Data-driven Minimum Entropy Control for Stochastic Nonlinear Systems using the Cumulant-Generating Function

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Abstract

This paper presents a novel minimum entropy control algorithm for a class of stochastic nonlinear systems subjected to non-Gaussian noises. The entropy control can be considered as an optimization problem for the system randomness attenuation, but the mean value has to be considered separately. To overcome this disadvantage, a new representation of the system stochastic properties was given using the cumulant-generating function based on the moment-generating function, in which the mean value and the entropy was reflected by the shape of the cumulant-generating function. Based on the samples of the system output and control input, a time-variant linear model was identified, and the minimum entropy optimization was transformed to system stabilization. Then, an optimal control strategy was developed to achieve the randomness attenuation, and the boundedness of the controlled system output was analyzed. The effectiveness of the presented control algorithm was demonstrated by a numerical example. In this paper, a data-driven minimum entropy design is presented without pre-knowledge of the system model; entropy optimization is achieved by the system stabilization approach in which the stochastic distribution control and minimum entropy are unified using the same identified structure; and a potential framework is obtained since all the existing system stabilization methods can be adopted to achieve the minimum entropy objective.

Index Terms

Non-Gaussian stochastic systems, cumulant-generating function, data-driven design, kernel density estimation, stabilization

I. Introduction

The information entropy has been presented by Claude Shannon as a measurement of the randomness for dynamic stochastic processes [1]. For Gaussian noise, the entropy is equivalent to the variance of the random variable, which implies that the minimum entropy design is an extension of the minimum variance design for non-Gaussian cases. In other words, the existing Gaussian-based results cannot be applied to the dynamic systems, which are subjected to non-Gaussian noises.

Since the probability density function (PDF) of the system output reflects the full information of the stochastic properties, the minimum entropy optimization has been introduced [1] where the entropy value of the system output can be obtained once the system output PDF is known. The PDF of the system output can be adjusted by the control design [2], [3], so the minimum randomness of the system output can be achieved by developing the control signal to minimize the system output entropy. Based on the various definitions of the entropy, the optimization problem can be formulated with an entropy-based cost function [1]. Moreover, the information potential can be used to simplify the cost function [4], and the data-driven method has been investigated [5] using kernel density estimation (KDE). Currently, most of the existing minimum entropy design approaches focus on the stochastic system with a known system model, and the entropy value evolution depends on an accurate system model [6]. For example, the state-space solution is given [7] and the entropy minimum is achieved for time-vary system models [8]. However, an accurate model is difficult to obtain in practice. Meanwhile, databased estimation approaches focus on direct optimization, which leads to the difficulty of choosing the controller structure. In addition, the convexity of the optimization problem is another challenge.

The minimum entropy optimization has been a powerful design tool for stochastic systems, such as non-Gaussian filtering [9], fault diagnosis [10], error minimum tracking [11], and so on. In practice, minimum entropy optimization is widely used for industrial system modeling [12] and networked control systems [13]. All of the mentioned results can be redesigned if the mentioned problems are fixed, which means that a new framework of minimum entropy optimization would generate real impacts in many aspects.

Motivated by the stochastic distribution control [14], the mentioned problems would be solved by transforming the stochastic system design into the deterministic system design in which the neural network weighting vector is used as the identified deterministic model state. The PDF tracking can be realized via the state assignment, and the minimum entropy is difficult to implement because the optimal PDF with minimum entropy is the Dirac delta function (δ function). In this paper, the moment-generating function is adopted to transform the PDF. Following the definition of the moment-generating function, the minimum entropy control problem can be redescribed as the cumulant-generating function tracking problem. When replacing the neural network weighting vector for stochastic distribution control, the sampling operation for cumulant-generating function should be considered with collected data in which the sampled values are used as the pseudo states of the system dynamics. Based upon these pseudo states, a time-variant linear model can be identified. The moment-generating function is equal to 1 if and only if all the moments of the random variable are equal to 0, and the minimum entropy can be achieved if all the pseudo states converge to 0. Therefore, the minimum entropy problem can be further converted to a system stabilization problem. Following the transformation, all the existing control theory about stabilization or states assignment can be inserted into this framework, and the described shortcomings can be resolved. Furthermore, a bridge between the stochastic distribution control and minimum entropy control can be established.

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The rest of the paper has been organized as follows: Section II presents the problem description, entropy, KDE, and moment-generating function. Section III describes the main algorithm, and Section IV provides the system stability analysis. To validate the presented algorithm, Section V demonstrates the effectiveness of the control algorithm using a numerical example. Section VI provides potential extension and further discussions. Section VII summarizes the conclusions and future work.

II. Formulation

A. Problem description

Consider the following single-input-single-output (SISO) discrete-time stochastic nonlinear system:

$$\begin{aligned} x_{k+1} &= f\left(x_k, u_k\right) + g\left(x_k\right) w_k \\ y_k &= h\left(x_k\right) \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^1$ is the control input, $y_k \in \mathbb{R}^1$ is the system output, and $w_k \in \mathbb{R}^1$ is the process noise. $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(0) = 0$, $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^1$ stand for the unknown differentiable nonlinear functions, and k is the sampling index in terms of time.

Based upon this system model, the control objective can be described as minimizing the randomness of the system output y_k.

To describe the randomness of the investigated system output y_k, the following assumption is given for Eq. (1).

Assumption 1: The moment-generating function of y_k in Eq. (1) always exists.

The continuous PDF can be obtained for most of the practical system variables. The moment-generating function normally exists since it can be produced by Laplace transformation.

B. Entropy and KDE

Since the system output can be considered as a random process, the randomness can be measured by Shannon's entropy, defined as follows:

$$H_k = -\int_{-\infty}^{\infty} \gamma_k\left(\alpha\right) \log \gamma_k\left(\alpha\right) d\alpha \tag{2}$$

where α denotes the random variable of the system output y_k, and y_k is the sample point of the random variable α. $\gamma_k\left(\alpha\right)$ stands for the PDF, and the value of entropy will be attenuated if $\gamma_k\left(\alpha\right)$ becomes sharper.

Using the samples of the system output y_k, the PDF can be approximated by sliding window KDE as follows:

$$\hat{\gamma}_k\left(\alpha\right) = \frac{1}{k_c \bar{h}} \sum_{i=k-k_c+1}^{k} G\left(\frac{\alpha - y_i}{\bar{h}}\right) \tag{3}$$

where $G\left(\cdot\right)$ is the Gaussian kernel function, and $\bar{h}$ is bandwidth. $k_c \in \mathbb{Z}^+$ is the prespecified length of the sliding window.

Based on the estimated PDF, the entropy value $\hat{H}_k$ for each k can be calculated based on the definition, and the data-based optimization can be achieved directly following the performance criterion as

$$J_k = \min_{u_k} \hat{H}_k \tag{4}$$

Notably, Eq. (4) does not contain the information of the mean value.

