Quantum circuit analysis using analytic functions

Mona Rmadan Hamed ABOBAKR

Submitted for the Degree
of Doctor of Philosophy

Faculty of Engineering and Informatics
School of Electrical Engineering and Computer Science

University of Bradford

2019

Abstract

Mona Abobakr.Quantum circuit analysis using analytic functions.

Keywords: Quantum circuits, analytic representations.

In this thesis, classical computation is first introduced. Finite quantum systems are considered with D-dimensional Hilbert space, and position x and momentum p taking values in $\mathbb{Z}(D)$ (the integers modulo D). An analytic representation of finite quantum systems that use Theta function is presented and considered. The first novel part of this thesis is contribution to study reversible classical CNOT gates and their binary inputs and outputs with reversible circuits. Furthermore, a reversible classical Toffoli gates are considered, as well as implementation of a Boolean expression with classical CNOT and Toffoli gates. Reversible circuits with classical CNOT and Toffoli gates are also considered. The second novel part of this thesis the study of quantum computation in terms of CNOT and Toffoli gates. Analytic representations and their zeros are considered, while zeros of the inputs and outputs for quantum CNOT and Toffoli gates are studied. Also, approximate computation of their zeros on the output are calculated. Finally, some quantum circuits are discussed.

Dedicated to my loving husband (Idris Hewedy) who keeps encouraging me and taking care of me. Also I would like to dedicate this thesis to my lovely daughters (Faiza and Layan) who are the light of my life.

Acknowledgements

The greatest thanks to God who gave me the strength and ability to reach this stage of my studies and continue my life.

I would like to thank my first supervisor Professor Apostolos Vourdas for his helpful advice and great support and encouragement he provided to me from the beginning of my research to the end, as well as kind understanding of my circumstances. I am very lucky and happy to have worked with his guidance throughout my research. I would like to thank my second supervisor Dr.C.Lei for his total support and cooperation. I would also like to thank all members of the Quantum Information Research Group in the School of Electrical Engineering and Computer Science at the University of Bradford.

Mum and Dad, I am very thankful for everything you have done and still do for me. There are no words that express how much I love and respect you.

I would like to also thank my soulmate, my lovely husband who has supported, encouraged and taken care of me, and I would like to thank him for his patience.

My father in low, mother in low, daughters, brothers, sisters and nephews are flowers of my life; thank you for your encouragement throughout my studies.

Finally, I would like to thank my Government for their financial support.

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Chapter 1

Introduction

General formalisms of quantum systems in quantum mechanics tend to be in the context of the harmonic oscillator. Here the variables are continuous. The position and momentum in this case take values in $\mathbb{R}$ (real numbers). The harmonic oscillator is a special case of quantum mechanics, where analytic functions have been used in different contexts [1, 2, 3]. The important analytic representation is the Bargmann representation for the harmonic oscillator in the complex plane [3, 4, 5].

The position and momentum take values inside $\mathbb{Z}(D)$ (the integers modulo D), in finite quantum systems with D-dimensional Hilbert space. The analytic representations of finite systems have used Theta function [6, 7, 8, 9]. Since, a Gaussian process works on a discretized circle, the Theta function becomes very important. It has been demonstrated that these analytic functions representing a quantum state has exactly D zeros in a square cell $\mathbf{S}$. Subsequently, when the D zeros are known, then the state of the quantum system is found.

Boolean algebra describes classical gates used in classical computation [10, 11].

Classical gates have subsets of a finite set B as their inputs and outputs. $B = \{x\}$ is the simplest case, and this corresponds to the binary system. The reversible gates are a special case of classical gates [12, 13], where CNOT and Toffoli gates are such examples.

In this thesis, we study reversible circuits with classical and quantum gates. In terms of CNOT and Toffoli gates we show how it is possible to implement an arbitrary Boolean expression. In the simple case of circuits that include only CNOT gates with binary inputs and outputs, they can be described with matrices. Tis is done by using operations of multiplication of matrices and direct sums of matrices. Other circuits that include both CNOT and Toffoli gates with general inputs and outputs (which may not be binary), are also studied. The general methodology is explained throughout using many examples.

This work can be generalized into a quantum context, because these circuits are reversible. Moreover, quantum CNOT gates and quantum Toffoli gates are considered, as are more general circuits that include quantum CNOT and quantum Toffoli gates.

There are several formulas of quantum mechanics. One is in terms of analytic functions and their zeros [4, 6, 7, 3, 14, 15, 16]. This method has been used extensively in different branches of quantum physics. Here we use it in the context of quantum gates.

1.1 Structure of the thesis

This thesis contains seven chapters. This first chapter provides a brief introduction to the topic, and delivers summary of the thesis.

The second chapter provides a brief introduction to classical computation. Here, Boolean algebra and Boolean rings are introduced. Also, reversible gates are considered.

In the third chapter, we review infinite quantum systems (including the quantum harmonic oscillator). We also discuss Hermite polynomials and their orthogonality and properties. Next, the topic of a commutator operator and its properties are introduced. Creation and annihilation operators and their properties and displacement operators and their properties are considered. Finally, coherent states and their properties and operators in quantum systems are given, while the Bargmann analytic representation is introduced.

The fourth chapter presents finite quantum systems. Aspects considered in this chapter include Jacobi theta functions, Fourier transform, and position and momentum states and displacement operators. Analytic representation and their zeros are also discussed and presented, as well as displacement operators in analytic representations.

In chapter five, we contribute to study circuits with reversible classical gates. We also study reversible classical CNOT gates and their binary inputs and outputs with reversible circuits. A reversible classical Toffoli gates are studied, as well as implementation of a Boolean expression with classical CNOT and Toffoli gates. Reversible circuits with classical CNOT and Toffoli gates are also considered.

The sixth chapter contribute to study quantum CNOT and Toffoli gates. Analytic representations and their zeros are considered, while zeros of the inputs and outputs for quantum CNOT and Toffoli gates are studied. Approximate computation of their zeros on the output are calculated. A number of quantum circuits are discussed to conclude the chapter.

In the seventh chapter, concludes the thesis with a discussion of the results.

