function, which is a special case for the sum of two gamma random variables, confirms that this formula is most general for a gamma convolution.

In the case of three correlated random variables Z_i, Z_j, Z_k of the form $Z_j = X_j + c_j Y$, we obtain the following density, where without loss of generality we assume $Z_i = \min(Z_i, Z_j, Z_k)$, $i, j, k \in \{1, 2, 3\}$

$$f(z_i, z_j, z_k, \alpha_0, \alpha_i, \alpha_j, \alpha_k, \beta_0, \beta_i, \beta_j, \beta_k) = \frac{z_i^{\alpha_i - 1} z_j^{\alpha_j - 1} z_k^{\alpha_k - 1} e^{-\frac{z_i}{\beta_i}} e^{-\frac{z_j}{\beta_j}} e^{-\frac{z_k}{\beta_k}} \beta_i^{-\alpha_i} \beta_j^{-\alpha_j} \beta_k^{-\alpha_k}}{\Gamma(\alpha_0)\Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_k)} \times \Phi_2\left(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3, z_1, z_2, z_3\right) \tag{27}$$

where Φ_2 is a semi-closed form representation of the density whose derivation is given in Appendix A.3. This joint distribution, which is a composition of equation (25) and equation (26), will be used for the modelling of co-dependence between default and instrument prices.

Our next objective is to derive the European option price implied by this model which requires the use of density given by equation (23). Furthermore, in order to keep the mathematical tractability, we also approximate this density with another gamma variable using moment matching where we utilize Satterthwaite (1946) approximation method for χ^2 variables. We obtain this approximation by first writing the first two moments of the gamma distribution:

$$\begin{aligned} \mu_\zeta &= c_1 \beta_1 \nu_1 + c_2 \beta_2 \nu_2, \\ V_\zeta &= c_1^2 \beta_1^2 \nu_1 + c_2^2 \beta_2^2 \nu_2. \end{aligned} \tag{28}$$

Then we use the following general relationship between the first two moments of the gamma distribution and its shape parameters ν, β to obtain ν_ζ and β_ζ parameters which are the pseudo-gamma distribution's parameters used for the approximation:

$$\nu = \frac{\mathsf{E}(X)^2}{\mathsf{V}(X)}, \quad \beta = \frac{\mathsf{V}(X)}{\mathsf{E}(X)}.$$

Solving these two equations with two unknowns given the exact first two moments we obtain,

$$\begin{aligned} \nu_\zeta &= \frac{(c_1 \beta_1 \nu_1 + c_2 \beta_2 \nu_2)^2}{c_1^2 \beta_1^2 \nu_1 + c_2^2 \beta_2^2 \nu_2}, \\ \beta_\zeta &= \frac{c_1^2 \beta_1^2 \nu_1 + c_2^2 \beta_2^2 \nu_2}{c_1 \beta_1 \nu_1 + c_2 \beta_2 \nu_2}. \end{aligned} \tag{29}$$

Here, the variable ζ corresponds to the gamma convolution random variable in equation (23) and using the approximation formula in equation (29), we obtain a gamma approximated version of this variable. Using these parameters, we finally write the following density:

$$f\left(\zeta, \nu_1, \beta_1, \nu_2, \beta_2\right) \approx \frac{\zeta^{\nu_\zeta - 1} e^{\frac{-\zeta}{\beta_\zeta}} \sigma_\zeta^{-\nu_\zeta}}{\Gamma\left(\nu_\zeta\right)}, \tag{30}$$

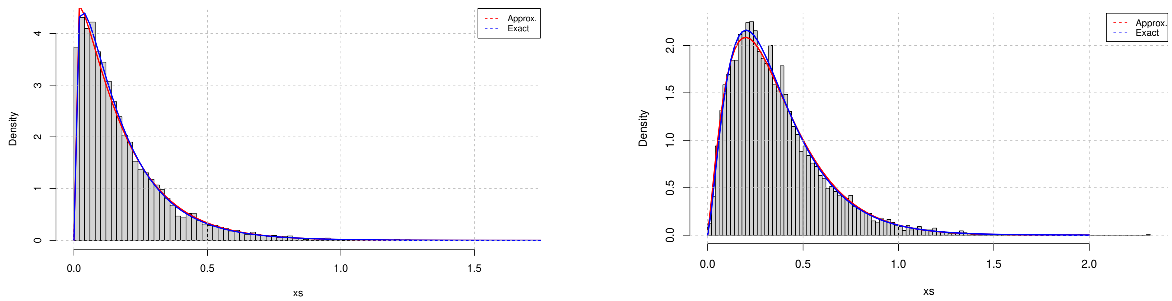
and conclude that $\zeta(\nu_1, \beta_1, \nu_2, \beta_2) \approx \gamma(\nu_\zeta, \beta_\zeta)$ is a gamma random variable with parameters ν_ζ, β_ζ.

In order to conduct a robustness check, we generate 10,000 simulated two gamma random variables with low shape parameters $\nu_{i,j} = \{1.4, 1.1\}$ and high shape parameters $\nu_{i,j} = \{2.8, 2.2\}$ and then low scale parameters $\beta_{i,j} = \{0.1, 0.05\}$ and high scale parameters $\beta_{i,j} = \{0.2, 0.5\}$ with $(i, j) = 1, 2$. From Figures 2 and 3, we observe that both exact densities of $\Gamma(\nu i, \beta_i) + \Gamma(\nu_j, \beta_j)$ convolution distribution and approximated densities fit quite well the simulated variables' histogram despite changes in parameters. We then conduct Kolmogorov-Smirnov and Wilcox non-parameteric tests. The results are presented in Table 1, and they further statistically show the robustness of the approximation.

Table 1: KS and Wilcox Non-Parametric Tests

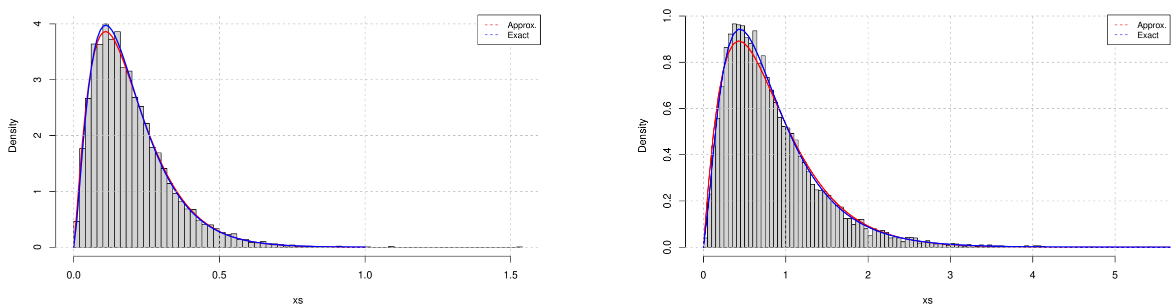
Parameter	KS-test p-val	Wilcox test p-val
Low-Shape	0.2510	0.2430
High-Shape	0.1595	0.3722
Low-Scale	0.6637	0.5391
High-Scale	0.1545	0.8209

Overall, we see that both visually and statistically, the gamma approximation and exact Gamma convolution formula work well numerically given fit the histogram, despite the necessary use of special functions. The only observable effect of parameters might be seen in Panel (b) of Figure 3, which shows a slightly upward-move effect of the shifted scale parameter.



(a) gamma density with low-shape (b) gamma density with high-shape

Figure 2: Approximate Gamma Convolution Density and Exact Gamma Convolution Density with Shape Shifts



(a) gamma density with low-scale (b) gamma density with higher-scale

Figure 3: Approximate Gamma Convolution Density and Exact Gamma Convolution Density with Scale Shifts

Now, we will first derive the exact density of $X(t)$ by using equation (23) and then derive the approximated density by using the gamma approximation given by equation (30).

Since $X(t) = \theta_X \gamma_X(t) + \sigma_X(I(t) + cZ(t))$ and $\gamma_X(t) = \gamma_I(t) + c^2\gamma_Z(t)$, we observe that, $X(t) \stackrel{d}{=} \theta_X \gamma_X(t) + \sigma_X W(\gamma_X(t))$ then we first write,

$$\mathsf{P}\left(X(t) < x\right) = \Psi(x) = \int_0^\infty \mathcal{N}\left(\frac{(x-\mu) - \theta_X \zeta}{\sigma_X \sqrt{\zeta}} \middle| \gamma_X = \zeta\right) \zeta^{\left(\frac{t}{\nu_I} + \frac{t}{\nu_Z} - 1\right)} e^{-\frac{\zeta}{c^2 \nu_Z}} \nu_I^{\frac{-t}{\nu_I}} (c^2 \nu_Z)^{\frac{-t}{\nu_Z}} \times \frac{{}_1F_1\left(\frac{t}{\nu_I}, \frac{t}{\nu_I} + \frac{t}{\nu_Z}, \left(\frac{1}{\nu_I} - \frac{1}{c^2\nu_Z}\right)\zeta\right)}{\Gamma\left(\frac{t}{\nu_I} + \frac{t}{\nu_Z}\right)} d\zeta. \tag{31}$$

Then for the gamma approximation case we have:

$$\mathsf{P}\left(X(t)<x\right)=\hat{\Psi}(x)=\int_{0}^{\infty}\mathcal{N}\left(\frac{(x-\mu)-\theta_X g}{\sigma_X\sqrt{g}}\big|\gamma_X=g\right)\frac{g^{\nu-1}e^{\frac{-g}{\beta}}\beta^{-\nu}}{\Gamma\left(\nu\right)}dg, \tag{32}$$

where $\nu=\frac{T\left(1+c^2\right)^2}{\nu_I+c^4\nu_Z}$ and $\beta=\frac{\nu_I+c^4\nu_Z}{(1+c^2)}$ which are obtained by setting $\beta_1=\beta_2=1$ in equation (29).

The above equation can further be written in terms of the confluent hypergeometric function of two variables as the following[4] (see Appendix A.5 for the details of the derivation):

$$\begin{aligned}\hat{\Psi}\left(x,\theta,\sigma,\nu,\beta,T\right)= \quad & D\Bigg[\frac{\Phi_1\left(\alpha,1-\alpha,1+\alpha,\frac{1+q}{2},c(1+q)\right)K_{\alpha+\frac{1}{2}}\left(n\sqrt{m^2+\frac{2}{\nu}}\right)}{\alpha\Gamma(\alpha)}\\ & -\frac{\Phi_1\left(1+\alpha,1-\alpha,2+\alpha,\frac{1+q}{2},c(1+q)\right)K_{\alpha-\frac{1}{2}}\left(n\sqrt{m^2+\frac{2}{\beta}}\right)}{(1+\alpha)\Gamma(\alpha)}\\ & +\frac{\Phi_1\left(\alpha,1-\alpha,1+\alpha,\frac{1+q}{2},c(1+q)\right)K_{\alpha-\frac{1}{2}}\left(n\sqrt{m^2+\frac{2}{\beta}}\right)}{\alpha\Gamma(\alpha)}\Bigg],\end{aligned} \tag{33}$$

where

$$\alpha=\nu, n=\frac{(x-\mu)}{\sigma}, m=\frac{\theta}{\sigma}, C=n\sqrt{m^2+\frac{2}{\beta}},$$
$$q=\frac{\sqrt{\left(\frac{(x-\mu)}{\sigma}\right)^2-2/\beta}}{\frac{(x-\mu)}{\sigma}}, D=e^{-C}\frac{\nu^{\alpha}C^{\alpha+\frac{1}{2}}}{\sqrt{2\pi\sigma^2}}.$$

It is also important to note that the construction in equation (17) leads to the following correlation coefficient for any two assets (k,j) in the above framework:

$$\rho^X_{k,j}=\frac{(c_jc_k)^2\theta_k\theta_j\nu_Z+\sigma_k\sigma_jc_kc_j}{\sqrt{\theta_k^2(\nu_k+\nu_Zc_k^2)+\sigma_k^2(1+c_k^2)}\sqrt{\theta_j^2(\nu_j+\nu_Zc_j^2)+\sigma_j^2(1+c_j^2)}}, \tag{34}$$

and the stochastic clock construction in equation (20) leads to the following correlation coefficient,

$$\rho^X_{k,j}=\frac{(a_ja_k)\theta_k\theta_j\nu_Z}{\sqrt{\theta_k^2(\nu_k+\nu_Za_k)+\sigma_k^2(1+a_k)}\sqrt{\theta_j^2(\nu_j+\nu_Za_j)+\sigma_j^2(1+a_j)}}. \tag{35}$$

[4]The sole purpose of the gamma approximation is to write equation (32) in terms of this special function. It is useful and will later be used in the derivation of another semi-closed form formula.

Now we are ready to derive the option price implied by the linear symmetric VGC factor process $X(t)$.