C. Moment-generating function

The PDF is uniquely determined by moment-generating function, which can be expressed as follows:

$$M_k\left(t\right) = \int_{-\infty}^{\infty} e^{t\alpha} \gamma_k(\alpha) d\alpha \tag{5}$$

where $t \in \mathbb{R}^1$.

Expanding the moment-generating function results in

$$M_k\left(t\right) = 1 + t m_1 + \frac{t^2 m_2}{2!} + \cdots + \frac{t^n m_n}{n!} + \cdots \tag{6}$$

where m_n denotes the nth-moment.

The randomness is equal to 0 if the random signal becomes deterministic signal where the ith-order moment is equal to 0, $i = 2, 3, \ldots, n$. Then, the value of the moment-generating function is only determined by mean value of the variable. Without loss of generality, if the mean value and randomness are both equal to 0, then $m_1 = 0$, and

$$M_k\left(t\right) = 1, \forall t \in \mathbb{R}^1 \tag{7}$$

The curves of the moment-generating functions compared with other distributions are demonstrated in Fig. 1.

Motivated by the stochastic distribution control, the minimum entropy can be achieved if the estimated moment-generating function $\hat{M}\left(t\right)$ for each k is tracking to a straight line where the random signal has been stabilized to a deterministic signal. Furthermore, this deterministic signal can be obtained with value 0 if and only if the moment-generating function is a horizontal straight line that goes across $(0, 1)$ in Fig. 1.

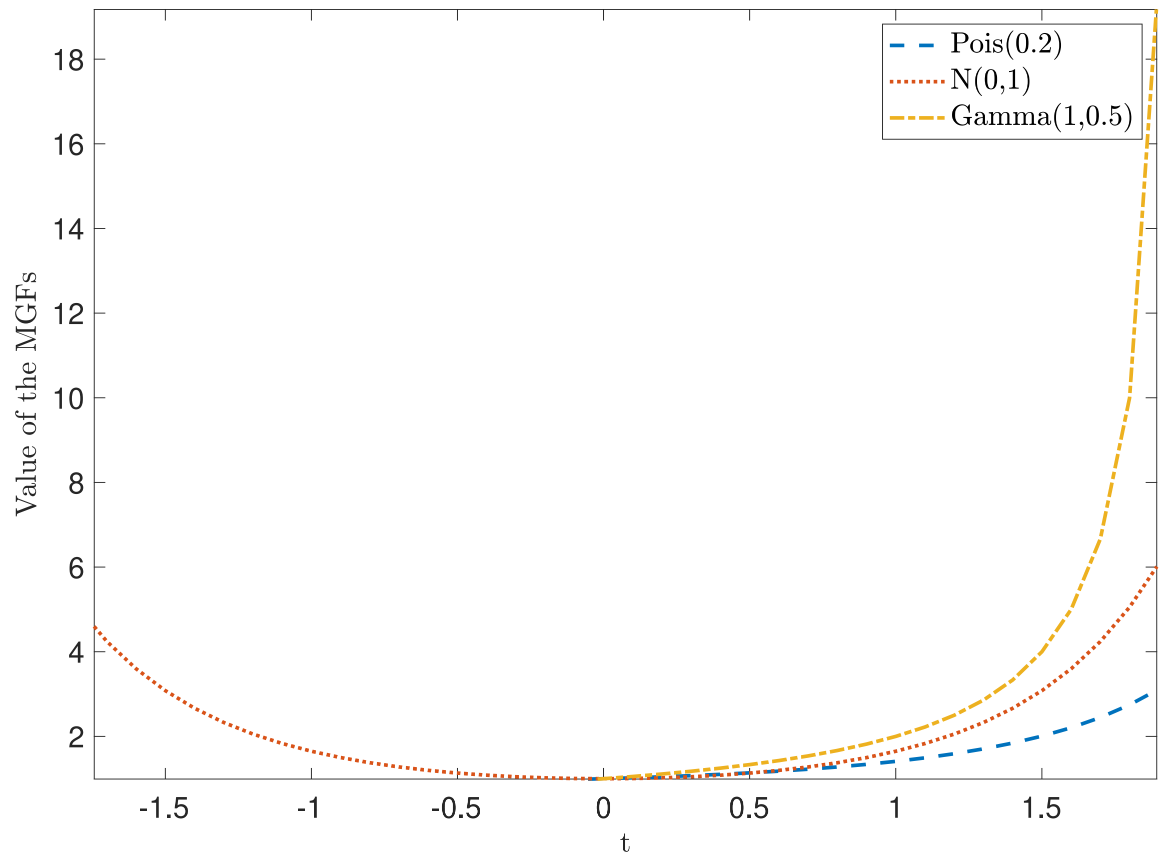

Fig. 1. The moment-generating functions for Poisson distribution, normal distribution, gamma distribution, and deterministic signal constant $\mathbf{0}$. All the moment-generating functions go across $(\mathbf{0},\mathbf{1})$ because the integral of any PDF is equal to $\mathbf{1}$.

III. Control Algorithm

The cumulant-generating function can be defined as follows using moment-generating function:

$$\phi_k(t) = \log M_k(t) \tag{8}$$

where $\phi_k(0) = 0$ and $\phi_k : \mathbb{R}^1 \to \mathbb{R}^1$.

Based on the estimated PDF, the estimated value of the cumulant-generating function $\phi_k(t)$ can be obtained for each k using the data that can be denoted as $\hat{\phi}_k(t)$. Then, this function can be sampled using the prespecified t. In particular, $t_1, t_2, \ldots, t_m$ can be selected, and the sampled $\hat{\phi}_k(t)$ can be re-expressed as

$$z_k := \left[\hat{\phi}_k(t_1), \hat{\phi}_k(t_2), \ldots, \hat{\phi}_k(t_m)\right]^T \tag{9}$$

where z_k denotes the pseudo state vector in terms of stochastic distribution, and m is the dimension of z_k. z_k is a function of t. The complete information of the stochastic distribution can be estimated by spline interpolation if m is sufficient large.

Remark 1: m is prespecified based on the complexity of the estimated PDF [15]. Because the moment-generating function value is sensitive to high-order moments, a neighbourhood around the origin should be selected for t to avoid the large-value of z_k. The more knots selected within the fixed interval, the shorter distance between each knot would be. Thus, the approximation of the spline interpolation can be achieved with the uniformly selected knot vector [16], [17].

Since the dynamics of the output PDF of the stochastic system are governed by a partial differential equation given as Kolmogorov equations, z_k satisfies the following assumption.