Chapter 2

Introduction to reversible Classical Computation

2.1 Introduction

This chapter opens with a review of Boolean algebra, which is very important because it demonstrates the logic of gates and circuits. Moreover, we provide some important rules of Boolean algebra in the first section of this chapter. In the second section, we introduce the concept of a Boolean ring and provide the general definition for it.

In the final section, we consider reversible gates with their equations, tables and circuits. For example, CNOT gates and Toffoli gates which are 2×2 and 3×3 reversible gates, respectively. These two gates will be considered later in classical and quantum computations. This chapter concludes with details of a Sayem gate which is a 4×4 reversible gate.

2.2 Boolean algebra

Boolean algebra is important because it describes the logical gates and circuits [17].

A power set has three operations, two are binary operations. These operations are intersection($\wedge$)(logical AND) and union ($\vee$)(logical OR). The third operation is complementation($\neg$)(logical NOT). This structure is called Boolean algebra [18].

The power set 2^B, where B is a general finite set. $(B = I)$ is the greatest element and $(\phi = 0)$ is the Least element. In the binary system, $(B = 1)$ and $2^B = \{\phi, B\}$, that will be used in this thesis. Here, provided some important rules of Boolean algebra in terms of intersection ($\wedge$), union ($\vee$) and complementation ($\neg$) operations. Let $M, N, P \in B$ [18]:

1.

$$\neg\phi = B, \qquad \neg B = \phi. \tag{2.1}$$

2.

$$M \wedge \phi = \phi, \qquad M \vee B = B. \tag{2.2}$$

3.

$$M \wedge B = M, \qquad M \vee \phi = M. \tag{2.3}$$

4.

$$M \wedge \neg M = \phi, \qquad M \vee \neg M = B. \tag{2.4}$$

5.

$$\neg(\neg M) = M. \tag{2.5}$$

6. Idempotent condition

$$M \vee M = M, \qquad M \wedge M = M. \tag{2.6}$$

7. De Morgan's Theorem.

$$\neg(M \vee N) = \neg M \wedge \neg N, \qquad \neg(M \wedge N) = \neg M \vee \neg N. \tag{2.7}$$

8. commutativity

$$M \vee N = N \vee M, \qquad M \wedge N = N \wedge M. \tag{2.8}$$

9. Associativity.

$$M \wedge (N \wedge P) = (M \wedge N) \wedge P \tag{2.9}$$

$$M \vee (N \vee P) = (M \vee N) \vee P. \tag{2.10}$$

10. Distributivity.

$$M \wedge (N \vee P) = (M \wedge N) \vee (M \wedge P) \tag{2.11}$$

$$M \vee (N \wedge P) = (M \vee N) \wedge (M \vee P). \tag{2.12}$$

Boolean algebra is used to simplify logic circuits. For example, the Boolean algebraic expression (final output) for the circuit shown in Figure 2.1, is $M \vee N \wedge [\neg(M \wedge N)]$. This circuit contains the AND, NOR and OR gates. The boolean expression for the AND gate is $M \vee N$. The NOR gate expression is $\neg(M \wedge N)$ and the OR gate expression is $M \wedge N$. If we use De Morgan's Theory, then the final output will change to $M \vee N \wedge (\neg M \vee \neg N)$. Here we used a binary system.

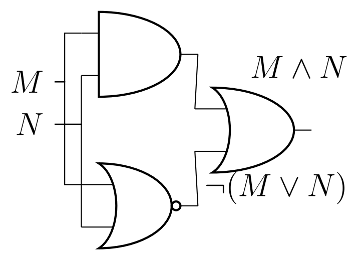

Figure 2.1: The circuit containing the AND, NOR and OR gates.

The truth table for this example is:

(M, N)	$M \vee N$	$\neg(M \wedge N)$	$M \vee N \wedge \neg(M \wedge N)$
$(0, 0)$	0	1	1
$(0, 1)$	0	0	0
$(1, 0)$	0	0	0
$(1, 1)$	1	0	1

Table 2.1: The truth table for the circuit containing the AND, NOR and OR gates.

2.3 Boolean ring

A Boolean ring is a ring with a unit that must satisfy an idempotent condition [18]. Let R be a set with two binary operations $(\oplus, .)$. It can be written as $(\mathsf{R}, \oplus, .)$ which is known as algebraic structure. In general, these operations are known as addition and multiplication correspondingly. The union operation (logical OR)of Boolean algebra can be replaced by the addition operation(logical XOR)of the Boolean ring [19, 20]. This operation is called the symmetric difference of sets which define as [21, 10]:

$$N \oplus M = (N \setminus M) \vee (M \setminus N) = \left[M \wedge (\neg N)\right] \vee \left[N \wedge (\neg M)\right]. \qquad (2.13)$$

Boolean ring multiplication is the intersection operation (logical AND) as in Boolean algebra which is defined as:

$$N.M = M \wedge N. \qquad (2.14)$$

The Boolean ring is the power set 2^B with the addition and the multiplication. We have some important relations which are:

$$M \oplus M = 0, \quad M \oplus 0 = M, \;\; M \oplus I = \neg M, \tag{2.15}$$
$$M.M = M, \;\; M.0 = 0, M.I = M.$$

Also, we can write the union in terms of the multiplication and addition as follows:

$$M \vee N = M \oplus N \oplus (M.N) \tag{2.16}$$

The addition and multiplication operations should satisfy the following conditions, and let $M, N, P \in \mathsf{R}$ [18, 19]:

i. Additive associativity.

$$(M \oplus N) \oplus P = M \oplus (N \oplus P), \quad \forall M, N, P \in \mathsf{R}. \tag{2.17}$$

ii. Addition is commutative.

$$M \oplus N = N \oplus M, \quad \forall M, N \in \mathsf{R}. \tag{2.18}$$

iii. Additive identity. There is an existing element $0 \in \mathsf{R}$ such that $\forall M \in \mathsf{R}$.

$$M \oplus 0 = 0 \oplus M = M. \tag{2.19}$$

iv. Additive inverses. $\forall M \in \mathsf{R}$ there is an existing element $-M$.

$$M \oplus -M = -M \oplus M = 0. \tag{2.20}$$

v. Distributivity.