4.1 European Option Price Formula for the VGC Factor Model

Using the framework above, we can obtain the factor-extended variance-gamma option price. We start by defining the asset price process $S(t)$ under risk neutral measure,

$$S(T) = S(t)e^{(r-\phi(-i))\tau + X(\tau)}, \tag{36}$$

where $X(t)$ is defined as in equation (17) and $\tau = T - t$. Then we proceed by writing the general pricing equation for a call option,

$$C(K, r, T-t, \sigma) = \mathsf{E}\left(e^{-r(T-t)}(S(T)-K)^{+}|\mathcal{F}(t)\right).$$

Using the gamma convolution result in equation (17), we can write

$$C(K, r, T-t, \sigma) = \mathsf{E}\left(e^{-r\tau}\left(S(t)e^{\left((r-\phi(-i))\tau + \theta_X\gamma_{X(\tau)} + \sigma W\left(\gamma_{X(\tau)}\right)\right)} - K\right)^{+}\Bigg|\mathcal{F}(t)\right). \tag{37}$$

We then decompose this pay-off as

$$C(K, r, T-t, \sigma) = S(t)\mathsf{E}^{s}\left(\mathsf{I}_{S(T)>K}|\mathcal{F}(t)\right) - Ke^{-r\tau}\mathsf{E}\left(\mathsf{I}_{S(T)>K}|\mathcal{F}(t)\right). \tag{38}$$

Here let us define

$$\begin{aligned} A :&= \{\log(S(T)) > \log(K)\}, \\ &= \{\log\left(\frac{S(t)}{K}\right) + (r - \phi_X(-i))\tau + \theta\gamma_{X(\tau)} > -W\left(\gamma_{X(\tau)}\right)\}. \end{aligned}$$

Without loss of generality, conditioning on $\gamma_X = \zeta$, we can rewrite A as

$$A := \left\{\frac{\log\left(\frac{S(t)}{K}\right) + (r - \phi_X(-i))(T-t) + \theta\zeta}{\sigma\sqrt{\zeta}} > \frac{-W_\zeta}{\sigma\sqrt{\zeta}}\Big|\gamma_X = \zeta\right\}.$$

Then, given set A we obtain B-S type parameters d_1, d_2 for VGC factor model as

$$\begin{aligned} v_+ &= \frac{\log\left(\frac{S(t)}{K}\right) + (r - q - \phi_X(-i))\tau + \theta_s\zeta}{\sigma\sqrt{\zeta}}, \\ v_- &= \frac{\log\left(\frac{S(t)}{K}\right) + (r - q - \phi_X(-i))\tau + \theta\zeta}{\sigma\sqrt{\zeta}}, \end{aligned}$$

where $\theta_s = \theta + \sigma^2$, $\nu_s = \frac{\nu}{\kappa}$ and $\phi_X(-i)$ and κ are derived in Appendix A.3.

Using these parameters and adding the constant dividend term, we obtain our VGC factor call option formula in B-S model option price representation,

$$C(K, r, T-t, \theta_s, \theta, \sigma, \nu, \nu_s) = S(t)e^{-q(T-t)}\Psi\left(v_+\right) - Ke^{-r(T-t)}\Psi\left(v_-\right). \tag{39}$$

where $\Psi(x)$ is as defined in equation (31). We can further present the same structure using equation (32) which is a re-formulation of equation (31) by approximating gamma convolution using another gamma random variable,

$$C(K, r, T-t, \theta_s, \theta, \sigma, \nu, \nu_s) = S(t)e^{-q(T-t)}\hat{\Psi}\left(v_+\right) - Ke^{-r(T-t)}\hat{\Psi}\left(v_-\right). \tag{40}$$

where $\hat{\Psi}(x)$ is defined in equation (32) and further could be written in a closed-form representation as in equation (33).

Both of the representations above are compact which contain a factor structure and allow idiosyncratic and systematic factors to be embedded in a single pricing formula. However, the latter formulation yields efficient calibration since the numerical evaluation of confluent hypergeometric function of the first kind is more time consuming than the gamma density. As Figures 2 and 3 confirm, the latter representation does not lose any considerable precision because of its approximation nature.

5 Probability of Default Estimation and Parameter Calibration

Our methodology to estimate the PD requires simultaneous use of equations (15) and (16). In the first step, we use the Merton-type VG model in the context of calibration and estimation. For this purpose, we refer to the semi-closed form in equation (2) to estimate the PD. The semi-closed form is more convenient and numerically faster than using the product of the normal cumulative distribution function and gamma density function. The second and final step involves calibrating the PD model to the real market data using real-world and risk-neutral parameters.

5.1 Calibration of Parameters under Affine Factor Structure

In our empirical setup, we consider two firms: Deutsche Bank (DB), a financial services company, and ENI, an Italian oil and gas company. We use two methods to filter-out the VGC Merton structural model to find the market value of assets and related parameters. This approach will produce a real-world probability measure parameter set. The method of Ronn and Verma (1986), which is the first method that we employ, uses the solution to the system of equations given by equation (15) by using the functions $\Psi(x)$ in equation (39). The second method we use is given by Vassalou and Xing (2004) which relies on the iterative procedure to estimate parameters

Table 2: VG parameter estimates for DB and ENI using two different methods

Method	θ_{ENI}	θ_{DB}	σ_{ENI}	σ_{DB}
Ronn-Verma (RV-Simultaneous Equations)	-0.0111	-0.0126	0.0687	0.02105
Vassalou-Xing (VX)	-0.0142	-0.0139	0.0416	0.0189

based on moment matching. Here we slightly modify the original method to include VGC model factor parameters using equation (41).[5] The advantage of these calibrations is that the calibrated parameters can be written in terms of a real-world probability measure. This is also in line with estimating the factor structure coefficients using the correlation matrix. Our return data, which include the data for DB, ENI and Brent oil futures, cover the time period from June 26, 2013 to June 26, 2014. We apply our VGC model in a structural context in order to estimate the market value of assets and correponding parameters.[6] In the application of the Vassalou and Xing (2004) iterative method, we use moment matching as follows,

$$\mathsf{E}\left(X(t)\right) = (\theta + r)t, \tag{41}$$

$$Variance(X(t)) = \left(\theta^2\nu + \sigma^2\right)t. \tag{42}$$

Therefore, in the iterative calculation procedure to obtain the market value of assets, $A(t)$, we match sample moments and iterate until convergence. Table 2 presents the parameter values coming from two different methods. We will then use the market value of assets $A(t)$ which are plotted in Figures 5 and 6 to calculate implied PDs which are illustrated in Figures 7 and 8. In Figures 5 and 6, the calculation involves the latent asset values of the companies $A(t)$. In the corresponding PD Figures 7 and 8, there is a positive and negative correlation between the financial instrument and its underlying asset, respectively. These figures also help us to observe the effect of the sign of correlation on the valuations adjustments which will be explained in the following section.

The second approach for the calibration is to determine the risk-neutral parameters[7]. This method also helps to measure the calibration performance of the VGC factor construction to option price data and credit spread data. Furthermore, it provides necessary coefficients to calculate CVA in different cases of dependence. To implement this calibration method, we use non-linear least squares to match the market spreads or prices with that of the proposed model. In our empirical application, we use the CDS term structure data from Markit for 1-, 2-, 3-, 4-, 5-, 7- and 10-year maturities on June 26, 2014 for DB and ENI.[8]

[5]Ericsson and Reneby (2005) note that the method of Ronn and Verma (1986) might be biased. Therefore, we also use the iterative estimation procedure of Vassalou and Xing (2004).

[6]We use these two companies (DB and ENI) and the specific sample period to compare our results with those of Ballotta and Fusai (2015).

[7]However, this is not completely consistent with finding factor structure coefficients using the correlation matrix since the correlation matrix is a real-world probability measure parametrization.

[8]These companies are used to represent a dependence structure where the price of the financial instrument's

As a benchmark market variable, we use the bootstrapped PDs derived from the CDS market quotations to calculate the credit spread (CS) in equation (43)[9] as suggested by Ballotta and Fusai (2015). In this setup, the implied CS and the optimization problem is the following:

$$CS(t,T) = \frac{-1}{T-t} \ln(PD(t,T) + (1-RR)PD(t,T)), \tag{43}$$

$$\min_{\nu,\theta,\sigma} \sqrt{\frac{\sum_{i=1}^{n}(CS^i_{Mkt} - CS^i_{Mdl})^2}{n}}, \tag{44}$$

where RR is the recovery rate, n indicates the number of maturities in the term structure, CS_{Mdl} is the credit spread generated by the model, and CS_{Mkt} is the corresponding market implied credit spread. The spread is derived from equation (43) using the implied default probabilities from CDS quotes.

Table 3: Implied Credit Spreads and VG Merton Model Credit Spreads

Tenor	DB Spread	ENI Spread	VG Merton Fit for DB	VG Merton Fit for ENI
6m	0.4423%	0.2739%	0.4347%	0.2710%
1y	0.5783%	0.4071%	0.5812%	0.4175%
2y	0.8907%	0.77%	0.8930%	0.7486%
3y	1.1637%	1.0189%	1.1856%	1.0453%
4y	1.4373%	1.2511%	1.4265%	1.2556%
5y	1.6443%	1.4228%	1.6073%	1.3872%
7y	1.7715%	1.4853%	1.8138%	1.5139%
10y	1.8625%	1.5179%	1.8483%	1.5086%
RMSE			0.02243%	0.02081%
MAE			0.01738%	0.01739%

In order to implement non-linear least squares optimization, we apply the Levenberg-Marquardt and Nelder-Mead algorithms interchangeably for a better fit.[10] As reported in Table 3, our VG Merton model provides a good fit to the credit spread term structure with root mean square error (RMSE) values of 0.022% and 0.021% for ENI and DB, respectively. Moreover, the implied market value of assets which is included as an additional parameter in the model is obtained after model calibration to the credit spreads of DB and ENI. In particular, given the liability values of 820.2 and 39.5, the final risk-neutral market value of assets for DB and ENI are found to be 1453.03 and 73.2, respectively.

underlying has a direct effect on the revenues of ENI. DB is a systemically important counterparty in the setting.

[9] Here we assume a constant recovery rate since the market CDS pricing data are given.

[10] As indicated and used by Luciano and Schoutens (2006), Nelder-Mead is a derivative-free optimization method that is generally a successful tool for fitting to market variables, whereas Levenberg-Marquardt is a widely used optimization algorithm for curve fitting.

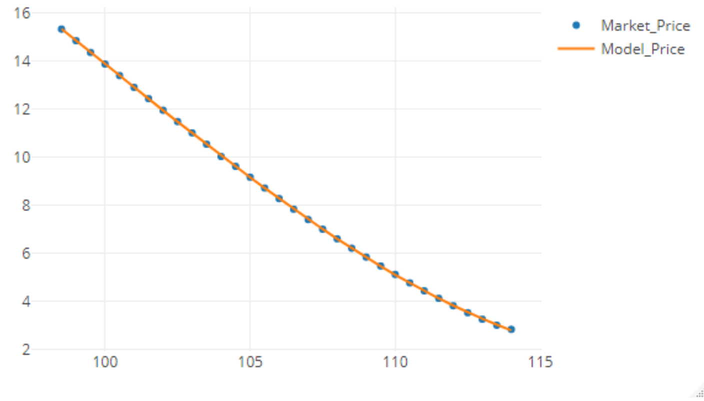


Figure 4: Fitting VG call option pricing model to the call options written on Brent oil futures: ν_I=2.192, ν_Z=1.222,σ=0.0595, θ =0.00072, r =0.0045, q =0.0037.

The risk-neutral parameters of the financial instrument are extracted using call option price data for Brent oil futures[11] on the same date (June 26, 2014) using equation (39). The result of this calibration will be used in CVA estimation (Section 6). The calibration procedure is similar to the one described above such that the optimization problem is,

$$\min_{\nu,\theta,\sigma} \sqrt{\frac{\sum_{i=1}^{n}(Call^{j}_{Mkt} - Call^{j}_{Mdl})^2}{n}}, \tag{45}$$

with ν, θ, σ being the risk-neutral parameters of call option price fit. In equation (45), n shows the number of available calls for various strikes. We observe from Figure 4 that the calibration produces an almost perfect fit for the call prices with an RMSE value of 0.014%.

5.2 Calibration of VGC Factor Components

Our next task is the calibration of the VGC factor model parameters of $I(t)$ and $Z(t)$ by matching the correlation in equation (34) implied by the VGC model to the empirical correlation matrix of the related companies. To do so, it is sufficient to use three different time series data, however, it is also possible to extend this procedure to more than three assets. In our case, we use the same data set for DB, ENI, and Brent oil described in Section 5.1.[12] The first step in matching helps us to find the parameters $c_k, c_j, \theta_Z, \nu_Z$, and σ_Z. While the first two parameters are

[11]The Brent oil futures call option data represent the derivative instrument in the setting which has a dependence to the counterparty ENI and used for the valuation adjustment calculations. Moreover, Brent oil futures are chosen to obtain robust parameters since these are liquid instruments and relevant to our dependence setting.