Assumption 2: An unknown differentiable nonlinear function $f_z(\cdot)$ exists such that the pseudo state vector z_k holds for the following discrete-time equation.

$$z_{k+1} = f_z(z_k, u_k) \tag{10}$$

Because the dynamics of the investigated stochastic system can be redescribed using Eq. (10), the following model structure can be adopted to simplify the formulation via dynamic linearization. To simplify the expression, (t) has been neglected for z_k, which results in

$$z_{k+1} = A_k z_k + B_k u_k + \bar{f}_z(z_k, u_k) \tag{11}$$

where

$$\{A_k, B_k\} = \left\{ \frac{\partial f_z}{\partial z}, \frac{\partial f_z}{\partial u} \right\}\bigg|_{z=z_k, u=u_k} \tag{12}$$

and A_k and B_k stand for the unknown time-variant coefficient matrices. Since the data set $\{z_k, z_{k-1}, \ldots, z_1, u_k, u_{k-1}, \ldots, u_1\}$ is measurable, the coefficient matrices $\{A_k, B_k\}$ can be identified using the well-know recursive least square algorithm. $\bar{f}_z$ denotes the combination of the linearization error, estimation error of KDE, identification error, and the unmodeled dynamics.

If the randomness of the system output is ideally eliminated and the mean value shifts to 0, then $\hat{\phi}_k(t) = 0$, which leads to the transformed control objective as follows:

$$\lim_{k \to \infty} z_k = \bar{0} \tag{13}$$

and the randomness attenuation is equivalent to system stabilization. $\bar{0}$ denotes a m-dimensional vector with all the elements being equal to 0.

Based on this discussion, any existing method for system stabilization can be adopted. In this case, the following performance criterion can be used.

$$J_z = \min_{u_i} h_z(z_{i+1}, u_i) \tag{14}$$

where the cost function is defined as follows:

$$h_z(z_{i+1}, u_i) := z_{i+1}^T Q_i z_{i+1} + R_i u_i^2 \tag{15}$$

and $Q_i \in \mathbb{R}_+^{m \times m}$ and $R_i \in \mathbb{R}_+^1$ denote the weights. $i = 1, 2, \ldots, k$ stands for the index of the sampling instant.

Substituting the pseudo state dynamics in Eq. (11) into the cost function results in

$$\begin{aligned}
&h_z(z_{i+1}, u_i) = R_i u_i^2 \\
&+ \left(A_i z_i + B_i u_i + \bar{f}_z(z_i, u_i)\right)^T Q_i \left(A_i z_i + B_i u_i + \bar{f}_z(z_i, u_i)\right) \\
&= z_i^T A_i^T Q_i A_i z_i + \left(R_i + B_i^T Q_i B_i\right) u_i^2 + 2 z_i^T A_i^T Q_i B_i u_i \\
&+ 2 z_i^T A_i^T Q_i \bar{f}_z(z_i, u_i) + 2 u_i^T B_i^T Q_i \bar{f}_z(z_i, u_i) \\
&+ \bar{f}_z^T(z_i, u_i) Q_i \bar{f}_z(z_i, u_i)
\end{aligned} \tag{16}$$

To minimize this performance criterion in Eq. (14) subjected to the pseudo state dynamics in Eq. (11), the following equation can be obtained.

$$\frac{\partial h_z(z_{i+1}, u_i)}{\partial u_i} = 0 \tag{17}$$

which results in

$$\begin{aligned}
&\left(R_i + B_i^T Q_i B_i\right) u_i + \bar{f}_z^T(z_i, u_i) Q_i \frac{\partial \bar{f}_z(z_i, u_i)}{\partial u_i} + z_i^T A_i^T Q_i B_i \\
&+ z_i^T A_i^T Q_i \frac{\partial \bar{f}_z(z_i, u_i)}{\partial u_i} + u_i^T B_i^T Q_i \frac{\partial \bar{f}_z(z_i, u_i)}{\partial u_i} = 0
\end{aligned} \tag{18}$$

The weight R_i can be selected to meet the following inequality:

$$\frac{\partial^2 h_z(z_{i+1}, u_i)}{\partial u_i^2} > 0 \tag{19}$$

which means that the optimal control law can be obtained by solving Eq. (18).

Since the unmodeled dynamics $\bar{f}_z(z_i, u_i)$ can be estimated numerically as the difference between the model predicted value and the real measurement value, the derivative term in Eq. (18) can be replaced by

$$\frac{\partial \bar{f}_z(z_i, u_i)}{\partial u_i} \approx \frac{\bar{f}_z(z_i, u_i) - \bar{f}_z(z_i, u_{i-1})}{u_i - u_{i-1}} \tag{20}$$

where

$$\begin{aligned}
\bar{f}_z(z_i, u_i) &\approx z_{u_i, i+1} - A_i z_i - B_i u_i \\
\bar{f}_z(z_i, u_{i-1}) &\approx z_{u_{i-1}, i+1} - A_i z_i - B_i u_{i-1}
\end{aligned} \tag{21}$$

and z_i is obtained by Eq. (9). An estimated state for the $i+1$ instant can be obtained using A_i, B_i, z_i and u_i. Then, the new model-based $(i+1)$-th state will be combined with the existing state $z_1, z_2, \ldots, z_i$. Using the mixed data set, $z_{u_i, i+1}$ can be obtained as the data-based estimation of the $(i+1)$-th state using Eq. (9). Similarly, $z_{u_{i-1}, i+1}$ can be obtained by replacing u_i as u_{i-1}.

Thus, Eq. (18) can be rewritten as

$$a_i u_i^2 + b_i u_i + c_i = 0 \tag{22}$$

where

$$
\begin{aligned}
a_i &= \left(R_i + B_i^T Q_i B_i\right) \\
b_i &= z_i^T A_i^T Q_i B_i - \left(R_i + B_i^T Q_i B_i\right) u_{i-1} \\
&\quad + \left(\bar{f}_z\left(z_i, u_i\right) - \bar{f}_z\left(z_i, u_{i-1}\right)\right)^T Q_i B_i \\
c_i &= \left(\bar{f}_z^T\left(z_i, u_i\right) Q_i + z_i^T A_i^T Q_i\right) \\
&\quad \times \left(\bar{f}_z\left(z_i, u_i\right) - \bar{f}_z\left(z_i, u_{i-1}\right)\right) - z_i^T A_i^T Q_i B_i u_{i-1}
\end{aligned} \tag{23}
$$

Solving this equation, the control law for instant k can be obtained in the following format.

$$
\begin{aligned}
u_k &= \frac{-b_k \pm \sqrt{b_k^2 - 4a_k c_k}}{2a_k} \\
&= -\sigma_k\left(K_k z_k + \tau\left(z_k\right)\right)
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
\sigma_k &= \frac{1}{2}\left(R_k + B_k^T Q_k B_k\right)^{-1} \\
K_k &= B_k^T Q_k A_k \\
\tau\left(z_k\right) &= b_k - z_k^T A_k^T Q_k B_k \pm \sqrt{b_k^2 - 4a_k c_k}
\end{aligned} \tag{25}
$$

In particular, Q_k and R_k can be sufficiently large to guarantee the real solution of the equation, σ_k adjusts the convergence rate, and the sign in the control law can be determined by satisfying Eq. (19). The control law with the smaller absolute value will be selected to minimize the performance criterion in Eq. (5) if both real solutions meet Eq. (19).

The following lemma is summarized to illustrate the procedure of the presented randomness attenuation.