$$M.(N \oplus P) = (M.N) \oplus (M.P) \tag{2.21}$$

$$(N \oplus P).M = (N.M) \oplus (P.M),\ \ \forall M, N, P \in \mathsf{R}. \tag{2.22}$$

vi. Multiplicative associativity.

$$(M.N).P = M.(N.P),\ \ \forall M, N, P \in \mathsf{R}. \tag{2.23}$$

vii. Multiplicative identity. There is an existing element $1 \in \mathsf{R}$ such that $\forall M \in \mathsf{R}$.

$$1.M = M.1 = M. \tag{2.24}$$

viii. Idempotent condition. $\forall M \in \mathsf{R}$ such that:

$$M.M = M. \tag{2.25}$$

2.4 Reversible gate

The reversible function is very important [22]. For example, the Boolean function is reversible if it satisfies the following conditions:

- The number of inputs is equal to the number of outputs.
- And any output pattern has a completely unique input pattern [22, 23, 24].

There are many examples of reversible logical gates. The simplest reversible logical gate is a NOT gate [22, 23]. Also there are other reversible logical gates, such as the CNOT gate and Toffoli gate [22, 23]. We can provide a number of examples of reversible gates with circuits and truth tables with binary systems, as follows:

2.4.1 NOT gate

Figure 2.2 shows a NOT gate as a block circuit. It is the simplest reversible gate. We can describe the input and output of the NOT gate as follows [22, 23].

$$A = (M), \quad O = [\neg M] \tag{2.26}$$

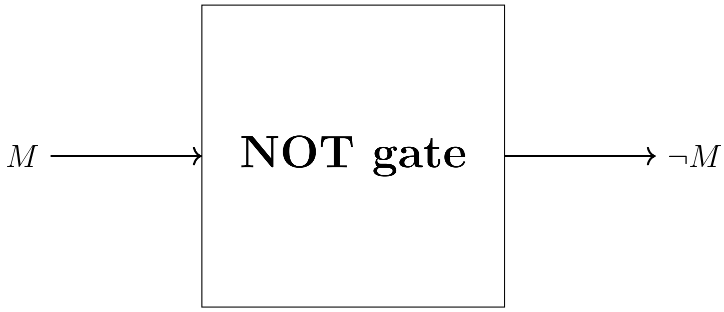

Figure 2.2: The NOT gate.

The NOT gate truth table is given in Table 2.2 [25].

Input(M)	output ($\neg M$)
0	1
1	0

Table 2.2: The NOT gate truth table.

2.4.2 CNOT gate

The CNOT gate, as shown in Figure 2.3, is a 2×2 reversible gate. Also called Feynman gate. We can describe the input and output of a CNOT gate as follows[22]:

$$A = (M, N), \quad O = \big[M, (M \oplus N)\big]. \tag{2.27}$$

Where A, O are the input and output vectors of a CNOT gate, respectively. And $\oplus$ is known as the symmetric difference [22, 23, 25].

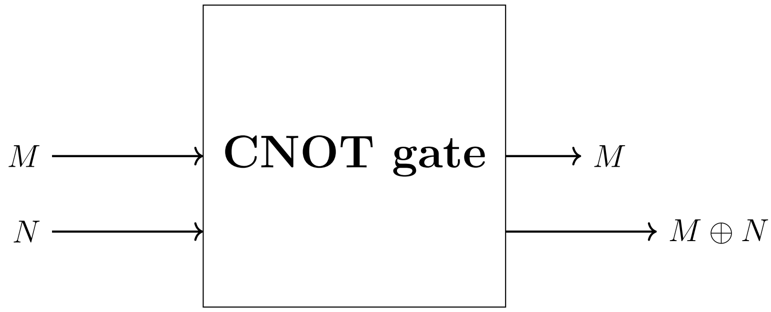

Figure 2.3: The CNOT gate.

The CNOT gate truth table is given in Table 2.3 [25].

Input(M, N)	output $\big[M, (M \oplus N)\big]$
$(0, 0)$	$(0, 0)$
$(0, 1)$	$(0, 1)$
$(1, 0)$	$(1, 1)$
$(1, 1)$	$(1, 0)$

Table 2.3: The CNOT gate truth table.

2.4.3 Double Feynman gate

The Double Feynman gate, as shown in Figure 2.4, is a 3×3 reversible gate. We can describe the input and output vectors of the Double Feynman gate, respectively, as follows [22, 23, 25]:

$$A = (M, N, S), \quad O = \big[M, (M \oplus N), (M \oplus S)\big]. \tag{2.28}$$

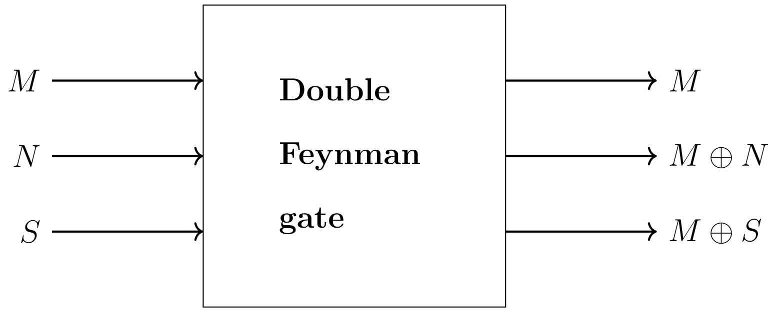

Figure 2.4: The Double Feynman gate.

The Double Feynman gate truth table is given in Table 2.4 [25].

Input(M, N, S)	output $\left[M, (M \oplus N), (M \oplus S)\right]$
$(0, 0, 0)$	$(0, 0, 0)$
$(0, 0, 1)$	$(0, 0, 1)$
$(0, 1, 0)$	$(0, 1, 0)$
$(0, 1, 1)$	$(0, 1, 1)$
$(1, 0, 0)$	$(1, 1, 1)$
$(1, 0, 1)$	$(1, 1, 0)$
$(1, 1, 0)$	$(1, 0, 1)$
$(1, 1, 1)$	$(1, 0, 0)$

Table 2.4: The Double Feynman gate truth table.