[12]The empirical correlation matrix is estimated using historical returns of DB, ENI, and Brent Crude Oil over the period between June 26, 2013 and June 26, 2014.

idiosyncratic, the last three parameters belong to the systematic component.

$$\begin{aligned}
\theta_j &= \mathsf{E}\left(X_j(t)\right) - r \qquad j = 1,2,3, \\
\sigma_j^2 &= \mathsf{V}\left(X_j(t)\right) - \theta_j^2 \nu_j \qquad j = 1,2,3, \\
\rho_{ij}\sqrt{\left(\theta_i^2 + \sigma_i^2(1+c_i^2)\right)\left(\theta_j^2 + \sigma_j^2(1+c_j^2)\right)} &= \theta_i\theta_j\nu_Z(c_ic_j)^2 + \sigma_i\sigma_j c_i c_j \qquad i,j = 1,2,3.
\end{aligned} \tag{46}$$

The empirical correlation calibration procedure leads to the equation system (46). After solving this system of equations, we obtain all the required parameters for the VGC model. In case of very small drift coefficients, we can use the following approximation for the correlation coefficient,

$$\rho_{ij} = \frac{c_i c_j}{\sqrt{1+c_i^2}\sqrt{1+c_j^2}}. \tag{47}$$

The above formula could be useful during the calibration procedure if the equation system (46) is not convergent.

The calibration procedure is implemented using the standard least squares method together with Levenberg-Marquardt and the Nelder-Mead solvers interchangeably for a better fit. The empirical correlation matrix and its estimation via calibrated parameters using equation system (46) are provided in Panel (a) of Table 4 and Table 5 where the latter considers the case of negative correlation between the Brent oil price and counterparties. Also the parameters c_j, θ_j, ν_j corresponding to the systematic factor sensitivity and σ_j for the idiosynratic part are given in Panel (b) of Table 4[13] and Table 5, the case of negative correlation. Panel(c) corresponds to the systematic factor's shape parameter ν_Z. The final step in the calibration process involves deriving the call option prices on Brent oil futures using the VGC option pricing model derived in Section 4.1 using the two-factor structures.

6 CVA Estimation with Dependent Factors

In this section, we estimate the CVA using the definition provided by Ballotta et al. (2016), however we slightly change the definition under constant interest rate environment but dependence assumption between default probability and the derivative. Under these assumptions, CVA is defined as

$$CVA(t) = \mathsf{E}_t\left(e^{-r(T-t)}\int_{\mathsf{R}}(1-RR)V_+(T,z)PD(t,T,z)dF(z)\right) \tag{48}$$

where RR denotes the recovery rate and PD stands for the probability of default as usual, $V_+(T) = max(V(T),0)$ is the positive exposure and $e^{-\int_s^T r(u)du}V_+(T)$ is the positive exposure discounted

[13]The calibration at this stage and in the rest of the paper is done via the iterative procedure of Vassalou and Xing (2004) as this iterative method is less biased and stable.

Table 4: VG Idiosyncratic and Systematic Component Parameter Estimates

(a) Calibrated Correlation Table

Company	$\hat{\rho}$	ρ	error
DB & ENI	0.5270582	0.5270582	4.126e-10
DB & Brent	0.217548	0.217548	1.09e-09
ENI & Brent	0.3489687	0.3489687	1.63-09

(b) Calibrated Idiosynratic VGC Term Parameters

Company	c_j	ν_I	σ_I	θ_I
DB	0.6995	1.0731	0.0188	-0.0139
ENI	2.338	0.8511	0.0416	-0.01422
Brent	0.4101	1.89	0.054	0.00184

(c) Calibrated Systematic VGC Term Parameters

Parameter	Value
ν_Z	1.269

Table 5: VG Idiosyncratic and Systematic Component Parameter Estimates(Negative Correlation)

(a) Calibrated Correlation(Negative) Table

Company	$\hat{\rho}$	ρ	error
DB & ENI	0.5270582	0.5270582	4.126e-10
DB & Brent	-0.217548	-0.217548	1.09e-09
ENI & Brent	-0.3489687	-0.3489687	1.63-09

(b) Calibrated Idiosynratic VGC Term Parameters(Negative Correlation)

Company	c_j	ν_I	σ_I	θ_I
DB	0.704	1.108	0.0219	0.0058
ENI	2.133	2.822	0.037	0.00548
Brent	-0.429	1.89	0.054	0.00184

(c) Calibrated Systematic VGC Term Parameters(Negative Correlation)

Parameter	Value
ν_Z	1.1284

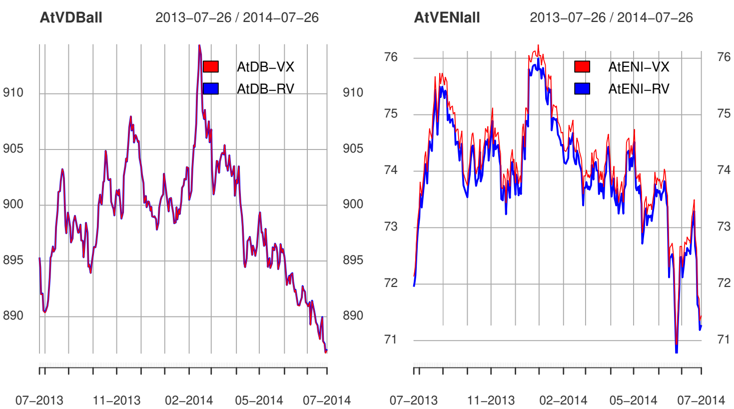


Figure 5: VGC Merton model Implied Asset Values using Two Calibration Methods (Ronn-Verma and Vassalou-Xing) with Positive Correlation.

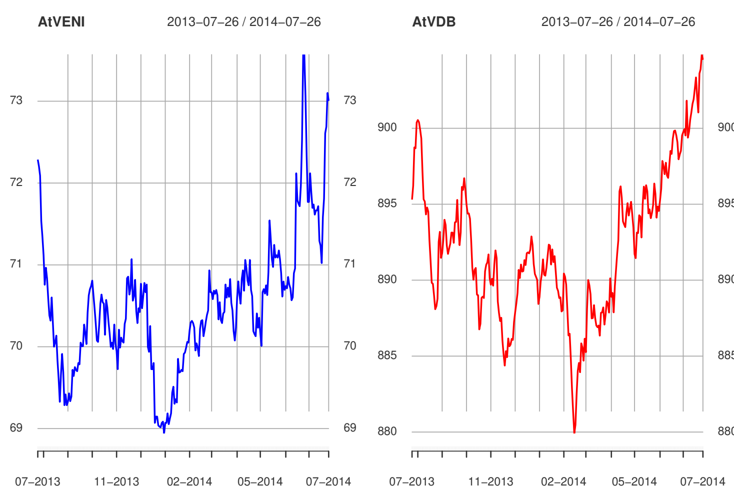


Figure 6: VGC Merton model Implied Asset Values with Negative Correlation.

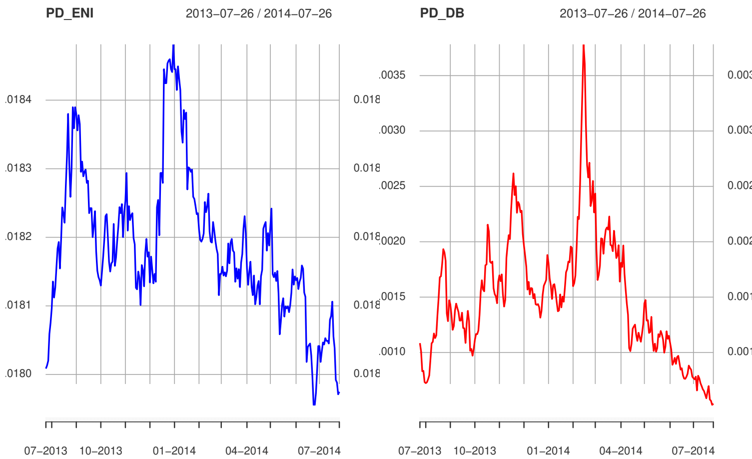


Figure 7: VGC Merton model PD with Negative Asset Correlation.

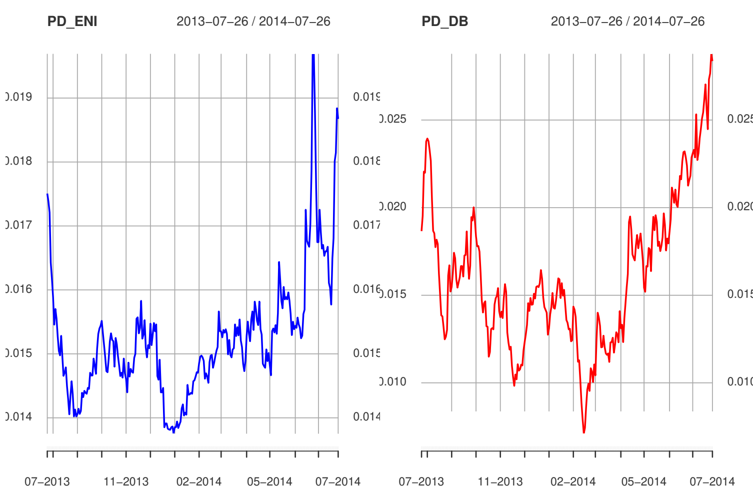


Figure 8: VGC Merton model PD with Empirical Asset Correlation Matrix

under non-constant interest rate r.

For Gaussian specifications we can obtain a closed-form solutions using the multivariate normal distribution,

$$f_{\mathbf{X}}(x_1,\ldots,x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^{\mathrm{T}}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k \mid \mathbf{\Sigma}|}}.$$

For that purpose, we first start with CVA_{u1} which is the univariate $CVA(t)$ for counterparty one (contract holder) involving the default of the other counterparty and price of the instrument.

Then using the fact that,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{N}_Y(z,\mu_Y,\sigma_Y)\mathcal{N}_X(z,\mu_X,\sigma_X)\mathcal{N}_Z(z,\mu_Z,\sigma_Z)dz &= \mathsf{E}_z\left(\mathcal{N}_Y(z,\mu_Y,\sigma_Y)\mathcal{N}_X(z,\mu_X,\sigma_X)\right), \\ &= \mathsf{P}\left(Y-Z\leq 0, X-Z\leq 0\right), \qquad (49)\\ &= \mathcal{N}_{\mu,\Sigma}\left(\mu_X-\mu_Z,\mu_Y-\mu_Z,\sigma_Y^2+\sigma_Z^2,\sigma_Z^2,\sigma_X^2+\sigma_Z^2\right),\end{aligned}$$

where $\mathcal{N}_{\mu,\Sigma}(\overrightarrow{x})$ is the cumulative distribution function for the multivariate normal distribution and $\mathcal{N}(x)$ is the standart normal distribution. Now in subsequent formulations, we will follow the definition of univariate and bivariate CVA and calculate the components: default probability of second counterparty, survival probability of first counterparty (contract holder), and the payoff of the instrument. We will cover the case of the standard call/put options, since they are more adaptable to our dependence setting. However, other instruments are also adaptable. Following calculations provide the formulas for univariate CVA_{u1}, in the case of convoluted Brownian motions using (49),

$$CVA_{u1}(t) = (1-RR_2)\left[S(t)P^{\mathsf{S}}\left(A_2(T)\leq L_2, S(T)\geq K\right) - e^{-r\tau}KP^{\mathsf{Q}}\left(A_2(T)\leq L_2, S(T)\geq K\right)\right] \qquad (50)$$

$$= (1-RR_2)\left[S(t)\mathcal{N}^S\left(\overrightarrow{\mu}^{\mathsf{S}},\Sigma\right) - Ke^{-r\tau}\mathcal{N}^Q\left(\overrightarrow{\mu},\Sigma\right)\right] \qquad (51)$$

$$\begin{aligned}\Sigma &= \begin{bmatrix}\left(c_1^2\sigma_z^2+\sigma_1^2\right)\tau & c_1c_2\sigma_z^2\tau \\ c_2c_1\sigma_z^2\tau & \left(c_2^2\sigma_z^2+\sigma_2^2\right)\tau\end{bmatrix} \\ \overrightarrow{\mu}^{\mathsf{S}} &= \begin{bmatrix}\mu_2 & \mu_3^{\mathsf{S}}\end{bmatrix} \\ \overrightarrow{\mu} &= \begin{bmatrix}\mu_2 & \mu_3\end{bmatrix} \qquad (52)\end{aligned}$$