Lemma 1: For Eq. (1), the randomness attenuate can be achieved for system output y_k using controller design in Eq. (24), where the performance criterion in Eq. (14) has been minimized.

Proof: Using the obtained control law in Eq. (24), the minimum of the performance criterion in Eq. (4) is achieved, which results in the minimum absolute value of z_k. It implies that the moment-generating function is as close as possible to the noise-free case following the spline interpolation convergence discussed in Remark 1. Therefore, the randomness is attenuated after applying the presented algorithm. ∎

Remark 2: When the values of the unmodeled dynamics are very closed to zero, the second term of the control law can be eliminated, which leads to a linear quadratic regulator.

Remark 3: The pseudo states have been selected using the sample values of the cumulant-generating function rather than the moment values because such sampled values include all the information of the moment, even with infinite order.

When the pseudo states z_i converge to the equilibrium $\bar{0}$, at the same time, the coefficient matrices A_i and B_i converge to real fixed-value constant matrices. Then, the gain K_k will become a constant matrix, which guarantees the control objective with the given weights R and Q. σ_k has been used to link two variable spaces x_k and z_k. The analysis of σ_k will be further given in the system stability analysis.

The block diagram in Fig. 2 illustrates the presented algorithm. In particular, the initialisation block sets up the stimulation signal and weights before obtaining the identified model. The data block shows the storage of the collected system output data. Using the cumulant-generating function, the pseudo state is obtained, and the identification process can be performed with the pseudo state and the control input signal. Using the identified matrices A, B, and numerical estimated $\bar{f}_z$, the control law can be calculated. Then, the control input u will be used to drive the unknown stochastic system, which produces the controlled system output in a closed loop.

To summarize the design procedure, the following pseudo code is demonstrated.

Furthermore, the presented algorithm can be extended as stochastic distribution control. Any desired PDF can be rewritten as the desired function $\phi_{ref}\left(t\right)$, and the desired pseudo states can be sampled as z_{ref}. The stochastic distribution control can be achieved by either a state assignment or an optimisation problem with the following performance criterion.

$$
J_k = \min\left\|z_{ref} - z_k\right\| \tag{26}
$$

This criterion indicates that the presented control framework unifies the stochastic distribution control and minimum entropy control. In particular, the presented minimum entropy control algorithm is a specified case of Eq. (26) with $z_{ref} = \bar{0}$.

IV. Stability

To analyze the convergence of the system output, the following assumption is used.

Assumption 3: A real positive upper bound L_k exists such that the following inequality holds.

$$
\left\|\bar{f}_z\left(z_k, u_k\right) - \sigma_k B_k \tau\left(z_k\right)\right\| \leq L_k \left\|z_k\right\| \tag{27}
$$

First, the stabilization of the pseudo state dynamics should be given. Substituting the control law in Eq. (24) into Eq. (11) provides

$$
\begin{aligned}
\left\|z_{k+1}\right\| &\leq \left\|A_k - \sigma_k B_k K_k\right\| \left\|z_k\right\| \\
&\quad + \left\|\bar{f}_z\left(z_k, u_k\right) - \sigma_k B_k \tau\left(z_k\right)\right\|
\end{aligned} \tag{28}
$$

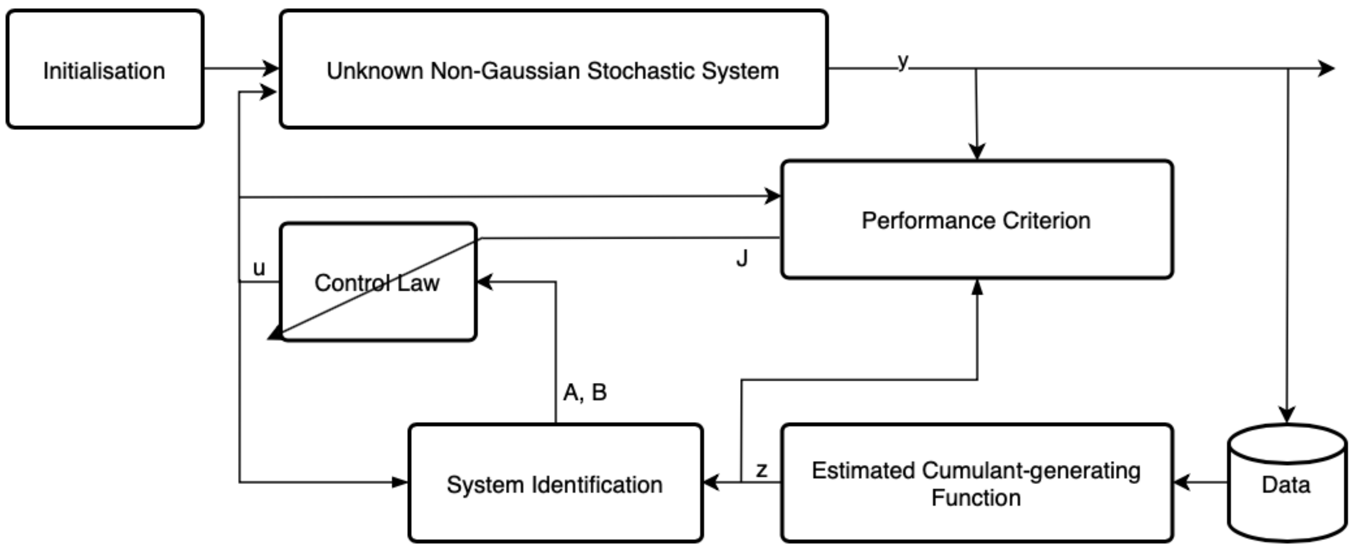

Fig. 2. Block diagram of the presented data-driven control algorithm.

Algorithm 1 Data-driven minimum entropy control based on the cumulant-generating function

Input: t_s: simulation time, k_c: bandwidth for KDE, u_0: default stimulus, m: dimension of z, w_k: system noise, R_0 and Q_0: weights
Output: y_k

Initialisation: Create variable z_k, y_k, u_k, x_k, R_k, Q_k. Setup value x_k, w_k, $u_1 = u_0$, $[t_1, \ldots, t_m]$. (t_i is uniformly selected within the interval.)
for $k \le t_s$ **do**
  $Q_k, R_k \leftarrow Q_{k-1}, R_{k-1}$
  if $k \ge k_c$ **then**
    $\phi_k(t) \leftarrow$ Eqs. (3,5,8)
    $z_k \leftarrow$ Eq. (9)
    $A, B \leftarrow$ Eq. (11)
    $\bar{f}_z \leftarrow$ Eq. (21)
    $a_k, b_k, c_k \leftarrow$ Eq. (23)
    while $b_k^2 < 4a_k c_k$ **do**
      Reselect weights Q_k and R_k. (In practice, Q_k can be pre-selected as a constant positive-defined matrix; R_k can be reselected by increasing the value, such as $R_k \leftarrow 2 * R_k$.)
    end while
    $u_k \leftarrow$ Eq. (24)
  else
    $u_k \leftarrow u_0$
  end if
  $x_{k+1}, y_k \leftarrow$ Eq. (1)
  $k \leftarrow k+1$
end for