2.4.4 Toffoli gate

The Toffoli gate, as shown in Figure 2.5, is a 3×3 reversible gate. We can describe the input and output vectors of the Toffoli gate, respectively, as follows [22, 23, 25]:

$$A = (M, N, S), \quad O = \big[M, N, (M.N \oplus S)\big]. \tag{2.29}$$

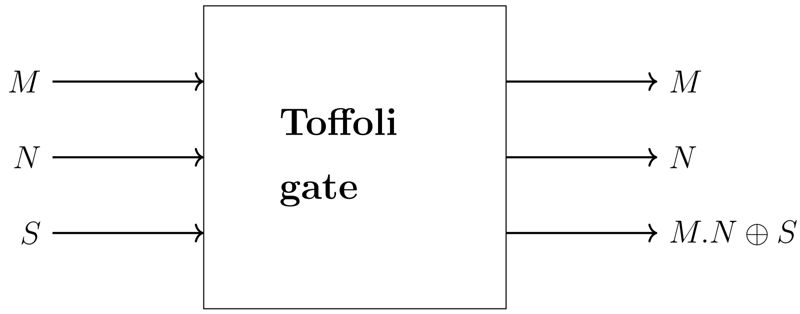

Figure 2.5: The Toffoli gate.

The Toffoli gate truth table is given in Table 2.5 [25].

Input(M, N, S)	output $\big[M, N, (M.N \oplus S)\big]$
$(0, 0, 0)$	$(0, 0, 0)$
$(0, 0, 1)$	$(0, 0, 1)$
$(0, 1, 0)$	$(0, 1, 0)$
$(0, 1, 1)$	$(0, 1, 1)$
$(1, 0, 0)$	$(1, 0, 0)$
$(1, 0, 1)$	$(1, 0, 1)$
$(1, 1, 0)$	$(1, 1, 1)$
$(1, 1, 1)$	$(1, 1, 0)$

Table 2.5: The Toffoli gate truth table.

2.4.5 Fredkin gate

The Fredkin gate, as shown in Figure 2.6, is a 3×3 reversible gate. We can describe the input and output vectors of a Fredkin gate, respectively, as follows [22, 23, 25]:

$$A = (M, N, S), \quad O = \left[M, (M.\neg N \oplus M.S), (M.\neg S \oplus M.N)\right]. \tag{2.30}$$

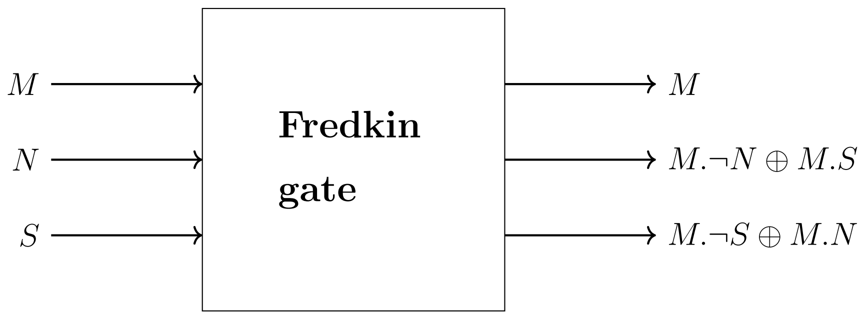

Figure 2.6: The Fredkin gate.

The Fredkin gate truth table is given in Table 2.6 [25].

Input (M, N, S)	output $\left[M, (M.\neg N \oplus M.S), (M.\neg S \oplus M.N)\right]$
$(0, 0, 0)$	$(0, 0, 0)$
$(0, 0, 1)$	$(0, 0, 1)$
$(0, 1, 0)$	$(0, 1, 0)$
$(0, 1, 1)$	$(0, 1, 1)$
$(1, 0, 0)$	$(1, 0, 0)$
$(1, 0, 1)$	$(1, 1, 0)$
$(1, 1, 0)$	$(1, 0, 1)$
$(1, 1, 1)$	$(1, 1, 1)$

Table 2.6: The Fredkin gate truth table.

2.4.6 Peres gate

The Peres gate, as shown in Figure 2.7, is a 3×3 reversible gate. We can describe the input and output vectors of the Peres gate, respectively, as follows [22, 23, 25]:

$$A = (M, N, S), \quad O = \left[M, (M \oplus N), (M.N \oplus S)\right]. \tag{2.31}$$

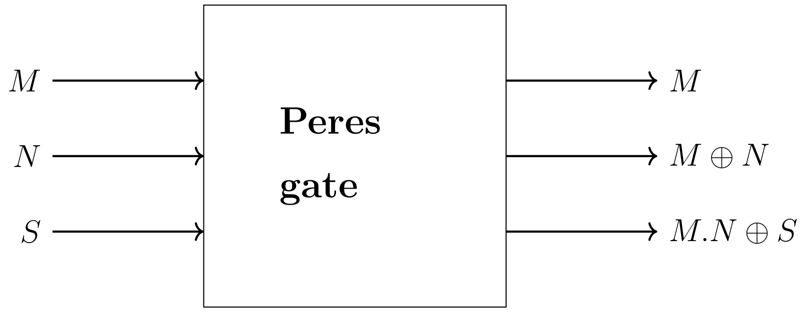

Figure 2.7: The Peres gate.

The Peres gate truth table is given in Table 2.7 [25].

Input(M, N, S)	output $\left[M, (M \oplus N), (M.N \oplus S)\right]$
$(0, 0, 0, 0)$	$(0, 0, 0, 0)$
$(0, 0, 0, 1)$	$(0, 0, 1, 1)$
$(0, 0, 1, 0)$	$(0, 0, 0, 1)$
$(0, 0, 1, 1)$	$(0, 0, 1, 0)$
$(0, 1, 0, 0)$	$(0, 1, 1, 0)$
$(0, 1, 0, 1)$	$(0, 1, 0, 1)$
$(0, 1, 1, 0)$	$(0, 1, 1, 1)$
$(0, 1, 1, 1)$	$(0, 1, 0, 0)$
$(1, 0, 0, 0)$	$(1, 0, 0, 0)$
$(1, 0, 0, 1)$	$(1, 0, 1, 1)$
$(1, 0, 1, 0)$	$(1, 1, 1, 0)$
$(1, 0, 1, 1)$	$(1, 1, 0, 1)$
$(1, 1, 0, 0)$	$(1, 0, 0, 1)$
$(1, 1, 0, 1)$	$(1, 0, 1, 0)$
$(1, 1, 1, 0)$	$(1, 1, 1, 1)$
$(1, 1, 1, 1)$	$(1, 1, 0, 0)$

Table 2.7: The Peres gate truth table.