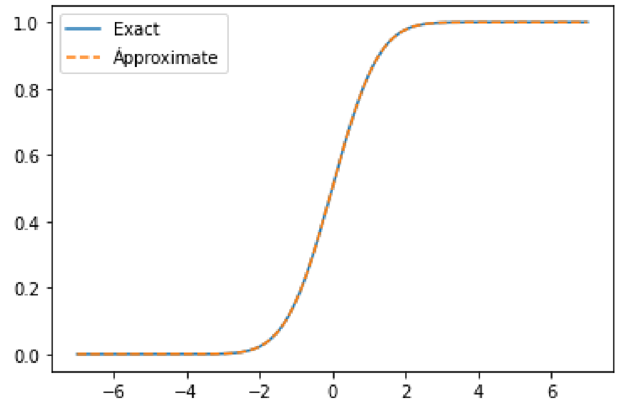


Figure 9: Approximate Normal CDF (equation (54)) vs exact Normal cumulative distribution

where

$$\mu_2 = \frac{\log\left(\frac{A_2(t)}{L_2}\right) + (r - 0.5\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}$$
$$\mu_3^{\mathsf{S}} = \frac{\log\left(\frac{S(t)}{K}\right) + (r + 0.5\sigma_3^2)\tau}{\sigma_3\sqrt{\tau}}$$
$$\mu_3 = \frac{\log\left(\frac{S(t)}{K}\right) + (r - 0.5\sigma_3^2)\tau}{\sigma_3\sqrt{\tau}}$$

where $c_1 c_2 \sigma_z^2 \tau$ is the covariance between $Y_1 = X_1 + c_1 Z$ and $Y_2 = X_2 + c_2 Z$. It is further possible to use the approximations in Hong (1999) for bivariate and trivariate normal distributions:

$$\begin{aligned} P\left(Y_1 > L_1, Y_2 > L_2\right) &= \mathcal{N}(L_{1|2})\mathcal{N}(-L_2) \\ P\left(Y_1 > L_1, Y_2 \leq L_2, Y_3 > L_3\right) &= \mathcal{N}(-L_{1|2,3})\mathcal{N}(L_{2|3})\mathcal{N}(-L_3) \end{aligned} \tag{53}$$

where $\phi(.)$ is normal pdf, $\mathcal{N}(.)$ is the normal cdf, and $L_{1|2} = L_1 - \rho_{12}\frac{\phi(L_2)}{\mathcal{N}(L_2)\sqrt{1-\rho_{12}^2}}$. Moreover, $d_{i|3} = \frac{\left(d_i - \rho_{i3}\frac{\phi(d_3)}{\mathcal{N}(d_3)}\right)}{\sqrt{1-\rho_{i3}^2}}$ for $i = 1, 2$, $d_{1|2,3} = \frac{\left(d_{1|3} - \rho_{12|3}\frac{\phi(d_{2|3})}{\mathcal{N}(d_{2|3})}\right)}{\sqrt{1-\rho_{12|3}^2}}$ and $\rho_{12|3} = \frac{(\rho_{12} - \rho_{13}\rho_{23})}{\sqrt{1-\rho_{13}^2}\sqrt{1-\rho_{23}^2}}$.

The following approximation of Vazquez-Leal et al. (2011) for the normal cdf is very useful to derive more compact expressions which reduces the need for additional numerical integrations in our case,

$$F(x) = \frac{1}{\exp\left(\frac{-358x}{23} + 111 \arctan\frac{37x}{294}\right) + 1}. \tag{54}$$

As seen from Figure 9, equation (54) is very successful in approximating exact normal cdf. By using equation (54), we can rewrite equation (53) in a compact form,

$$P\left(Y_1 < L_1, Y_2 > L_2\right) = \frac{1}{A} \tag{55}$$

where

$$\begin{aligned} A^{14} = \quad & \exp\left(\frac{358\left(L_{1|2}+L_2\right)}{23} + 111\arctan\left(\frac{37\left(L_{1|2}+L_2\right)}{(294 - \frac{L_{1|2}L_2 37^2}{294})}\right)\right) \\ & + \exp\left(\frac{358 L_{1|2}}{23} + 111\arctan\frac{37 L_{1|2}}{294}\right) \\ & + \exp\left(\frac{358 L_2}{23} + 111\arctan\frac{37 L_2}{294}\right) + 1 \end{aligned} \tag{56}$$

$$\begin{aligned} CVA_{u1} &= (1 - RR_2)\left[S(t)\mathcal{N}^S\left(\overrightarrow{\mu}^{\mathsf{S}}, \Sigma\right) - Ke^{-r\tau}\mathcal{N}^Q\left(\overrightarrow{\mu}, \Sigma\right)\right], \\ &= (1 - RR_2)\left[S(t)\mathcal{N}(d_{1|2})\mathcal{N}(-d_+) - Ke^{-r\tau}\mathcal{N}(d_{1|2})\mathcal{N}(-d_-)\right] \end{aligned}$$

$$CVA_{u1} = \frac{S(t)}{\exp\left(\frac{358\left(-d_{1|2}+d^+\right)}{23} + 111\arctan\left(\frac{37\left(d_{1|2}-d^+\right)}{294 - \frac{d_{1|2}d^+ 37^2}{294}}\right)\right) + \exp\left(\frac{358 d_{1|2}}{23} + 111\arctan\frac{37 d_{1|2}}{294}\right) + \exp\left(\frac{358 d^+}{23} + 111\arctan\frac{37 d^+}{294}\right) + 1} -$$

$$\frac{Ke^{-r\tau}}{\exp\left(\frac{358\left(-d_{1|2}+d^+\right)}{23} + 111\arctan\left(\frac{37\left(d_{1|2}-d^-\right)}{294 - \frac{d_{1|2}d^- 37^2}{294}}\right)\right) + \exp\left(\frac{358 d_{1|2}}{23} + 111\arctan\frac{37 d_{1|2}}{294}\right) + \exp\left(\frac{358 d^-}{23} + 111\arctan\frac{37 d^-}{294}\right) + 1}$$

where $d^+ = \frac{\frac{\log(S(t)}{K} + (r+0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and $d^- = \frac{\frac{\log(S(t)}{K} + (r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$ as usual in B-S framework and $d_{1|2} = \frac{\frac{\log(A_2(t)}{D_2} + (r-\sigma_A^2)\tau}{\sigma_A\sqrt{\tau}} - \rho_{12}\frac{\phi(d^+)}{\mathcal{N}(d^+)\sqrt{1-\rho_{12}^2}}$.

In the case of a bivariate CVA as well as the Gaussian systematic and idiosyncratic factor case, using equation (49) we can write a closed-form formula with a B-S structure and in terms of

[14] This result is the simple application of $\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}$

trivariate normal distribution as follows,

$$CVA_1(t) = (1 - RR_2)\Big[S(t)P^{\mathsf{S}}\left(A_1(T) \geq L_1, A_2(T) \leq L_2, S(T) \geq K\right) - e^{-r\tau}KP^{\mathsf{Q}}\left(A_1(T) \geq L_1, A_2(T) \leq L_2, S(T)\right. \tag{57}$$

$$= (1 - RR_2)\Big[S(t)\mathcal{MVN}^S\left(\overrightarrow{\mu}^{\mathsf{S}}, \Sigma\right) - Ke^{-r\tau}\mathcal{MVN}^Q\left(\overrightarrow{\mu}, \Sigma\right)\Big] \tag{58}$$

$$\begin{aligned}
\Sigma &= \begin{bmatrix} \left(c_1^2\sigma_z^2 + \sigma_1^2\right)\tau & c_1c_2\sigma_z^2\tau & c_1c_3\sigma_z^2\tau \\ c_1c_2\sigma_z^2\tau & \left(c_2^2\sigma_z^2 + \sigma_2^2\right)\tau & c_2c_3\sigma_z^2\tau \\ c_1c_3\sigma_z^2\tau & c_2c_3\sigma_z^2\tau & \left(c_3^2\sigma_z^2 + \sigma_3^2\right)\tau \end{bmatrix} \\
\overrightarrow{\mu}^{\mathsf{S}} &= \begin{bmatrix} \mu_1 & \mu_2 & \mu_3^{\mathsf{S}} \end{bmatrix} \\
\overrightarrow{\mu} &= \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix}
\end{aligned}$$

where

$$\mu_1 = \frac{\log\left(\frac{A_1(t)}{L_1}\right) + (r - 0.5\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}$$

$$\mu_2 = \frac{\log\left(\frac{A_2(t)}{L_2}\right) + (r - 0.5\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}$$

$$\mu_3^{\mathsf{S}} = \frac{\log\left(\frac{S(t)}{K}\right) + (r + 0.5\sigma_3^2)\tau}{\sigma_3\sqrt{\tau}}$$

$$\mu_3 = \frac{\log\left(\frac{S(t)}{K}\right) + (r - 0.5\sigma_3^2)\tau}{\sigma_3\sqrt{\tau}}$$

Finally, we find the result below using the approximation for trivariate normal distribution in equation (57),

$$CVA_1(t) = (1 - RR_2)\left[S(t)\mathcal{N}(d_{1|2,3})\mathcal{N}(-d_{2|3})\mathcal{N}(-d_+) - Ke^{-r\tau}\mathcal{N}(d_{1|2,3})\mathcal{N}(-d_{2|3})\mathcal{N}(-d_-)\right] \tag{59}$$

Using the same idea in equation (55) we can write equation (57),

$$CVA_1 = (1 - RR_2)\left[\frac{S(t)}{A_1 + 1} - \frac{Ke^{-r\tau}}{A_2 + 1}\right] \tag{60}$$

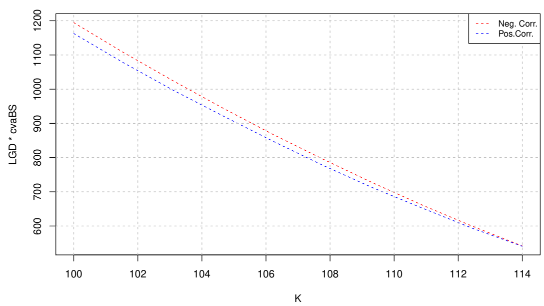
where

$$
\begin{aligned}
A_1 = \exp\left(\frac{358\left(-d_{1|2,3}+d_{2|3}+d^{+}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d_{2|3}-d^{+}-d_{1|2,3}d_{2|3}d^{+}\right)}{294+\frac{d_{1|2,3}d^{+}37^2-d_{2|3}d^{+}37^2+d_{1|2,3}d_{2|3}37^2}{294}}\right)\right)+ \\
\exp\left(\frac{358\left(-d_{1|2,3}+d^{+}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d^{+}\right)}{294+\frac{d_{1|2,3}d^{+}37^2}{294}}\right)\right) \\
+\exp\left(\frac{358\left(d_{2|3}+d^{+}\right)}{23}+111\arctan\left(\frac{37\left(-d_{2|3}-d^{+}\right)}{294-\frac{d_{2|3}d^{+}37^2}{294}}\right)\right) \\
+\exp\left(\frac{358\left(-d_{1|2,3}+d_{2|3}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d_{2|3}\right)}{294+\frac{d_{1|2,3}d_{2|3}37^2}{294}}\right)\right) \\
\exp\left(\frac{-358d_{1|2,3}}{23}+111\arctan\frac{37d_{1|2,3}}{294}\right) \\
+\exp\left(\frac{358d_{2|3}}{23}+111\arctan\frac{-37d_{2|3}}{294}\right) \\
+\exp\left(\frac{358d_{+}}{23}+111\arctan\frac{-37d_{+}}{294}\right) \\
A_2 = \exp\left(\frac{358\left(-d_{1|2,3}+d_{2|3}+d^{-}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d_{2|3}-d^{-}-d_{1|2,3}d_{2|3}d^{-}\right)}{294+\frac{d_{1|2,3}d^{-}37^2-d_{2|3}d^{-}37^2+d_{1|2,3}d_{2|3}37^2}{294}}\right)\right)+ \\
\exp\left(\frac{358\left(-d_{1|2,3}+d^{-}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d^{-}\right)}{294+\frac{d_{1|2,3}d^{-}37^2}{294}}\right)\right) \\
+\exp\left(\frac{358\left(d_{2|3}+d^{-}\right)}{23}+111\arctan\left(\frac{37\left(-d_{2|3}-d^{-}\right)}{294-\frac{d_{2|3}d^{-}37^2}{294}}\right)\right) \\
+\exp\left(\frac{358\left(-d_{1|2,3}+d_{2|3}\right)}{23}+111\arctan\left(\frac{37\left(d_{1|2,3}-d_{2|3}\right)}{294+\frac{d_{1|2,3}d_{2|3}37^2}{294}}\right)\right) \\
\exp\left(\frac{-358d_{1|2,3}}{23}+111\arctan\frac{37d_{1|2,3}}{294}\right) \\
+\exp\left(\frac{358d_{2|3}}{23}+111\arctan\frac{-37d_{2|3}}{294}\right) \\
+\exp\left(\frac{358d_{-}}{23}+111\arctan\frac{-37d_{-}}{294}\right).
\end{aligned}
$$

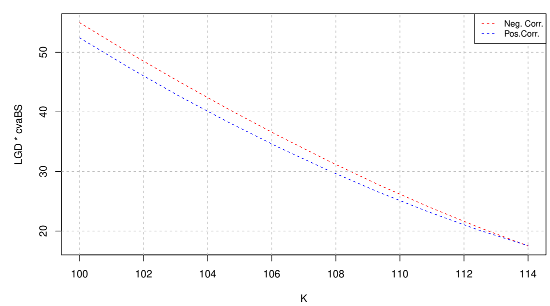
The application of these formulas yields the numerical outputs shown in Figure 10.