Thus, the following condition can be obtained for the stabilization of the dynamics based on Assumption 3.

$$\|A_k - \sigma_k B_k K_k\| + \|L_k\| \le 1 \tag{29}$$

Next, the boundedness of the original system, Eq. (1), should be further investigated. Particularly, for a general nonlinear model, the linearization can be adopted where the following linear model can be obtained.

$$x_{k+1} = F_k x_k + C_k u_k + G_k x_k w_k \tag{30}$$

where

$$\{F_k, C_k, G_k\} = \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial g}{\partial x} \right\} \Big|_{x=x_k, u=u_k} \tag{31}$$

The control law in Eq. (24) stabilizes z_k, which means

$$\lim_{k \to \infty} \bar{f}_z(z_k, u_k) = 0 \tag{32}$$

and the control law can be simplified as

$$u_k = -\sigma_k K_k z_k \tag{33}$$

Substituting the simplified control law into the system model results in

$$x_{k+1} = F_k x_k - \sigma_k C_k K_k z_k + G_k x_k w_k \tag{34}$$

For each z_k, there exists a constant matrix N_k such that,

$$z_k \leq N_k x_k \tag{35}$$

because the dimension of z_k is greater than the dimension of x_k.

Based on the given formula, the analysis can be started from the first sampling instant $k = 1$, where

$$x_2 \leq \left(F_1 - \sigma_1 C_1 K_1 N_1 + G_1 w_1\right) x_1 \tag{36}$$

The mean value can be obtained for the system states such that

$$E\left\{x_2\right\} \leq \left(F_1 - \sigma_1 C_1 K_1 N_1 + G_1 \varepsilon_w\right) E\left\{x_1\right\} \tag{37}$$

where ε_w denotes the mean value of the noise w_k.

In addition, Eq. (37) leads to

$$\left\|E\left\{x_2\right\}\right\| \leq \left\|F_1 - \sigma_1 C_1 K_1 N_1 + G_1 \varepsilon_w\right\| \left\|E\left\{x_1\right\}\right\| \tag{38}$$

This implies that a constant σ_1 exists such that the following inequality holds.

$$\left\|F_1 - \sigma_1 C_1 K_1 N_1 + G_1 \varepsilon_w\right\| \leq 1 \tag{39}$$

which means that the system states are convergent in the norm-mean sense from sampling instant $k = 1$ to $k = 2$.

Similarly, for the k-th sampling instant,

$$\left\|E\left\{x_{k+1}\right\}\right\| \leq \left\|F_k - \sigma_k C_k K_k N_k + G_k \varepsilon_w\right\| \left\|E\left\{x_k\right\}\right\| \tag{40}$$

and the system states converge to a constant if and only if

$$\lim_{k\to\infty} \left\|F_k - \sigma_k C_k K_k N_k + G_k \varepsilon_w\right\| = 1 \tag{41}$$

Thus,

$$F_* - \sigma_* C_* K_* N_* + G_* \varepsilon_w = \pm I \tag{42}$$

where $*$ means that k goes to infinity.

Using the norm operation again, the following equation can be obtained.

$$\left\|F_* + G_* \varepsilon_w\right\| \leq \left\|\sigma_*\right\| \left\|C_* K_* N_*\right\| + 1 \tag{43}$$

It has been shown that

$$\left\|\sigma_*\right\| \geq \frac{\left\|F_* + G_* \varepsilon_w\right\| - 1}{\left\|C_* K_* N_*\right\|} \tag{44}$$

N_* can be considered as the upper bounded value for $N_1, N_2, \ldots$, σ_* can be selected to hold this inequality, and the controller can be simplified to a linear quadratic regulator design. In other words, the coefficient of the control law σ_k can be selected, making the controller as the linear quadratic regulator design regarding the pseudo states when k goes to infinity regarding the pseudo states.

Since the state x_k converges to a constant, the system output y_k is also bounded, and y_k converges to 0 when x_k goes to 0 based on the module assumption $h(0) = 0$.

From the perspective of the moment values of the system output, the moment-generating function is bounded if the pseudo states are convergent where

$$M_k(t) < \infty \tag{45}$$

Based on the definition of $M_k(t)$ in Eq. (6),

$$m_i < \infty, \forall i \in \mathbb{Z}_+^1 \tag{46}$$

where m_i denotes the i-th order moment of the system output. It also implies that the moment values of the system output are bounded.

To summarize the convergence analysis, the following theorem is given, and the proof has been given in this section.

Theorem 1: The investigated system output in Eq. (1) and its moments are bounded if the proper σ_* is selected to satisfy the inequality in Eq. (44).

V. A Numerical Example

To validate the presented algorithm, the following system model can be considered.

$$\begin{aligned} x_{k+1} &= x_k \sin\left(x_k\right) + u_k + x_k w_k \\ y_k &= 0.5 x_k \end{aligned}$$

and the cumulant-generating function can be sampled by $t = [-2, -1, 1, 2]$. Then, the dimension of the pseudo state z is 4. The $\phi_k\left(0\right) = 0$. Moreover, w_k is non-Gaussian noise, and its distribution is given as follows:

$$\gamma_W\left(w'\right) = \begin{cases} \left(\int_0^1 w'^2\left(1-w'\right)dw'\right)^{-1} w'^2\left(1-w'\right), \\ w' \in \left[\begin{array}{cc} 0 & 1 \end{array}\right] \\ \qquad 0, otherwise \end{cases}$$

To start the numerical simulation, the KDE sliding window length k_c is prespecified as 20, and the model identification is started since $k = 20$. With the matrices A and B, the control input can be obtained and added into the system at $k = 25$. Before that, the control input is set up as $u_k = \sin\left(k\right)$ to sufficiently stimulate the investigated system. The results are shown in Figs. 3-7.