2.4.7 Sayem gate

The Sayem gate, as shown in Figure 2.8, is a 4×4 reversible gate. We can describe the input and output vectors of the Sayem gate, respectively, as follows [25]:

$$A = (M, N, S, R), \tag{2.32}$$

$$O = \big[M, (M.\neg N \oplus M.S), (M.\neg N \oplus M.S \oplus R), (M.N \oplus M.\neg S \oplus R)\big].$$

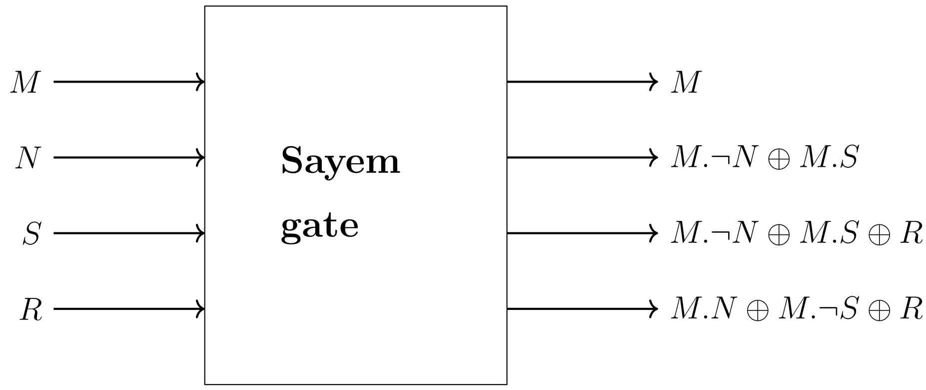

Figure 2.8: The Sayem gate.

The Sayem gate truth table is shown in Table 2.8 [26].

Input(A)	output (O)
$(0,0,0)$	$(0,0,0)$
$(0,0,1)$	$(0,0,1)$
$(0,1,0)$	$(0,1,0)$
$(0,1,1)$	$(0,1,1)$
$(1,0,0)$	$(1,1,0)$
$(1,0,1)$	$(1,1,1)$
$(1,1,0)$	$(1,0,1)$
$(1,1,1)$	$(1,0,0)$

Table 2.8: The Sayem gate truth table.

2.5 Summary

In sections two and three of this chapter, we reviewed Boolean algebra and introduced the Boolean ring. Moreover, reversible gates were reviewed in the fourth section. For instance, a NOT gate was considered, as was a CNOT gate, which is a 2×2 reversible gate. The Toffoli gate, which is a 3×3 reversible gate was also presented.

This chapter was concluded with truth table for the Sayem gate, which is 4×4 reversible gate.

Chapter 3

Analytic representations for infinite quantum systems

3.1 Introduction

A quantum harmonic oscillator is a special type of quantum system in one dimension. Therefore, a quantum harmonic oscillator considers the commutator of two operators as:

$$[A, B] = AB - BA \neq 0. \quad (3.1)$$

A quantum harmonic oscillator provides a very good introduction to a creation operator ($a^\dagger$) and annihilation (a) operator, and also introduces a position operator ($\hat{x}$) and momentum operator ($\hat{p}$) which satisfies important relations:

$$[\hat{x}, \hat{p}] = i \tag{3.2}$$

We take ($\hbar = m = w = 1$) to simplify our review. We study the displacement operator and their properties. In addition, we provide an introduction to Hermite polynomials, as well as we study some important operators of quantum systems. For example, Hermitian and displaced parity operators.

We conclude this chapter using a very important representation, namely the Bargmann analytic representation.

3.2 Hermite Polynomials

Here, we will explain physicists Hermite polynomials, which are defined as [27, 28, 29]

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n} \exp\left(-x^2\right), \tag{3.3}$$

Also called Rodrigues Formula the first five physicists' Hermite polynomials are:

$$H_0(x) = 1, \tag{3.4}$$

$$H_1(x) = 2x, \tag{3.5}$$

$$H_2(x) = 4x^2 - 2, \tag{3.6}$$

$$H_3(x) = 8x^3 - 12x, \tag{3.7}$$

$$H_4(x) = 16x^4 - 48x^2 + 12. \tag{3.8}$$

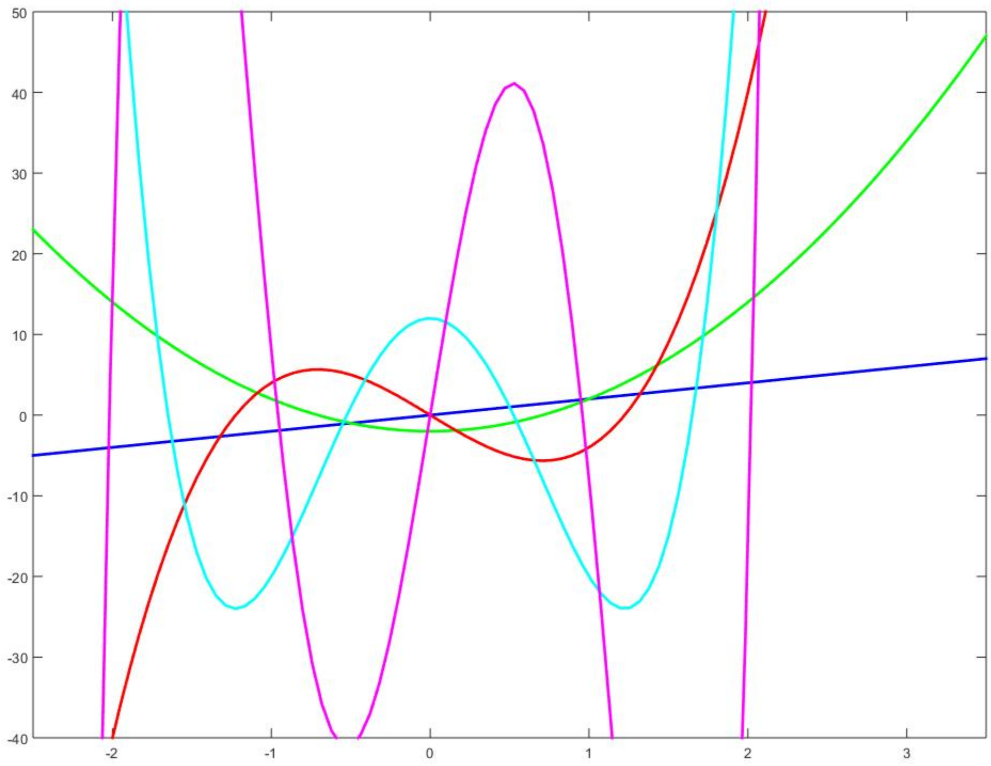

Figure 3.1: Hermite Polynomials

Kronecker function We can define the Kronecker function as:

$$\delta_{m,n} = \begin{cases} 0, & \text{for } n \neq m, \\ 1, & \text{for } n = m \end{cases}$$

3.2.1 Orthogonality of Hermite polynomials

The Hermite function can be defined as [27, 28, 29] :

$$u_n(x) = \frac{1}{\pi^{\frac{1}{4}}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{x^2}{2}\right) H_n(x). \tag{3.9}$$

H_n is an nth-degree polynomial for $n = 0, 1, 2, 3, ...$

These polynomials are orthogonal with respect to the weight function $w(x) = \exp(-x^2)$ then we have the general form of orthogonality depending on the Kronecker function

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x) = n! 2^n \sqrt{2\pi} \delta_{m,n}. \tag{3.10}$$

On the other hand, we have:

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x) = 0, \quad n \neq m \tag{3.11}$$

Also we have

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n^2(x) = 2^n n! \sqrt{\pi}, \quad n = m \tag{3.12}$$

3.2.2 Hermite polynomials properties

We have some important properties of Hermite polynomials, as follows [28]:

1.

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2). \tag{3.13}$$

2. Generating function.

$$w(x,t) = \exp(2tx - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \tag{3.14}$$

3.

$$H_{2n+1}(0) = 0. \tag{3.15}$$

4.

$$H'_{2n}(0) = 0. \tag{3.16}$$

5.

$$H'_{2n+1}(0) = (-1)^n \frac{(2n+2)!}{(n+1)!}. \tag{3.17}$$

6. Recurrence Relation.

$$\begin{aligned} &H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \qquad n = 1, 2, ... \\ \text{or } &H_{n+1}(x) - 2xH_n(x) + H'_n(x) = 0, \qquad n = 1, 2, ... \end{aligned} \tag{3.18}$$

$$H'_n(x) = 2nH_n - 1(x), \;\; n = 1, 2, ... \tag{3.19}$$

$$H'_{n+1}(x) = 2(2n+1)H_n(x), \;\; n = 1, 2, ... \tag{3.20}$$

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x), n = 0, 1, 2, ... \tag{3.21}$$

7. Hermite differential equation.

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0, \;\; n = 1, 2, ... \tag{3.22}$$

Some important relations between Hermite function, and creation and annihilation operators, which will be considered later.

$$au_n(x) = \sqrt{n}u_{n-1}(x), \quad a = \frac{x + \partial_x}{\sqrt{2}}. \tag{3.23}$$

3.3 Harmonic oscillator: Hamiltonian

In this section we will consider the special case of a quantum system on a real line, which called a one-dimensional linear harmonic oscillator. The Hamiltonian of the harmonic oscillator with position and momentum operators is defined as [30, 31]:

$$\hat{\mathrm{H}} = \frac{1}{2}\hat{x^2} + \frac{1}{2}\hat{p^2}, \tag{3.24}$$

3.3.1 Commutator operator

Before we explain the commutator operator, we should define the eigenfunction as follows [32]:

Definition 3.3.1. An operator performs on a function to give a new function,but this new function should be multiplied by a constant such that:
$Ag = \lambda g$.
Where A is a linear operator and g is an eigenfunction and λ is an eigenvalue

and a whole equation is called an eigenstate [32].

Now we can define the commutator of two operators A and B as follows:

$$[A, B] = AB - BA \neq 0 \tag{3.25}$$

3.3.2 Properties of commutator operator

Suppose that A, B, C are operators and a, b, c are numbers [32, 33].

1.

$$[A, A] = AA - AA = 0. \tag{3.26}$$

2.

$$[A, B] \quad = AB - BA = -(BA - AB) = -[B, A]. \tag{3.27}$$

3.

$$[A, BC] = B[A, C] + [A, B]C. \tag{3.28}$$

We can prove this as:

$$\begin{aligned} [A, BC] &= ABC - BCA = ABC - BCA + BAC - BAC \\ &= BAC - BCA + ABC - BAC \\ &= B[A, C] + [A, B]C. \end{aligned} \tag{3.29}$$

Or

$$B[A,C]+[A,B]C = BAC-BCA+ABC-BAC$$
$$= ABC-BCA=[A,BC]. \tag{3.30}$$

And in similar way, we can prove this:

$$[AB,C]=A[B,C]+[A,C]B. \tag{3.31}$$

4.

$$[A,bB+cC]=b[A,B]+c[A,C]. \tag{3.32}$$

We can prove this property as follows:

$$\begin{aligned}[A,bB+cC] &= A(bB+cC)-(bB+cC)A\\ &= AbB+AcC-bBA-cCA\\ &= AbB-bBA+AcC-cCA\\ &= b(AB-BA)+c(AC-CA)\\ &= b[A,B]+c[A,C]. \end{aligned}\tag{3.33}$$

Also, in the same way, we can prove this:

$$[aA+bB,C]=a[A,C]+b[B,C]. \tag{3.34}$$

5. The following property is called a Jacobi Identity [33]:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \tag{3.35}$$

Proof.