From Figure 10, we observe that in line with intuition, wrong-way risk, negative correlation between counterparty and the instrument requires additional risk-based value adjustment compared to the one with positive correlation, which is the right-way risk. Moreover as indicated in Section 3, the vanishing PD effect due to the Geometric Brownion Motion (GBM) setting of classical Merton model becomes clearly visible due to the difference between 6 months and 1 year CVA.

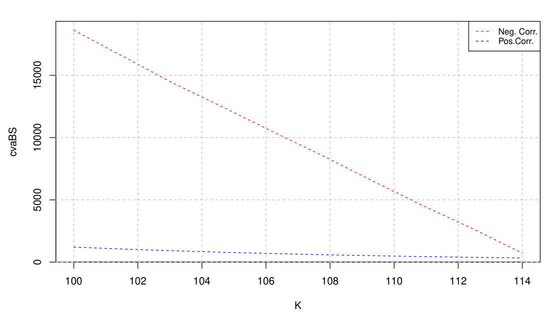
In the special case of our VGC factor model, we can first write the univariate CVA assuming dependence between defaults, market value of assets, and underlying derivative by means of the



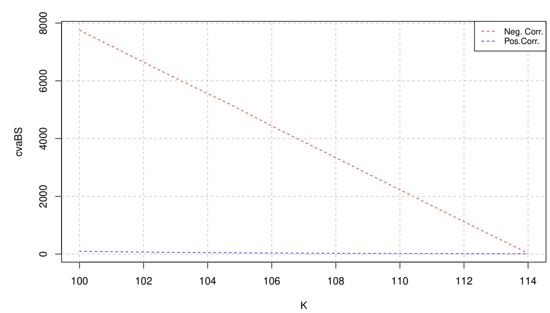
(a) CVA(bps) 1 year



(b) CVA(bps) 6 months



(c) CVA(bps) 1 year (%90 corr.)



(d) CVA(bps) 6 months (%90 corr.)

Figure 10: CVA(bps) with wrong-way and right-way risk in a Gaussian factor environment.

following formulas using equation (50),

$$CVA_{u1} = (1 - RR_2)\Big[S(t)\Big(\int_0^\infty \int_0^{\gamma_{X_1}} \mathcal{N}(d_{1|2}(\gamma_{X_1}))\mathcal{N}(-d_+(\gamma_{X_2}))f_{\gamma_{X_1}>\gamma_{X_2}}(\gamma_{X_1},\gamma_{X_2})d\gamma_{X_1}\gamma_{X_2} + \int_0^\infty \int_0^{\gamma_{X_2}} \mathcal{N}(d_{1|2}(\gamma_{X_1}))\mathcal{N}(-d_+(\gamma_{X_2}))f_{\gamma_{X_1}<\gamma_{X_2}}(\gamma_{X_1},\gamma_{X_2})d\gamma_{X_1}\gamma_{X_2}\Big) - Ke^{-r\tau}\Big(\int_0^\infty \int_0^{\gamma_{X_1}} \mathcal{N}(d_{1|2}(\gamma_{X_1}))\mathcal{N}(-d_-(\gamma_{X_2}))f_{\gamma_{X_1}>\gamma_{X_2}}(\gamma_{X_1},\gamma_{X_2})d\gamma_{X_1}\gamma_{X_2} + \int_0^\infty \int_0^{\gamma_{X_2}} \mathcal{N}(d_{1|2}(\gamma_{X_1}))\mathcal{N}(-d_-(\gamma_{X_2}))f_{\gamma_{X_1}<\gamma_{X_2}}(\gamma_{X_1},\gamma_{X_2})d\gamma_{X_1}\gamma_{X_2}\Big)\Big] \quad (61)$$

where using equation (25) and plugging related parameters, the first part of the multivariate gamma density becomes $f(\gamma_{X_1}, \gamma_{X_2})$,

$$f(\gamma_{X_1}, \gamma_{X_2}) = \frac{\gamma_{X_1}^{\nu_1-1}\gamma_{X_2}^{\nu_2-1}e^{\frac{-\gamma_{X_1}}{\beta_1}}e^{\frac{-\gamma_{X_2}}{\beta_2}}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_Z)}B(\nu_1,\nu_2) \times\Phi_1\left(\nu_1, 1-\nu_Z, \nu_1+\nu_2; \frac{\gamma_{X_1}}{\gamma_{X_2}}, \gamma_{X_1}\left(\frac{1}{\beta_2}+\frac{1}{\beta_Z}-\frac{1}{c_1\beta_1}\right)\right) \quad (62)$$

and in the same way using equation (26) and plugging related parameters in the second part of $f(\gamma_{X_2}, \gamma_{X_1})$, we obtain,

$$f(\gamma_{X_2}, \gamma_{X_1}) = \frac{\gamma_{X_1}^{\nu_1-1}\gamma_{X_2}^{\nu_2-1}e^{\frac{-\gamma_{X_1}}{\beta_1}}e^{\frac{-\gamma_{X_2}}{\beta_2}}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_Z)}B(\nu_1,\nu_2) \times\Phi_1\left(\nu_1, 1-\nu_Z, \nu_1+\nu_2; \frac{\gamma_{X_2}}{\gamma_{X_1}}, \gamma_{X_2}\left(\frac{1}{\beta_Z}+\frac{1}{\beta_2}-\frac{1}{c_2\beta_1}\right)\right) \quad (63)$$

We can then write the bivariate CVA assuming dependence between defaults, market value of assets, and underlying derivative by means of the following formulas using equation

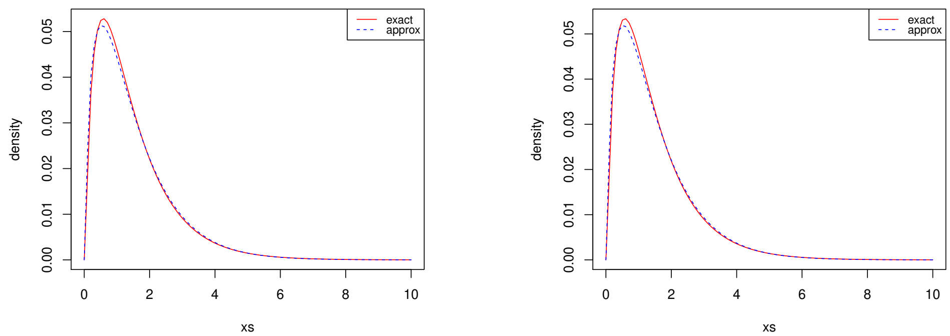
$$CVA_1 = (1 - RR_2)\int\cdots\int_{\mathsf{R}^3\times[0,\infty]} \mathcal{N}_{\mu,\Sigma}(\theta_1\gamma_{X_1}, \theta_2\gamma_{X_2}, \theta_3\gamma_{X_3}, \sigma_1, \sigma_2, \sigma_3, \gamma_{X_1}, \gamma_{X_2}, \gamma_{X_3}) \times f(\gamma_{X_1}, \gamma_{X_2}, \gamma_{X_3})d\gamma_{X_1}d\gamma_{X_2}d\gamma_{X_3}. \quad (64)$$

Using the approximation approach to equation (50), we can write equation (64) as,

$$CVA_1 = (1 - RR_2)\int\cdots\int_{\mathsf{R}^3\times[0,\infty]} \mathcal{N}(-d_{1|2,3})\mathcal{N}(d_{2|3})\mathcal{N}(-d_2)f(\gamma_{X_1}, \gamma_{X_2}, \gamma_{X_3})d\gamma_{X_1}d\gamma_{X_2}d\gamma_{X_3}. \quad (65)$$

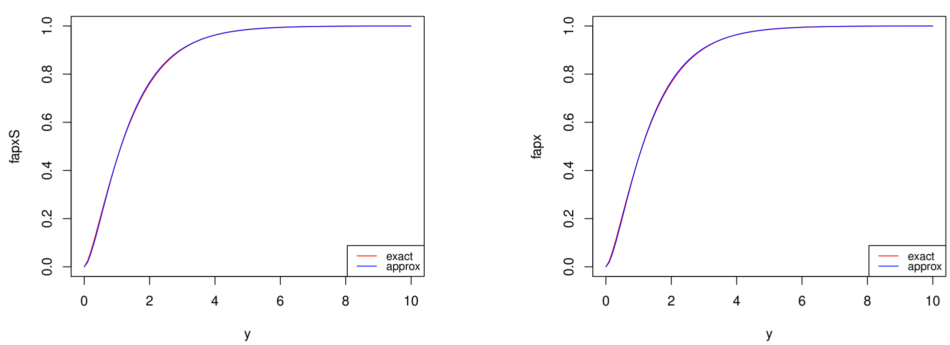
where now multivariate gamma density becomes the equation (27) due to the case of three correlated gammas.

Figures 11 and 12 show the robustness of approximations used for both the normal cdf and gamma convolution. Here, we can see that the approximations will not affect the CVA outcome significantly, as we observe that the approximate default probabilities and pricing probabilities do match the exact ones.



(a) density S measure (b) density Q measure

Figure 11: Approximate Gamma Convolution Density and Exact Gamma Convolution Density with calibrated parameters



(a) cumulative density S measure (b) cumulative density Q measure

Figure 12: Approximate Gamma Convolution Cumulative Density and Exact Gamma Convolution Cumulative Density with calibrated parameters

Using equation system (61) and (65) for various strike price levels, we calculate CVA for ENI, where it has an exposure to option seller DB, for the purpose of observing the bilateral effect and the impact of wrong-way risk. The first case is presented in Figure 13 with positive correlation scenario. In the second case negative correlation is imposed between Brent and equity prices of DB to see the wrong-way risk and it is presented in Figures 14 to 17. In that sense, we observe that as the inverse correlation between the counterparty DB and Brent prices increases, while default probability starts to rise dramatically, the exposure either starts to increase or remains at the same level. Moreover, we analyze the effect of dependence strucuture imposed by two models, i.e., stochastic clock factor decomposition model and full VGC factor decomposition model. In Figures 16 and 17, we observe the impact of these model differences and correlation sign change over CVA. We inevitably see a deeper effect of VGC factor dependence on CVA towards both directions. However, the stochastic clock dependence shows weaker but more stable impacts. In this dependence model, it is easier to integrate inverse relationships between counterparties or instruments simply by changing the sign of drift parameter whereas in VGC this process requires also a recalibation of parameters. However, this dependence only considers time dependence but not economic state dependence. On the other hand, the VGC factor dependence is more realistic since it considers both time dependence and economic state dependence through stochastic clock and systematic factor $Z(t)$ simultaneously.

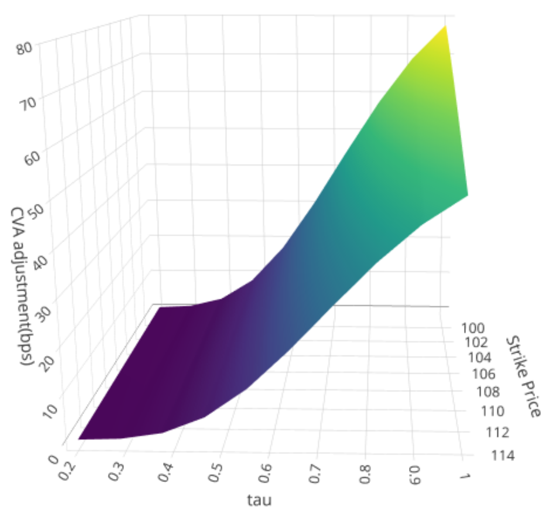


Figure 13: CVA under Stochastic Clock Dependence and Positive Correlation

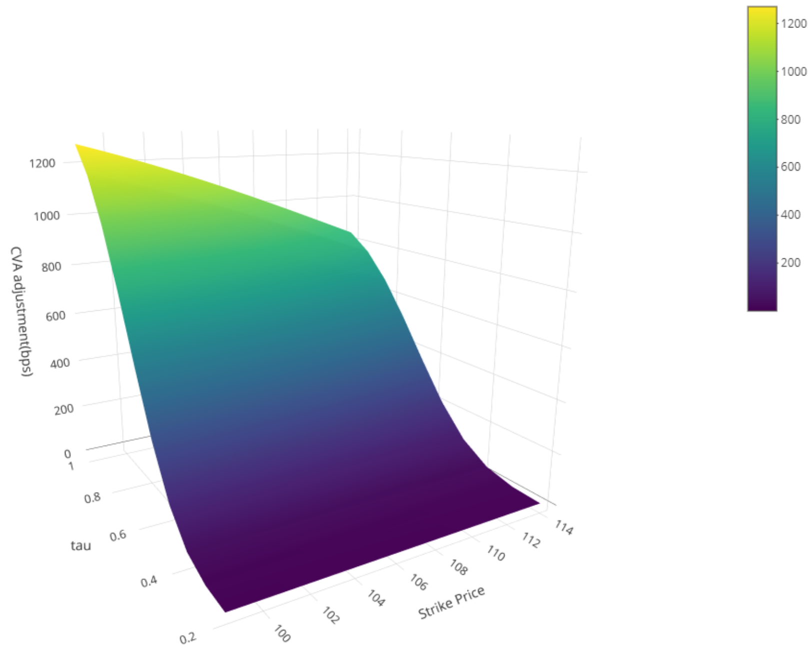


Figure 14: CVA under Stochastic Clock Dependence and Negative Correlation

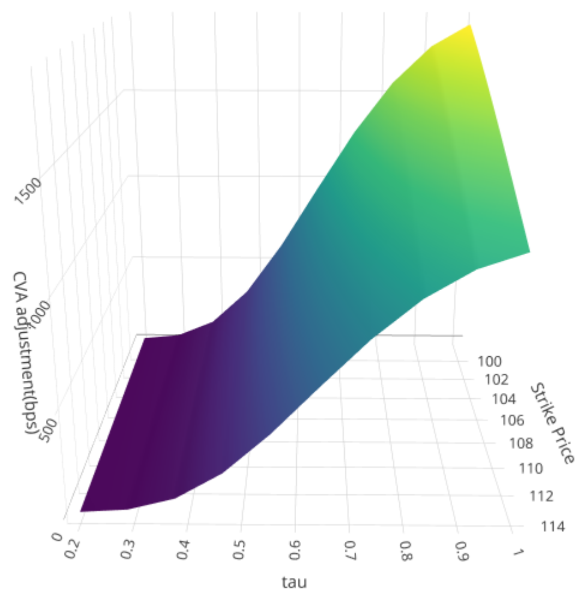


Figure 15: CVA under Stochastic Clock Dependence and Negative Correlation (wrong-way %90 corr.)