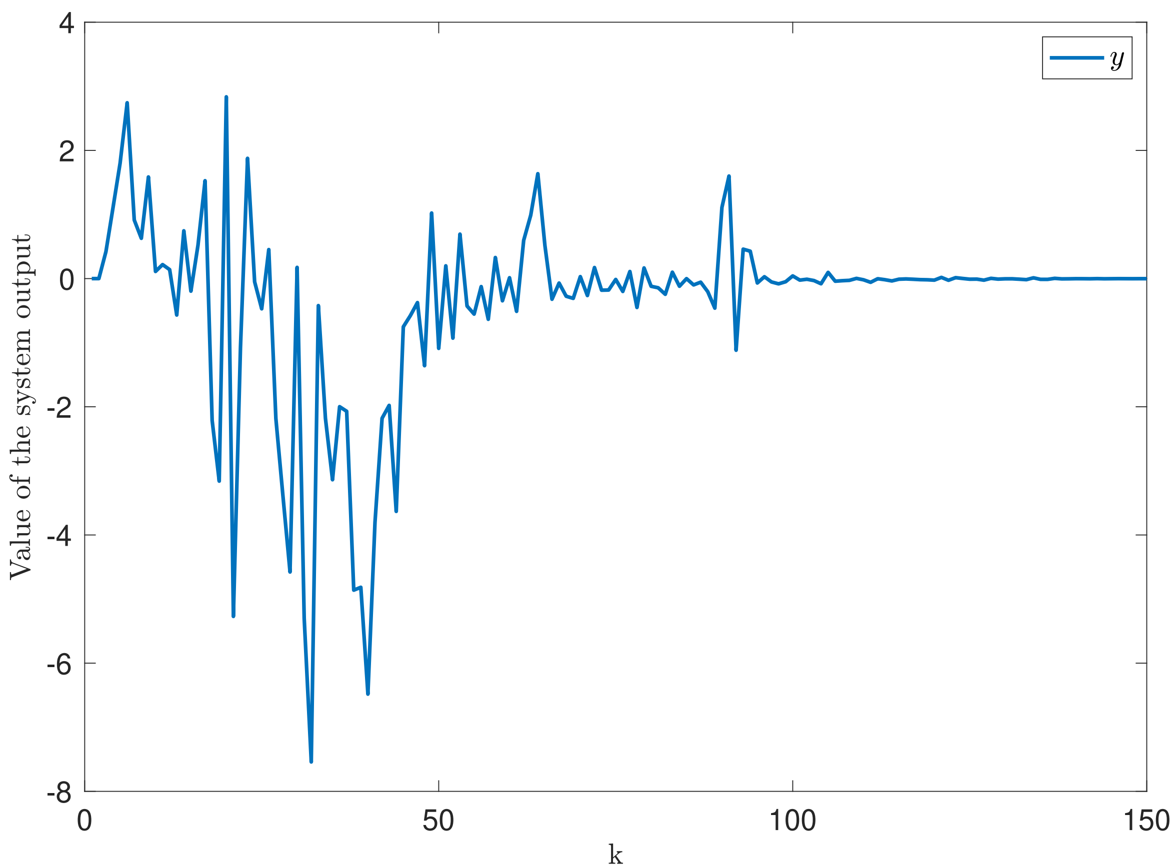

Fig. 3. The system output $\boldsymbol{y_k}$.

In particular, Fig. 3 shows that the system output randomness is attenuating, and Fig. 4 shows the value of the control input. The control input is selected as a harmonic signal before $k = 20$, and the control design affects the system performance once the cumulant-generating function can be estimated with sufficient data. Ultimately, u goes to 0. Fig. 5 indicates the curves for pseudo state vector z_k. All four pseudo states converge to 0. Similarly, the 3D mesh is obtained using the value of z_k, which is shown in Fig. 6. Once all the pseudo states go to 0, the system output y_k randomness should be minimized, and the PDF of the system output will become very sharp, as illustrated in Fig. 7.

For system stability, the coefficient σ_k in the control input can be selected closed to 1 when $\sigma_k = 0.9$, and the coefficient matrices A and B can be identified for each k, e.g. for $k = 150$, resulting in

$$A = \left[\begin{array}{cccc} 14.7 & -19.1 & 9.7 & -10.9 \\ 15.3 & -19.9 & 11.8 & -12.5 \\ -15.6 & 21.1 & -12 & 13.2 \\ -21.3 & 28.9 & -17.6 & 18.9 \end{array}\right]$$

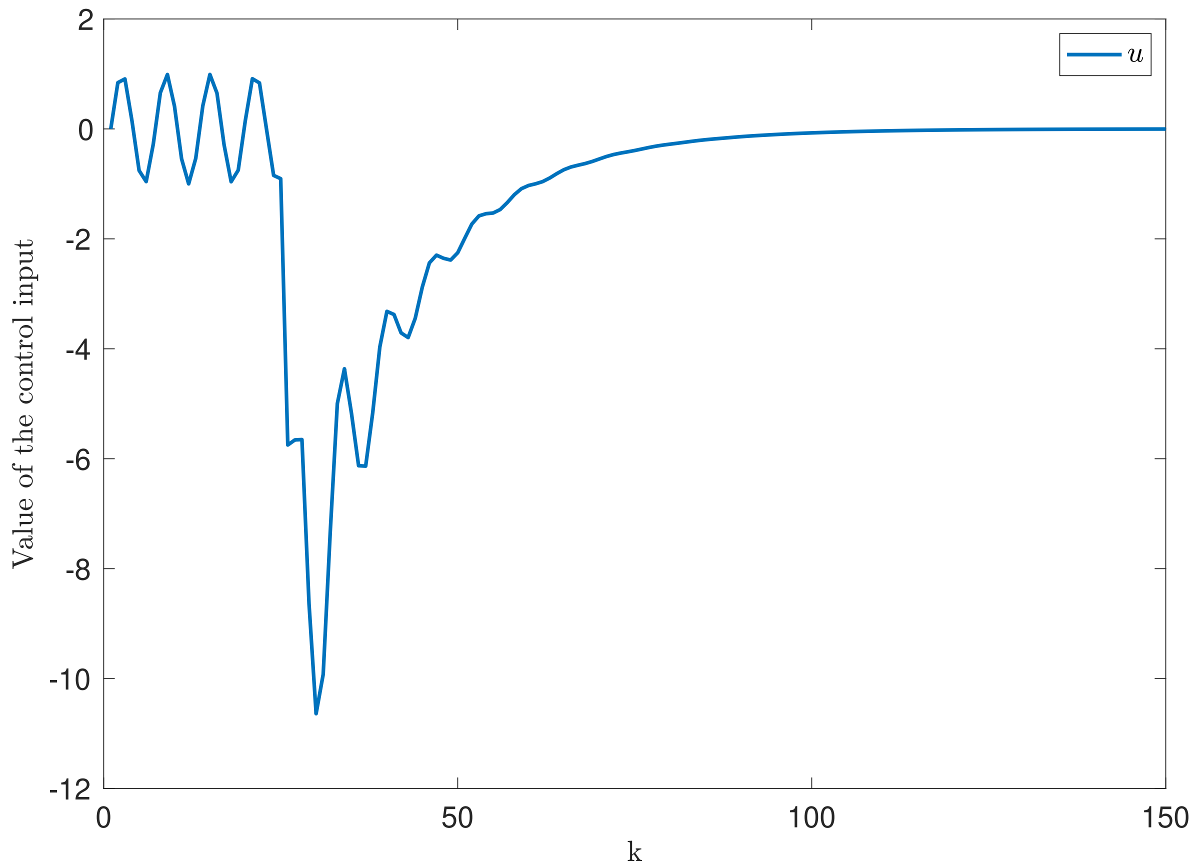

Fig. 4. The control input $\boldsymbol{u_k}$

and

$$B = \begin{bmatrix} -0.29 & -0.32 & 0.38 & 0.53 \end{bmatrix}^T$$

Moreover, the control gain K can be obtained using the described matrices at $k = 150$, resulting in

$$K = \begin{bmatrix} -42.8 & 56.8 & -36.3 & 38.14 \end{bmatrix}$$

and $Q = 100I$ and $R = 1$.