$$\begin{aligned}
[A,[B,C]] &= A[B,C] - [B,C]A \\
&= ABC - ACB - BCA + CBA \\
&= ABC - ACB - BCA + CBA - BAC + BAC - CAB + CAB \\
&= -BCA + BAC + CAB - ACB - CAB + CBA + ABC - BAC \\
&= -(BCA - BAC - CAB + ACB) - (CAB - CBA - ABC + BAC) \\
&= -[B,[C,A]] - [C,[A,B]] \\
&\Rightarrow [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \qquad (3.36)
\end{aligned}$$

The anti-commutator of two operators A and B are defined as:

$$\{A,B\} = AB + BA$$

3.3.3 Creation and annihilation operators and their properties

Let x, p position and momentum and $\hat{x}, \hat{p}$ be position and momentum operators, respectively, as mentioned before. And $a^\dagger, a$ creation and annihilation operators, respectively, and can define them as follows [30, 34, 35, 3, 31]:

$$a^{\dagger} = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}, \qquad a = \frac{\hat{x} + i\hat{p}}{\sqrt{2}}. \tag{3.37}$$

$$\begin{aligned} \hat{x} &= \frac{1}{\sqrt{2}}(a + a^{\dagger}), & \hat{p}, &= \frac{-i}{\sqrt{2}}(a - a^{\dagger}). \\ \hat{p} &= -i\partial_x, & \hat{x} &= i\partial_p. \end{aligned} \tag{3.38}$$

We have the following relation:

$$[x, \partial_x] = -1, \tag{3.39}$$

and we can prove it easily as follows:

Let $f(x)$ be a function of x, then:

$$\begin{aligned} [x, \partial_x] f(x) &= (x\, \partial_x - \partial_x\, x) f(x) \\ &= x\partial_x\, (f(x)) - \partial_x\, (x\, f(x)) \\ &= x\, g(x) - x\, g(x) - f(x) \\ &= -1 f(x) \Rightarrow [x, \partial_x] = -1. \end{aligned} \tag{3.40}$$

Where $\partial_x\, (f(x)) = g(x)$, and the same way we can prove that $[\partial_x, x] = 1$.

Properties of creation and annihilation operators.

We have some important properties of creation and annihilation operators, as follows [30, 34, 35, 3, 31]:

1.

$$[a, a] = 0, \qquad\qquad [a^\dagger, a^\dagger] = 0. \tag{3.41}$$

2.

$$[a, a^\dagger] = 1 \qquad\qquad [a^\dagger, a] = -1. \tag{3.42}$$

Before completing the properties of creation and annihilation we want to prove the next equation:

$$[\hat{x}, \hat{p}] = i. \tag{3.43}$$

Proof.

$$\begin{aligned}
[\hat{x}, \hat{p}] &= \frac{-i}{2}(a + a^\dagger)(a - a^\dagger) + \frac{i}{2}(a - a^\dagger)(a + a^\dagger) \\
&= \frac{-i}{2}(a.a - aa^\dagger + a^\dagger a - a^\dagger.a^\dagger - a.a - aa^\dagger + a^\dagger a + a^\dagger.a^\dagger) \\
&= \frac{-i}{2}(a.a - a.a + a^\dagger.a^\dagger - a^\dagger.a^\dagger + 2a^\dagger a - 2aa^\dagger) \\
&= \frac{-i}{2}[a, a] + \frac{-i}{2}[a^\dagger, a^\dagger] + \frac{-i}{2}2[a^\dagger, a],
\end{aligned} \tag{3.44}$$

from the previous properties we get:

$$[\hat{x}, \hat{p}] = i. \tag{3.45}$$

□

In a similar way, we proved this:

$$[\hat{p}, \hat{x}] = -i. \tag{3.46}$$

And from Eq (3.37) we can prove the following equation:

$$[a, a^\dagger] = 1. \tag{3.47}$$

We can see easily:

$$\begin{aligned} &[a, a^\dagger] = 1 \Rightarrow aa^\dagger - a^\dagger a = 1, \\ &\Rightarrow aa^\dagger = 1 + a^\dagger a. \end{aligned} \tag{3.48}$$

There is another way to prove the above equation, if we use these relations:

$$a^\dagger = \frac{\hat{x} - \partial_x}{\sqrt{2}}, \qquad a = \frac{\hat{x} + \partial_x}{\sqrt{2}}. \tag{3.49}$$

Proof.

$$\begin{aligned} [a, a^\dagger] &= (\frac{\hat{x} + \partial_x}{\sqrt{2}})(\frac{\hat{x} - \partial_x}{\sqrt{2}}) - (\frac{\hat{x} - \partial_x}{\sqrt{2}})(\frac{\hat{x} + \partial_x}{\sqrt{2}}) \\ &= \frac{1}{2}(\hat{x}.\hat{x} - \hat{x}.\partial_x + \partial_x.\hat{x} - \partial_x.\partial_x - \hat{x}.\hat{x} - \hat{x}.\partial_x + \partial_x.\hat{x} + \partial_x.\partial_x) \\ &= \frac{1}{2}[\hat{x}, \hat{x}] + [\partial_x, \partial_x] + 2[\partial_x, \hat{x}] = [\partial_x, \hat{x}] = 1. \end{aligned} \tag{3.50}$$

□

3.

$$[a^\dagger\, a, a] = -a. \tag{3.51}$$

Where $a^\dagger\, a$ is called an occupation number operator, and we can prove this easily as:

$$[a^\dagger\, a, a] = a^\dagger\, a\, a - a\, a^\dagger\, a = (a^\dagger\, a - a\, a^\dagger)a = -1\, a = -a. \tag{3.52}$$

We can extract the following:

$$\begin{aligned} [a^\dagger\, a, a] = -a \Rightarrow a^\dagger\, a\, a - a\, a^\dagger\, a = -a \\ \Rightarrow (a^\dagger\, a)\, a = -a + a\, a^\dagger\, a = a(a\, a^\dagger - 1) \\ \Rightarrow (a^\dagger\, a)\, a = a(a\, a^\dagger - 1). \end{aligned} \tag{3.53}$$

4.