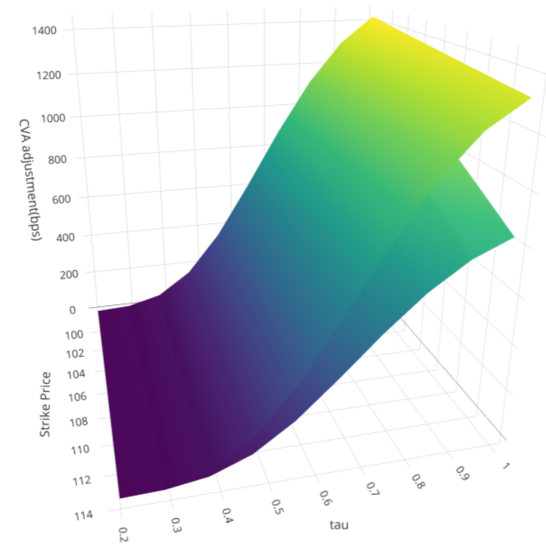


Figure 16: CVA under Stochastic Clock Dependence vs Linear Factor Dependence Negative Correlation (Wrong-way)

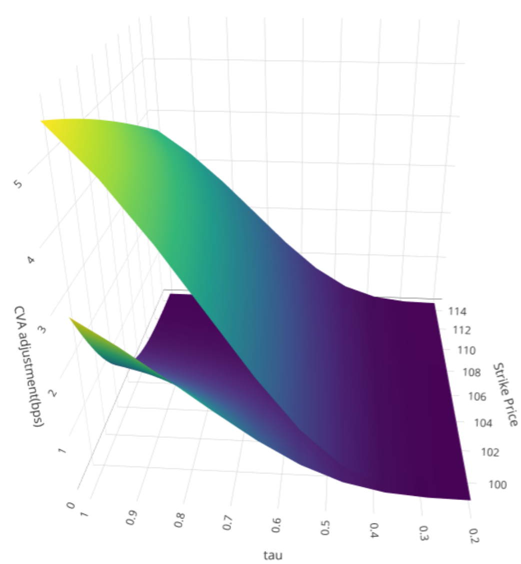


Figure 17: CVA under Stochastic Clock Dependence vs Linear Factor Dependence Positive Correlation (right-way)

7 Conclusion

In this study, we propose an extended structural credit risk model in the context of a modified version of the well-known variance gamma process, where we use a convoluted gamma time-change model. Our modelling framework involves the extension of the classical Merton model with a log variance gamma convolution asset value which is the result of a linear factor decomposition feature. In this framework, dependence can be taken into account without using classical tools such as copulas. The log-variance-gamma-convolution extension allows for a better estimation of the default probability in a Merton model environment.

Numerical experiments show that our model provides a considerable amount of flexibility and fitting ability to various CDS term structures and option prices for different strikes. Compared to other models, it works better for the short-term cases and solves the vanishing probability of default (PD) and credit spread problems. Moreover, our model can calculate the unobserved asset value of a financial company given only CDS spread and liability. Our methodology satisfies the parameter identification requirements of the Merton type structural model. Hence it can easily be used and extended to various cases for dynamic financial stability measurement purposes. For instance, our model can be employed to calculate asset (or equity) weighted PD for the entire banking sector and then to create system-wide risk indicators based on this PD. We further applied our proposed model to estimate a very popular counterparty credit risk measure, credit valuation adjustment (CVA), which is required by the Basel III accord and other related complementary regulatory rules. We demonstrate how realistically CVA responds to stressful conditions, unlike the case when Brownian motion is assumed.

Further analysis can be conducted by applying similar linear decomposition model structures with convolutions of other subordinated Lévy processes such as the normal inverse Gaussian and Meixner models where a closed-form convolution formula is feasible. Similarly, application of the same modelling architecture to Heston-Bates type stochastic volatility models can be promising, even with an extension to the case of barrier-type Lévy credit risk models.

References

Antonelli, F., Ramponi, A., Scarlatti, S., 2021. CVA and vulnerable options pricing by correlation expansions. Annals of Operations Research 299, 401–427.

Ballotta, L., Bonfiglioli, E., 2016. Multivariate asset models using Lévy processes and applications. European Journal of Finance 22, 1320–1350.

Ballotta, L., Deelstra, G., Rayee, G., 2017. Multivariate FX models with jumps: Triangles, Quantos and implied correlation. European Journal of Operational Research 260, 1181–1199.

Ballotta, L., Fusai, G., 2015. Counterparty credit risk in a multivariate structural model with jumps. Finance 36, 39–74.

Ballotta, L., Fusai, G., Loregian, A., Fabricio Perez, M., 2019. Estimation of multivariate asset models with jumps. Journal of Financial and Quantitative Analysis 54, 2053–2083.

Ballotta, L., Fusai, G., Marena, M., 2016. A gentle introduction to default risk and counterparty credit modelling. SSRN Working Paper URL: http://dx.doi.org/10.2139/ssrn.2816355.

BIS, 2016. OTC derivatives statistics at end-june 2016. Bank for International Settlements - Statistical Release URL: https://www.bis.org/publ/otc_hy1611.pdf.

Bonollo, M., De Persio, L., Oliva, I., Semmoloni, A., 2015. A quantization approach to the counterparty credit exposure estimation. SSRN Working Paper URL: http://dx.doi.org/10.2139/ssrn.2574384.

Bonollo, M., Di Persio, L., Mammi, L., Oliva, I., 2016. Estimating the counterparty risk exposure by using the Brownian motion local time. International Journal of Applied Mathematics and Computer Science 27, 435–447.

Brigo, D., Morini, M., Pallavicini, A., 2013. Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes. John Wiley & Sons, Hoboken, NJ.

Brigo, D., Vrins, F., 2018. Disentangling wrong-way risk: Pricing credit valuation adjustment via change of measures. European Journal of Operational Research 269, 1154–1164.

Chaudhry, M.A., Syed, M.Z., 2002. On a Class of Incomplete Gamma Functions with Applications. Chapman and Hall / CRC.

Cont, R., Tankov, P., 2004. Financial Modelling with Jump Processes. Chapman & Hall / CRC Financial Mathematics Series, Boca Raton, CRC Press UK.

Di Salvo, F., 2008. A characterization of the distribution of a weighted sum of gamma variables through multiple hypergeometric functions. Integral Transforms and Special Functions 19, 563–575.

Ericsson, J., Reneby, J., 2005. Estimating structural bond pricing models. Journal of Business 78, 707–735.

Fabozzi, F.J., Leccadito, A., Tunaru, R.S., 2014. Extracting market information from equity options with exponential Lévy process. Journal of Economic Dynamics and Control 38, 125–141.

Fabozzi, F.J., Yang, Y., Bianchi, M.L., 2015. Bilateral counterparty risk valuation adjustment with wrong-way risk on collateralized commodity counterparty. Journal of Financial Engineering 2, 1550001.

Fiorani, F., Luciano, E., Semeraro, P., 2010. Single and joint default in a structural model with purely discontinuous asset prices. Quantitative Finance 10, 249–263.

Geman, H., Ane, T., 1996. Stochastic subordination. Risk 9, 145–150.

Gemmil, G., Marra, M., 2019. Explaining CDS prices with Merton's model before and after the Lehman default. Journal of Banking & Finance 106, 93–109.

Gnoatto, A., Picarelli, A., Reisinger, C., 2020. Deep xva solver – a neural network based counterparty credit risk management framework. URL: https://arxiv.org/abs/2005.02633, doi:10.48550/ARXIV.2005.02633.

Gradshteyn, I.S., Ryzhik, I.M., 2007. Table of integrals, series, and products. Seventh ed., Elsevier/Academic Press, Amsterdam.

Gray, D., Malone, S., 2008. Macro Financial Risk Analysis. John Wiley & Sons, Hoboken, NJ.

Hirsa, A., 2012. Computational Methods in Finance. Chapman and Hall/CRC Financial Mathematics Series. 1st ed., CRC Press, Boca Raton, FL.

Hong, H.P., 1999. An approximation to bivariate and trivariate normal integrals. Civil Engineering and Environmental Systems 16, 115–127.

Hull, J., White, A., 2012. CVA and wrong-way risk. Financial Analysts Journal 68, 58–69.

Humbert, P., 1922. The confluent hypergeometric functions of two variables. Proceedings of the Royal Society of Edinburgh 41, 73–96.

Luciano, E., Schoutens, W., 2006. A multivariate jump-driven financial asset model. Quantitative Finance 6, 385–402.

Madan, D.B., 1998. The variance gamma process and option pricing. Review of Finance 2, 79–105.

Madan, D.B., Milne, F., 1991. Option pricing with V.G. martingale components. Mathematical Finance 1, 39–55.

Madan, D.B., Seneta, E., 1990. The variance gamma (V.G.) model for share market returns. Journal of Business 63, 511–524.

Mathai, A., Moschopoulos, P., 1991. On a multivariate gamma. Journal of Multivariate Analysis 39, 135–153.

Merton, R., 1974. On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance 29, 449–470.

Moosbrucker, T., 2006. Explaining the correlation smile using variance gamma distributions. Journal of Fixed Income 16, 71–87.

Ronn, E.I., Verma, A.K., 1986. Pricing risk-adjusted deposit insurance: An option-based model. Journal of Finance 41, 871–895.

Satterthwaite, F.E., 1946. An approximate distribution of estimates of variance components. Biometrics Bulletin 2, 110–114.

Vassalou, M., Xing, Y., 2004. Default risk in equity returns. Journal of Finance 59, 831–868.

Vazquez-Leal, H., Castañeda-Sheissa, R., Sarmiento-Reyes, A., Sanchez-Orea, J., 2011. High accurate simple approximation of normal distribution integral. Mathematical Problems in Engineering 2012, 124029.

Yang, H., Kanniainen, J., 2017. Jump and volatility dynamics for the S&P500: Evidence for infinite-activity jumps with non-affine volatility dynamics from stock and option markets. Review of Finance 21, 811–844.

Appendix:

A.0: The Merton Model

Merton (1974) assesses a company's ability to meet its obligations by evaluating the credit risk of that company's debt. The model assumes that the total value of the assets, $A(t)$, follows a geometric Brownian motion:

$$dA(t) = rA(t)dt + \sigma_A A(t)dW(t), \tag{A.1}$$

where r is the expected rate of return, σ is the volatility of the assets, and $W(t)$ is the Brownian motion at time t. The model further assumes that the financial market is frictionless so that the liquidation value is equal to firm value.[15] Denoting the company's total value of debt with maturity T by D and the total value of equity by $E(t)$ where $t \leq T$, from the fundamental accounting identity we have

$$E(T) = \max(A(T) - D, 0), \tag{A.2}$$

which shows that the equity is an implicit call option on the company's total value of assets with strike price D and maturity T (Gray and Malone, 2008). Therefore, one can use the B-S call option formula to calculate the market value of equity,

$$E(t) = A(t)\mathcal{N}(d_1) - De^{-r(T-t)}\mathcal{N}(d_2), \tag{A.3}$$

where

$$d_1 = \frac{\ln(A(t)/D) + (r + \frac{\sigma_A^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{A.4}$$

$$d_2 = d_1 - \sigma_A\sqrt{T-t}, \tag{A.5}$$

and $\mathcal{N}(x)$ is the cumulative standard normal distribution function. Under this setup, the default probability at maturity T under the risk-neutral probability measure is

$$P(A(T) < D|A(t)) = \mathcal{N}(-d_2). \tag{A.6}$$

A.1: Derivation of the VG Model Delta

Proof. First, the option value is written as usual:

$$\mathsf{E}(e^{-r\tau}(S(T) - K)^+) = \mathsf{E}(e^{-r\tau}(S(T) - K)\mathsf{I}_{\{S(T)>K\}}). \tag{A.7}$$

[15] For instance, dividend payments are ignored for the sake of simplicity.