VI. Further Discussion

A. Multiple input, multiple output system extension

In this paper, the investigated system, Eq. (1), is a SISO system. With the single control input, the control law calculation was simplified, and the performance of the presented algorithm was highlighted. However, multi-variable systems widely exist in practice, where the multiple input multiple output (MIMO) models are established for system description. Extending the SISO results to MIMO case is significant both in theory and in engineering applications.

Following the presented algorithm, the same procedure can be directly adopted to MIMO systems. In particular, the pseudo state can also be obtained where the cumulant-generating function is established based on the estimated joint PDFs. The first challenge of MIMO conversion is the multi-dimensional KDE, where the Gaussian kernel in Eq. (3) has to be replaced by multi-dimensional Gaussian kernel. Therefore, the increased estimation error will be introduced into the pseudo states z_k. Meanwhile, the complexity of identifying the matrices A_i and B_i in Eq. (11) increases as the dimension of the system output y_k increases. The second challenge is that the control law will be calculated by solving Eq. (22). In MIMO systems, u_i is a vector, which means that Eq. (22) is a matrix equation.

The presented algorithm can be extended to MIMO using the same procedure as for SISO. However, the complexity of estimation and model identification would increase, which depends on the dimension of the system output and control input. Because the pseudo state z_k rather than the system state x_k is used, the dimension of system state will not affect the results, even if the state space is on a large scale. In practice, the pseudo state z_k needs to be selected in a high dimension to sufficiently represent the cumulant-generating function, where m is normally selected to be greater than the dimension system state x_k, which means that the complexity will also be affected by the large-scale state space.

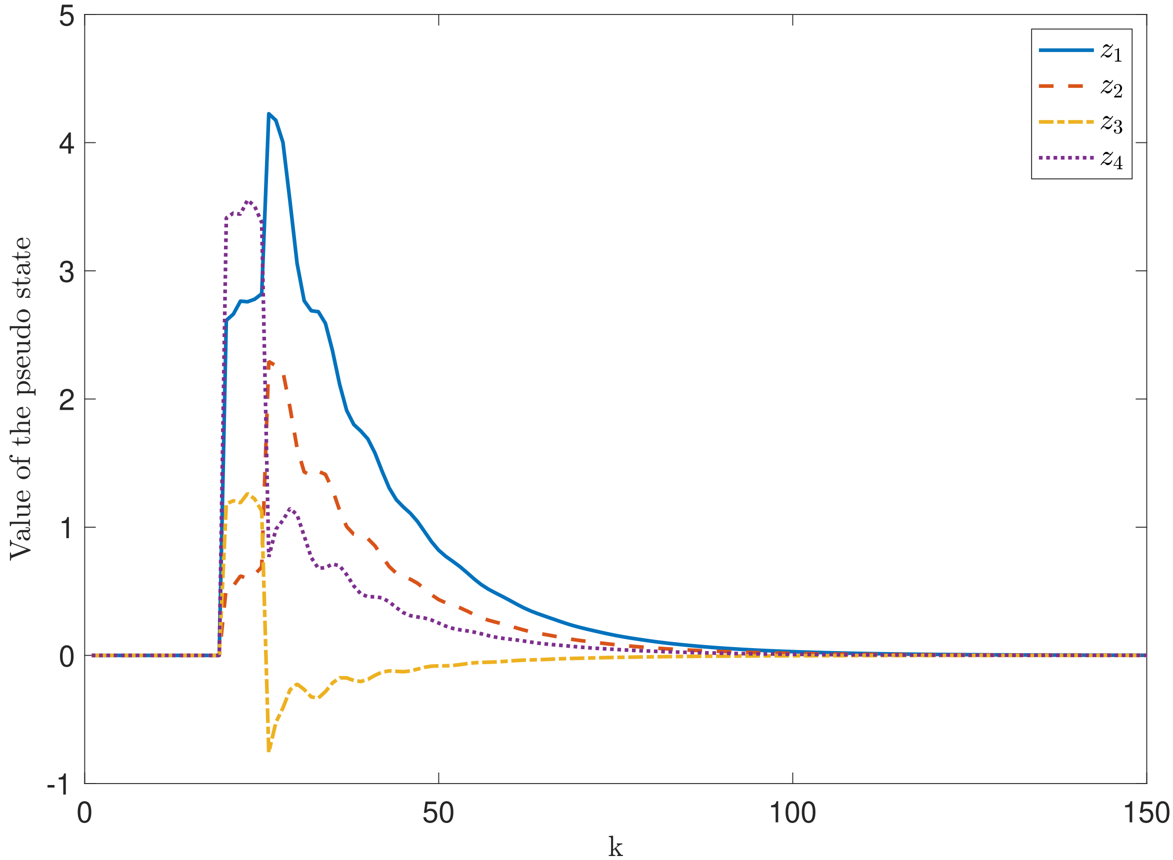

Fig. 5. The stabilization of the pseudo state $\boldsymbol{z_k}$.

B. Measurement noise

To demonstrate the presented approach clearly, the investigated system model, Eq. (1), was given without measurement noise. As an ideal case for optimization, nonlinear function g in the model will become 0 when the system state x_k converges to 0. Thus, the process noise w_k will be eliminated. The randomness of the system output would be 0 based on the ideal model using the presented algorithm. However, the measurement noise will never be ignored because of the sensor properties. In this context, the system output equation can be updated with an additional measurement noise v_k. Then, v_k will affect the collected system output data and change the shape of the estimated PDF, which implies that the additional randomness will be reflected by the pseudo state z_k. Therefore, the randomness cannot be 0, where a lower bound will be achieved for the randomness attenuation. However, the presented algorithm can be adopted directly because the measurement noise is introduced into the design via the KDE.

Most of the measurement noise is considered as Gaussian white noise with zero mean value. The stability analysis also holds because the mean operation will eliminate the measurement noise term.

VII. Conclusions

The minimum entropy control was investigated for a class of non-Gaussian stochastic nonlinear systems following a novel data-driven approach. In particular, the cumulant-generating function was considered as the performance criterion, and the dynamics of the system output were restated by a linear model using the pseudo states from the estimated cumulant-generating function. Based on the identified model, an optimal control design was obtained regarding to the pseudo states with an adjustable coefficient. Following this approach, the minimum entropy optimisation was transformed as either a state assignment or a system stabilization problem. In other words, the minimum entropy problem was uniformed with the stochastic distribution control. The adjustable coefficient can guarantee the convergence of the system output in norm-mean sense, which was demonstrated in stability analysis. To validate the presented algorithm, a numerical example was given, and the simulation results indicate the effectiveness of this framework.

Based on the presented data-driven approach, future work will focus on the combination of artificial intelligence and data mining to enhance the accuracy of the estimated system dynamics. In addition, the practical applications for industrial processes [12] and networked control systems [13] may be considered as a case study for the extension of this paper.