Then the derivative of this expectation with respect to the stock price is:

$$\frac{\partial \mathsf{E}((S(T)-K)\mathsf{I}_{\{S(T)>K\}})}{\partial S(t)} = \mathsf{E}^S(\mathsf{I}_{\{S(T)>K\}}|\mathcal{F}(t)) + \mathsf{E}(\delta((S(T)-K)e^{-r\tau})S_T e^{-r\tau}|\mathcal{F}(t)) \\ -\mathsf{E}(Ke^{-r\tau}\delta((S(T)-K)e^{-r\tau})|\mathcal{F}(t)), \quad \text{(A.8)}$$

where δ is the Dirac function which is the derivative of Heaviside function $\mathsf{I}_{\{S_T>K\}}$. Since Dirac terms are equal to 1 when $S(T)=K$, this leads to the following result:

$$\frac{\partial \mathsf{E}((S(T)-K)\mathsf{I}_{\{S(T)>K\}})}{\partial S} = \frac{\partial C}{\partial S} = \mathsf{E}^S(\mathsf{I}_{\{S(T)>K\}}|\mathcal{F}(t)) = \mathsf{Q}^S(S(T)>K) = F^S(x) \quad \text{(A.9)}$$

where $F^S(x)$ is the cumulative distribution function under Q^S. □

A.2 Sum of weighted Independent Gamma Random Variables with Different Scale and Shape Parameters

Proof. We start by writing,

$$\begin{aligned} \zeta_1 &= c_1\gamma_1 = \Lambda_1, \\ \zeta_2 &= c_1\gamma_1 + c_2\gamma_1 = \Lambda_1 + \Lambda_2, \end{aligned}$$

where $\Lambda_1 \sim Ga(\alpha_1, c_1\sigma_1)$ and $\Lambda_2 \sim Ga(\alpha_2, c_2\sigma_2)$. Then using $\Lambda_2 = \zeta_2 - \Lambda_1 = \zeta_2 - \zeta_1$, we calculate the Jacobian matrix and the determinant,

$$\begin{aligned} \zeta_{i,j} &= \begin{bmatrix} \frac{\partial \Lambda_1}{\partial \zeta_1} & \frac{\partial \Lambda_1}{\partial \zeta_2} \\ \frac{\partial \Lambda_2}{\partial \zeta_1} & \frac{\partial \Lambda_2}{\partial \zeta_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \\ |J| &= \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1. \end{aligned}$$

Then we can start by writing the density of ζ_2 in line with convolution of two random variables,

$$f(\zeta_2, \alpha_1, \alpha_2, \sigma_1, \sigma_2, c_1, c_2) = \int_0^{\zeta_2} \frac{\Lambda_1^{\alpha_1-1} e^{\frac{-\Lambda_1}{c_1\sigma_1} + \frac{\Lambda_1}{c_2\sigma_2}} (\zeta_2 - \Lambda_1)^{\alpha_2-1} e^{\frac{-\zeta_2}{c_2\sigma_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} d\Lambda_1,$$

after some tedious algebra and modifying the boundary of the integral we obtain,

$$f(\zeta_2,\alpha_1,\alpha_2,\sigma_1,\sigma_2,c_1,c_2)=\zeta_2^{\alpha_2+\alpha_1-1}e^{\frac{-\zeta_2}{c_2\sigma_2}}\underbrace{\int_0^1\frac{\Lambda_1^{\alpha_1-1}\left(1-\Lambda_1\right)^{\alpha_2-1}e^{-\zeta_2\Lambda_1\left(\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}\right)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}d\Lambda_1}_{\frac{{}_1F\left(\alpha_1,\alpha_2+\alpha_1,\zeta_2\left(\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}\right)\right)}{\Gamma(\alpha_1+\alpha_2)}}. \quad (A.10)$$

We see that the integral in (A.10) can be written in terms of confluent hypergeometric function of the second kind using Gradshteyn and Ryzhik (2007) (page 870, Equation-7.621-5).

Therefore final representation will be,

$$f(\zeta_2,\nu_1,\nu_2,\sigma_1,\sigma_2,c_1,c_2)=\zeta_2^{\alpha_2+\alpha_1-1}e^{\frac{-\zeta_2}{c_2\sigma_2}}(c_1\sigma_1)^{-\alpha_1}(c_2\sigma_2)^{-\alpha_2}\frac{{}_1F_1\left(\alpha_1,\alpha_2+\alpha_1,\zeta_2\left(\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}\right)\right)}{\Gamma\left(\alpha_1+\alpha_2\right)} \quad (A.11)$$

The characteristic function will be straightforward to derive. First we write characteristic function,

$$\begin{aligned}\Phi_{\zeta_2}(u,\alpha_1,\alpha_2,\sigma_1,\sigma_2,c_1,c_2) &= \int_0^\infty e^{iu\zeta_2}\zeta_2^{\alpha_2+\alpha_1-1}e^{\frac{-\zeta_2}{c_2\sigma_2}}(c_1\sigma_1)^{-\alpha_1}(c_2\sigma_2)^{-\alpha_2}\times\\ \frac{{}_1F_1\left(\alpha_1,\alpha_2+\alpha_1,\zeta_2\left(\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}\right)\right)}{\Gamma\left(\alpha_1+\alpha_2\right)}d\zeta_2\\ &= \int_0^\infty e^{-\left(\frac{1}{c_2\sigma_2}-iu\right)\zeta_2}\zeta_2^{\alpha_2+\alpha_1-1}(c_1\sigma_1)^{-\alpha_1}(c_2\sigma_2)^{-\alpha_2}\times\\ \frac{{}_1F_1\left(\alpha_1,\alpha_2+\alpha_1,\zeta_2\left(\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}\right)\right)}{\Gamma\left(\alpha_1+\alpha_2\right)}d\zeta_2\end{aligned} \quad (A.12)$$

Then using Gradshteyn and Ryzhik (2007) (page 822, Equation 4) we can write,

$$\Phi_{\zeta_2}(u,\alpha_1,\alpha_2,\sigma_1,\sigma_2,c_1,c_2)=D\left(2c_1\sigma_1-c_2\sigma_2-iuC\right)^{-\alpha_1-\alpha_2}\times F\left(\alpha_2,\alpha_1+\alpha_2,\alpha_1+\alpha_2,\frac{k}{k-\left(\frac{1}{c_2\sigma_2}-iu\right)}\right) \quad (A.13)$$

where $D=(c_1\sigma_1)^{\alpha_2}(c_2\sigma_2)^{\alpha_1}$, $k=\frac{1}{c_1\sigma_1}-\frac{1}{c_2\sigma_2}$ and $C=c_1\sigma_1c_2\sigma_2$. □

A.3 VGC Characteristic Function and Martingale Correction Factor

Proof. The characteristic function of VGC[16] random variable can be written as,

$$\begin{aligned}
\mathsf{E}\left(e^{iuX(t)}\right) &= \mathsf{E}\left(\mathsf{E}\left(e^{iu\theta g + iu\sigma W(g)}|\gamma = g\right)\right) \\
&= \mathsf{E}\left(e^{i\left(u\theta - \frac{u^2\sigma^2}{2}\right)g}\right) \\
&= D\left(2c_1\sigma_1 - c_2\sigma_2 - iuC\theta + C\frac{u^2\sigma^2}{2}\right)^{-\alpha_1-\alpha_2} \\
&\times \; F\left(\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2, \frac{k}{k - \left(\frac{1}{c_2\sigma_2} - iu\theta + \frac{u^2\sigma^2}{2}\right)}\right)
\end{aligned}$$

Then $\phi_X(-i)$, the natural logarithm of VGC characteristic function evaluated at $-i$, and also the martingale correction factor could be written as follows,

$$\phi_X(-i) = \log(D) + (-\alpha_1 - \alpha_2)\log\left(2c_1\sigma_1 - c_2\sigma_2 + A\right) + \log\left(F\left(\alpha_2, \alpha_1+\alpha_2, \alpha_1+\alpha_2, \frac{k}{k - \left(\frac{1}{c_2\sigma_2} + A\right)}\right)\right) \tag{A.14}$$

where $A = C\theta - \frac{C\sigma^2}{2}$ and $\kappa = \left(2c_1\sigma_1 - c_2\sigma_2 - C\theta - \frac{C\sigma^2}{2}\right)$ which is used in equation(39). □

A.4 Multivariate Gamma Density

Proof. Let $Z_1(\nu_1, \beta_1) = X_1 + c_1 Y$ and $Z_2(\nu_2, \beta_2) = X_2 + c_2 Y$ be correlated random variables then their joint density can be shown first by re-defining the linear relationships,

$$\begin{aligned}
Y &= Z_0, \\
X_1 &= Z_1 - c_1 Z_0, \\
X_2 &= Z_2 - c_2 Z_0.
\end{aligned}$$

[16]Setting $c_1\sigma_1 = c_2\sigma_2$, one can easily recover original VG process characteristic function and, therefore, the martingale correction factor.

Then we define the Jacobian

$$\zeta_{i,j} = \begin{bmatrix} \frac{\partial X_1}{\partial Z_0} & \frac{\partial X_1}{\partial Z_1} & \frac{\partial X_1}{\partial Z_2} \\ \frac{\partial X_2}{\partial Z_0} & \frac{\partial X_2}{\partial Z_1} & \frac{\partial Y}{\partial Z_2} \\ \frac{\partial Y}{\partial Z_0} & \frac{\partial Y}{\partial Z_1} & \frac{\partial Y}{\partial Z_2} \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$|J| = \begin{vmatrix} -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1.$$

We write the density of Z_1, Z_2 given that z_0, z_1, z_2 are all gamma distributed random variables with $Ga(\alpha_0, \beta_0)$ and $Ga(\alpha_1, \beta_1, \alpha_1, \beta_2)$ and then we integrate out z_0 to obtain $f(z_1, z_2)$. We begin by writing the component of the density (25),

$$f(z_1, z_2) = \frac{z_1^{\alpha 1-1} z_2^{\alpha 2-1} e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}} \beta_1^{-\alpha_1} \beta_2^{-\alpha_2}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\min(z_1,z_2)} z_0^{\alpha_0-1} e^{z_0\left(\frac{c_1}{\beta_1}+\frac{c_2}{\beta_2}-\frac{1}{\beta_0}\right)} \left(1 - c_1\frac{z_0}{z_1}\right)^{\alpha_1-1} \left(1 - c_2\frac{z_0}{z_2}\right)^{\alpha_2-1} dz_0.$$

Then we write the second component (26),

$$f(z_1, z_2) = \frac{z_1^{\alpha 1-1} z_2^{\alpha 2-1} e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}} \beta_1^{-\alpha_1} \beta_2^{-\alpha_2}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{\min(z_1,z_2)} z_0^{\alpha_0-1} e^{z_0\left(\frac{c_1}{\beta_1}+\frac{c_2}{\beta_2}-\frac{1}{\beta_0}\right)} \left(1 - c_1\frac{z_0}{z_2}\right)^{\alpha_1-1} \left(1 - c_2\frac{z_0}{z_1}\right)^{\alpha_2-1} dz_0.$$

Without loss of generality, assume that $z_1 > z_2$, and if we set $u = c_1 z_0$ together with the function defined in Humbert (1922) (page 79) yields

$$f(z_1, z_2) = \frac{z_1^{\alpha_1-1} z_2^{\alpha_2-1} e^{-\frac{z_1}{\beta_1}} e^{-\frac{z_2}{\beta_2}} \beta_1^{-\alpha_1} \beta_2^{-\alpha_2}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} \times \underbrace{\int_0^1 u^{\alpha_0-1} (1-u)^{\alpha_1-1} \left(1 - u\frac{c_2 z_1}{z_2 c_1}\right)^{\alpha_2-1} e^{u\left(z_1\left(\frac{1}{\beta_1}+\frac{c_2}{\beta_2 c_1}-\frac{1}{\beta_0 c_1}\right)\right)} du}_{\frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\Phi_1\left(\alpha_0, 1-\alpha_2, \alpha_0+\alpha_1; \frac{c_2 z_2}{z_1 c_1}, z_1\left(\frac{1}{\beta_1}+\frac{c_2}{c_1\beta_2}-\frac{1}{\beta_0 c_1}\right)\right)}{\Gamma(\alpha_0+\alpha_2)}}.$$

Finally regarding two cases $z_1 > z_2$ and $z_1 < z_2$, we obtain respectively the densities,

$$f(z_1, z_2, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2) = \frac{z_1^{\alpha_1-1} z_2^{\alpha_2-1} e^{\frac{-z1}{\beta_1}} e^{\frac{-z2}{\beta_2}}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_0, \alpha_1) \times \Phi_1\left(\alpha_0, 1-\alpha_2, \alpha_0+\alpha_1; \frac{c_2 z_2}{c_1 z_1}, z_1\left(\frac{1}{\beta_1}+\frac{c_2}{c_1\beta_2}-\frac{1}{c_1\beta_0}\right)\right), \tag{A.15}$$

$$f(z_1,z_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2) = \frac{z_1^{\alpha_1-1} z_2^{\alpha_2-1} e^{\frac{-z1}{\beta_1}} e^{\frac{-z2}{\beta_2}}}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_0,\alpha_1)$$
$$\times \Phi_1\left(\alpha_0, 1-\alpha_1, \alpha_0+\alpha_2; \frac{c_1 z_1}{c_2 z_2}, z_2\left(\frac{c_1}{c_2\beta_1} + \frac{1}{\beta_2} - \frac{1}{c_2\beta_0}\right)\right), \qquad \text{(A.16)}$$

where Φ_1 is confluent hypergeometric function of two variables in Humbert (1922).