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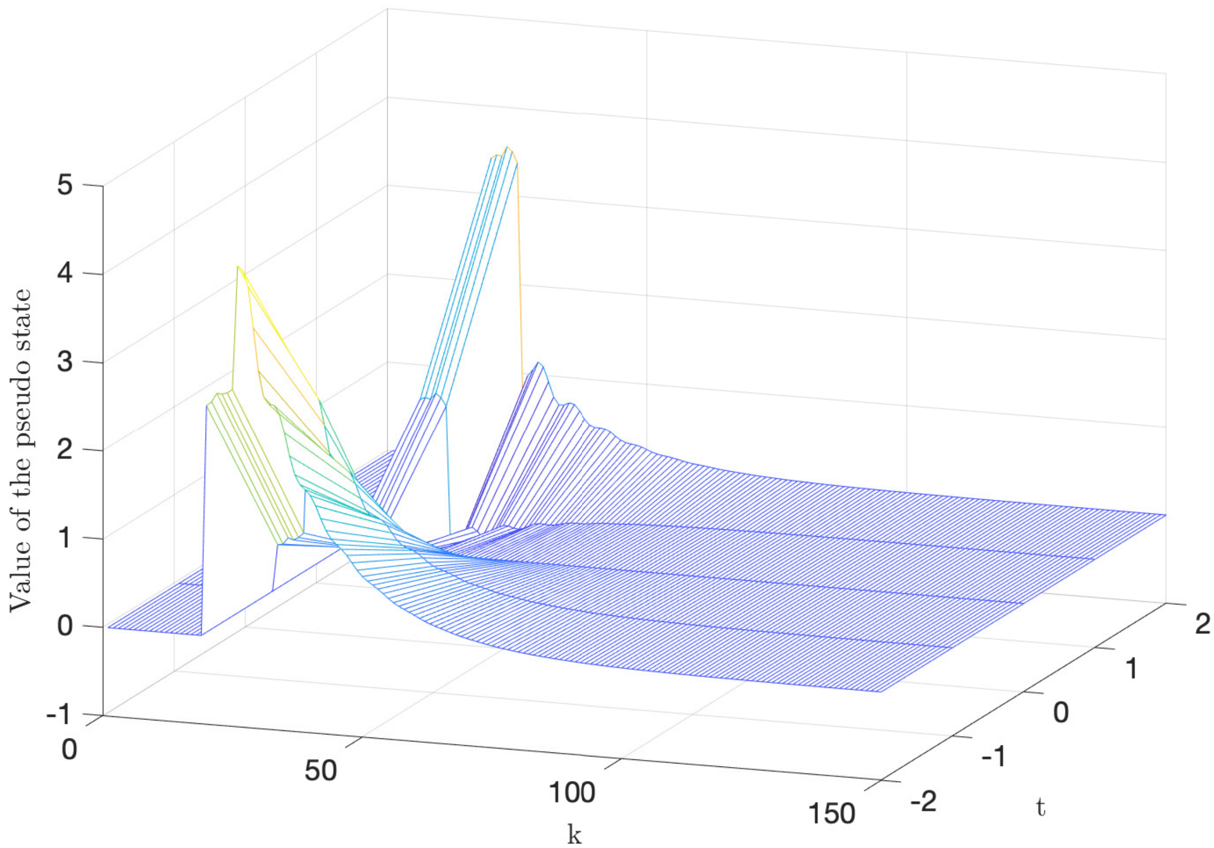

Fig. 6. The 3D mesh of the pseudo state $\boldsymbol{z_k}$, which is based on the cumulant-generating function of the system output $\boldsymbol{y_k}$.

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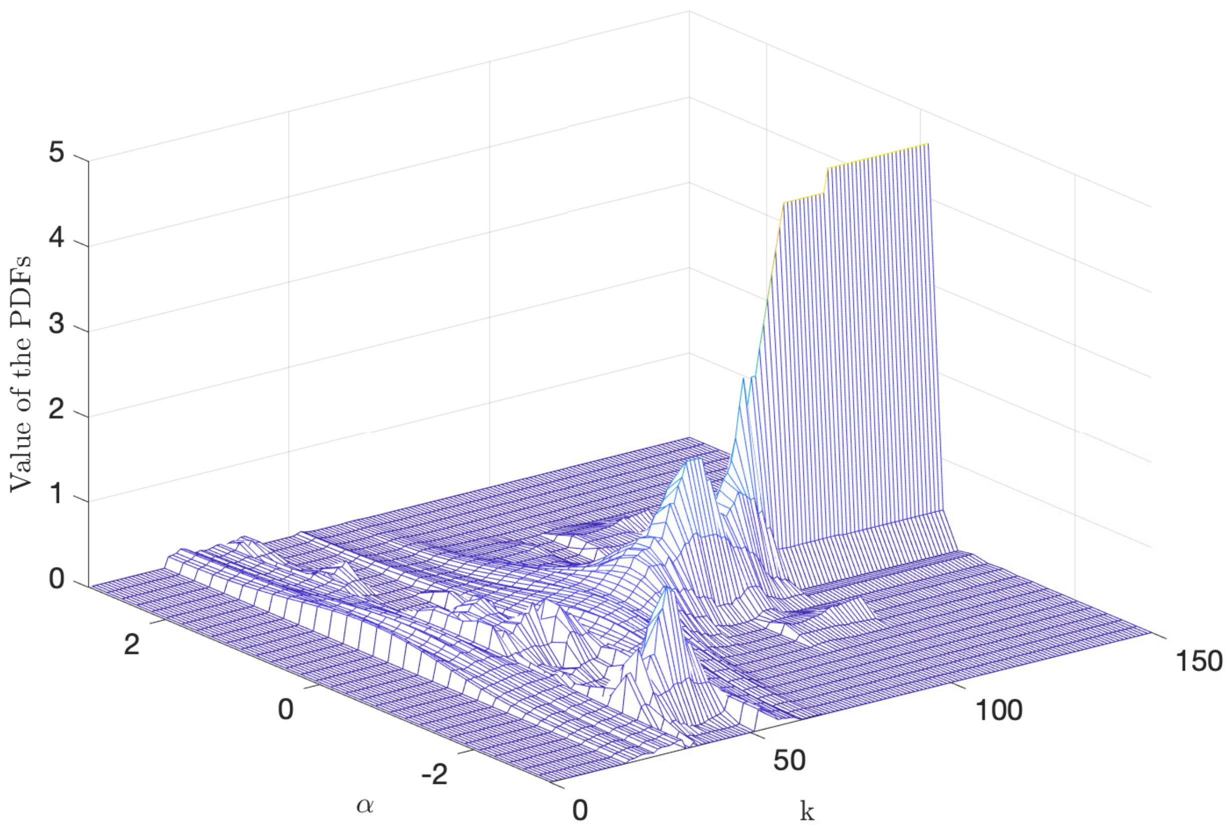

Fig. 7. The PDF of the system output y_k.