Using the derivations above and applying to the case of three correlated random variables Z_i, Z_j, Z_k of the form $Z_j = X_j + c_j Y$, we obtain the following density where we have the condition (without loss of generality) $Z_i = \min(Z_i, Z_j, Z_k)$,

$$f(z_i,z_j,z_k,\alpha_0,\alpha_i,\alpha_j,\alpha_k,\beta_0,\beta_i,\beta_j,\beta_k) = \frac{z_i^{\alpha_i-1} z_j^{\alpha_j-1} z_k^{\alpha_k-1} e^{-\frac{z_i}{\beta_i}} e^{-\frac{z_j}{\beta_j}} e^{-\frac{z_k}{\beta_k}} \beta_i^{-\alpha_i}\beta_j^{-\alpha_j}\beta_k^{-\alpha_k}}{\Gamma(\alpha_0)\Gamma(\alpha_i)\Gamma(\alpha_j)\Gamma(\alpha_k)} \qquad \text{(A.17)}$$
$$\times \underbrace{\int_0^1 u^{\alpha_0-1}(1-u)^{\alpha_i-1}\left(1-u\frac{c_i z_i}{c_j z_j}\right)^{\alpha_j-1}\left(1-u\frac{z_j}{z_k}\right)^{\alpha_k-1} e^{u\left(z_i\left(\frac{c_i}{c_j\beta_i}+\frac{1}{\beta_j}+\frac{1}{\beta_k}-\frac{1}{\beta_0 c_i}\right)\right)} du}_{\Phi_2(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\beta_0,\beta_1,\beta_2,\beta_3,z_1,z_2,z_3)}.$$

Furthermore, we can write Φ_2 in terms of special functions. Starting with a Taylor expansion of e^x and rewriting (A.17), we obtain

$$\Phi_2(\alpha_0,\alpha_i,\alpha_j,\alpha_k,\beta_0,\beta_i,\beta_j,\beta_k,z_i,z_j,z_k) = \sum_{m,n,l=0}^{\infty} F_D^{(2)}(\alpha_0+m,\alpha_j-1,\alpha_k-1,\alpha_i+\alpha_0+m)\frac{\Gamma(\alpha_0+m)\Gamma(\alpha_i)}{\Gamma(\alpha_0+\alpha_i+m)} \times$$
$$\frac{\left(z_i\left(\frac{c_i}{c_j\beta_i}+\frac{1}{\beta_j}+\frac{1}{\beta_k}-\frac{1}{\beta_0 c_i}\right)\right)^m (\alpha_j-1)^n(\alpha_k-1)^l}{m!n!l!}.$$

Multiplying equation (A.18) by $\Gamma(\alpha_0+\alpha_i)$ and dividing by $\Gamma(\alpha_0)$, we obtain Φ_2 in terms of hypergeometric series,

$$\Phi_2(\alpha_0,\alpha_i,\alpha_j,\alpha_k,\beta_0,\beta_i,\beta_j,\beta_k,z_i,z_j,z_k) = \sum_{m=0}^{\infty}\sum_{n=0,l=0}^{\infty}\frac{(\alpha_0)_{m+n+l}(\alpha_j-1)_n(\alpha_k-1)_l}{(\alpha_0+\alpha_i)_{m+n+l}} \times$$
$$\frac{\left(z_i\left(\frac{c_i}{c_j\beta_i}+\frac{1}{\beta_j}+\frac{1}{\beta_k}-\frac{1}{\beta_0 c_i}\right)\right)^m \left(\frac{c_i z_i}{c_j z_j}\right)^n \left(\frac{z_j}{z_k}\right)^l}{m!n!l!},$$

where $(q)_n$ is Pochhammer symbol and F_D^2 is Lauricella function of $n=2$ and type D. □

A.5 Approximate VGC Model CDF

Proof. The approximate VGC model CDF can be obtained by using the fact that,

$$\begin{aligned} F(x) &= \int_0^\infty \mathcal{N}\left(\frac{(x-\theta g)}{\sqrt{g}}\right)\frac{g^{\alpha-1}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg, && \text{(A.18)} \\ &= I(m,n) && \text{(A.19)} \\ &= \int_0^\infty \int_{-\infty}^V \left(\mathcal{N}_n\left(\frac{n(v)}{\sqrt{g}} - m(v)\sqrt{g}\right)n_v g^{-\frac{1}{2}} + \mathcal{N}_m\left(\frac{n(v)}{\sqrt{g}} - m(v)\sqrt{g}\right)m_v g^{\frac{1}{2}}\right)dv\frac{g^{\alpha-1}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg. \end{aligned}$$

Here we set $m = m(v)$, $n = n(v)$ and $m(V) = m$, $n(V) = n$. We will define following integrals and parametrizations,

$$\begin{aligned} I^1(v) &= \int_0^\infty \mathcal{N}_n\left(\frac{n(v)}{\sqrt{g}} - m(v)\sqrt{g}\right)\frac{g^{\alpha-\frac{3}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(n-mg)^2}{2g}\right)\frac{g^{\alpha-\frac{3}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \\ I^2(v) &= \int_0^\infty \mathcal{N}_n\left(\frac{n(v)}{\sqrt{g}} - m(v)\sqrt{g}\right)\frac{g^{\alpha-\frac{1}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(n-mg)^2}{2g}\right)\frac{g^{\alpha-\frac{1}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \end{aligned}$$

Using equation (10) in Chaudhry and Syed (2002) , we have

$$\begin{aligned} I^1(v) &= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(n-mg)^2}{2g}\right)\frac{g^{\alpha-\frac{3}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \\ &= \sqrt{\frac{2}{\pi}}e^{mn}\left(\frac{n}{\sqrt{\frac{2}{\beta}+m^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right) \\ I^2(v) &= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-(n-mg)^2}{2g}\right)\frac{g^{\alpha-\frac{1}{2}}e^{-\frac{g}{\beta}}\beta^{-\alpha}}{\Gamma(\alpha)}dg \\ &= \sqrt{\frac{2}{\pi}}e^{mn}\left(\frac{n}{\sqrt{\frac{2}{\beta}+m^2}}\right)^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right) \end{aligned}$$

In order to obtain $F(x)$ defined in (A.18) we need to compute the following

$$I^1 = \int_{-\infty}^{\infty} I^1(v)dv, \qquad \text{(A.20)}$$
$$I^2 = \int_{-\infty}^{\infty} I^2(v)dv,$$

since $F(x) = I^1 + I^2$. Here, we need a variable transformation and a domain which keeps the Bessel function $K(.)$ inside the integrals in equation (A.20) unchanged. We choose the domain $[-\infty, n]$ and apply the following transformations:

$$m(v) = \frac{n\sqrt{m^2 + \frac{2}{\beta}}}{v}, \qquad \text{(A.21)}$$
$$n(v) = \sqrt{v^2 - \frac{2}{\beta}}.$$

After these transformations, we first obtain

$$\int_{-\infty}^{n} I^1(v)n_v dv = K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)\int_{-\infty}^{n} \exp\left(\frac{m\sqrt{n^2+\frac{2}{\beta}}}{v}\sqrt{v^2-\frac{2}{\beta}}\right)\left(\frac{m\sqrt{n^2+\frac{2}{\beta}}}{v^2}\right)^{\alpha-1/2} \times \qquad \text{(A.22)}$$
$$v\left(v^2 - 2/\beta\right)^{\frac{-1}{2}} n_v dv$$

$$\int_{-\infty}^{n} I^2(v)m_v dv = K_{\alpha+\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)\int_{-\infty}^{m} \exp\left(\frac{m\sqrt{n^2+\frac{2}{\beta}}}{v}\sqrt{v^2-\frac{2}{\beta}}\right)\left(\frac{m\sqrt{n^2+\frac{2}{\beta}}}{v^2}\right)^{\alpha+1/2} \times \qquad \text{(A.23)}$$
$$\frac{m\sqrt{n^2+\frac{2}{\beta}}}{v^2} m_v dv$$

Then using the following,

$$u = \sqrt{\frac{v^2 - \frac{2}{\beta}}{v}},$$
$$v = \sqrt{\frac{2}{\beta(1-u^2)}},$$
$$dv = \sqrt{2}u(1-u^2)^{\frac{-3}{2}},$$

we obtain,

$$\int_{-\infty}^{n} I^1(v) n_v dv = K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)\int_{-1}^{\sqrt{\frac{n^2-\frac{2}{\beta}}{n}}} \sqrt{2}e^{cu}\left(\frac{(1-u^2)\beta}{2}\right)^{\alpha-1/2} u\left(1-u^2\right)^{-1/2} du,$$

$$\int_{-\infty}^{n} I^2(v) m_v dv = K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)\int_{-1}^{\sqrt{\frac{n^2-\frac{2}{\beta}}{n}}} \sqrt{2}e^{cu}\left(\frac{(1-u^2)\beta}{2}\right)^{\alpha-1/2} \left(1-u^2\right)^{-1/2} du.$$

Then setting $q = \sqrt{\frac{n^2-\frac{2}{\beta}}{n}}$, we obtain

$$\begin{aligned} I^1 &= \int_{-\infty}^{n} I^1(v) n_v dv = K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right) \\ &\times \int_0^1 \sqrt{2}e^{c(u(1+q)-1)}\left(\frac{1-(u(1+q)-1)^2)\beta}{2}\right)^{\alpha-1/2} (u(1+q)-1)\left(1-(u(1+q)-1)^2\right)^{-1/2} du, \\ &= K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right) \times \int_0^1 u(1+q)^{\alpha}\left(1-\frac{u(1+q)}{2}\right)^{\alpha-1} e^{c(u(1+q)-1)}\frac{C^{\alpha+\frac{1}{2}}}{\sqrt{2\pi}} du \\ I^2 &= \int_{-\infty}^{n} I^2(v) m_v dv = K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right) \times \int_0^1 u(1+q)^{\alpha-1}\left(1-\frac{u(1+q)}{2}\right)^{\alpha-1} e^{c(u(1+q)-1)}\frac{C^{\alpha+\frac{1}{2}}}{\sqrt{2\pi}} du \end{aligned}$$

Then in the context of confluent hypergeometric function of second kind (Humbert (1922), page 79 or Gradshteyn and Ryzhik (2007), formula 3.385), for I^1 we have

$$a = \alpha + 1, \quad \beta = 1 - \alpha, \quad \gamma = \alpha + 2$$

for I^2, we have

$$a = \alpha, \quad \beta = 1 - \alpha, \quad \gamma = \alpha + 1$$

After collecting all these and using the definition of confluent hypergeometric function of second kind we obtain,

$$F(x) = I^1 + I^2 = e^{-C}\beta^{\alpha}\frac{C^{\alpha+1/2}}{\sqrt{2\pi}}\Phi_1\left(\alpha, 1-\alpha, \alpha+1, \frac{1+q}{2}, c(1+q)\right) \times\frac{K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)}{\alpha\Gamma(\alpha)} -$$

$$e^{-C}\beta^{\alpha}\frac{C^{\alpha+1/2}}{\sqrt{2\pi}}\Phi_1\left(\alpha+1, 1-\alpha, \alpha+2, \frac{1+q}{2}, c(1+q)\right) \times \frac{K_{\alpha-\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)}{(1+\alpha)\Gamma(\alpha)} +$$

$$e^{-C}\beta^{\alpha}\frac{C^{\alpha+1/2}}{\sqrt{2\pi}}\Phi_1\left(\alpha, 1-\alpha, \alpha+1, \frac{1+q}{2}, c(1+q)\right) \times \frac{K_{\alpha+\frac{1}{2}}\left(|n|\sqrt{m^2+\frac{2}{\beta}}\right)}{\alpha\Gamma(\alpha)}.$$

where $C = n\sqrt{m^2+\frac{2}{\beta}}$. □