$$[a^\dagger\, a, a^\dagger] = a^\dagger, \tag{3.54}$$

then

$$[a^\dagger\, a, a^\dagger] = a^\dagger\, a\, a^\dagger - a^\dagger\, a^\dagger\, a = a^\dagger(a\, a^\dagger - a^\dagger\, a) = a^\dagger\, 1 = a^\dagger. \tag{3.55}$$

Also, we can get:

$$(a^\dagger\, a)a^\dagger = a^\dagger(a^\dagger\, a + 1). \tag{3.56}$$

5. We have the following relations among $\hat{\mathrm{H}}, a^\dagger, a$:

$$[a, \hat{\mathrm{H}}] = a, \qquad [a^\dagger, \hat{\mathrm{H}}] = -a^\dagger. \tag{3.57}$$

Let $N = a^\dagger a$ then:

$$\hat{\mathrm{H}} = (N + \frac{1}{2}). \tag{3.58}$$

We can represent a Hamiltonian operator with eigenstate $|n\rangle$ which called number states as:

$$\hat{\mathrm{H}}|n\rangle = (N + \frac{1}{2})|n\rangle = (n + \frac{1}{2})|n\rangle = E_n|n\rangle, \quad E_n = n + \frac{1}{2} \tag{3.59}$$

6.

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{3.60}$$

7.

$$a^\dagger a|n\rangle = n|n\rangle. \tag{3.61}$$

8.

$$\langle n|a^\dagger a|n\rangle = n. \tag{3.62}$$

We can prove this relation from the last property as:

$$\langle n|a^\dagger a|n\rangle = \langle n|a^\dagger \sqrt{n}|n-1\rangle = \sqrt{n}\langle n|a^\dagger|n-1\rangle, \tag{3.63}$$

from definition of † we get:

$$\langle n|a^\dagger a|n\rangle = \sqrt{n}\langle n-1|a|n\rangle^\dagger = \sqrt{n}\sqrt{n}\langle n-1|n-1\rangle. \tag{3.64}$$

Note that n real number and $\langle n-1|n-1\rangle^\dagger = 1$, then:

$$\langle n|a^\dagger a|n\rangle = n. \tag{3.65}$$

9. And similarly we can prove that:

$$\langle n|aa^\dagger|n\rangle = n+1. \tag{3.66}$$

$$\langle n|aa^\dagger|n\rangle = \langle n|a\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\langle n|a|n+1\rangle, \tag{3.67}$$

from definition of † we get:

$$\begin{aligned}\langle n|aa^\dagger|n\rangle &= \langle n|a\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\langle n+1|a^\dagger|n\rangle^\dagger \\ &= \sqrt{n+1}\langle n+1|\sqrt{n+1}|n+1\rangle^\dagger = \sqrt{n+1}\sqrt{n+1}\langle n+1|n+1\rangle^\dagger,\end{aligned} \tag{3.68}$$

then

$$\langle n|aa^{\dagger}|n\rangle = n+1. \tag{3.69}$$

10.

$$\langle n|m\rangle = \delta_{n,m}, \tag{3.70}$$

We know that $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ if $n \neq m$.

We have the following formulae which are useful to prove many relations in this thesis.

Baker-Campbell-Hausdorff (BCH) Formulae.

It is also called nested commutator expansion. Suppose there are two operators α, β then Baker-Campbell-Hausdorff defines generally the following relations:

$$\exp(\alpha)\beta\exp(-\alpha) = \beta + [\alpha,\beta] + \frac{1}{2!}[\alpha,[\alpha,\beta]] + \frac{1}{3!}[\alpha,[\alpha,[\alpha,\beta]]] + ... \tag{3.71}$$

$$\exp(\alpha)\exp(\beta)\exp(-\alpha) = \exp\left(\beta + [\alpha,\beta] + \frac{1}{2!}[\alpha,[\alpha,\beta]] + \frac{1}{3!}[\alpha,[\alpha,[\alpha,\beta]]] + ...\right). \tag{3.72}$$

In particular cases, depending on when the resulting the commutator $[\alpha, \beta]$ commutes with each operator α, β i.e ($[[\alpha,\beta],\alpha] = [[\alpha,\beta],\beta] = 0$), we obtain another relation which is defined as [35, 31]:

$$\exp(\alpha)\exp(\beta) = \exp\Big(\alpha + \beta + \frac{1}{2}[\alpha, \beta]\Big) = \exp(\alpha + \beta)\exp\Big(\frac{1}{2}[\alpha, \beta]\Big), \quad (3.73)$$

and we can obtain the next relation from Eq (3.73) as:

$$\exp(\alpha + \beta) = \exp\Big(-\frac{1}{2}[\alpha, \beta]\Big)\exp(\alpha)\exp(\beta). \quad (3.74)$$

.

3.3.4 Displacement operator and their properties

Displacement operator is defined as follows [35]:

$$\mathfrak{D}(B) = \exp\Big(Ba^{\dagger} - B^{*}a\Big), \quad (3.75)$$

where $a^{\dagger}, a$ are creation and annihilation operators and $B = |B|e^{i\varphi} \in \mathbb{C}$, where $\mathbb{C}$ is a complex number, and φ is a wave function.

Also, we have:

$$\mathfrak{D}(B) = \exp\Big(-\frac{1}{2}|B|^2\Big)\exp\Big(Ba^{\dagger}\Big)\exp(-B^{*}a), \quad (3.76)$$

then:

$$\mathfrak{D}^{*}(B) = \exp\Big(-\frac{1}{2}|B|^2\Big)\exp(-B^{*}a)\exp\Big(-Ba^{\dagger}\Big). \quad (3.77)$$

We can prove Eq (3.76) using Eqs (3.75), (3.74) as:

$$\begin{aligned}\mathfrak{D}(B) = \exp\Big(Ba^\dagger - B^*a\Big) &= \exp\Big(-\frac{1}{2}[Ba^\dagger, -B^*a]\Big)\exp\Big(Ba^\dagger\Big)\exp(-B^*a) \\ &= \exp\Big(-\frac{1}{2}(-Ba^\dagger B^*a + B^*aBa^\dagger)\Big)\exp\Big(Ba^\dagger\Big)\exp(-B^*a) \\ &= \exp\Big(-\frac{1}{2}(-|B|^2a^\dagger a + |B|^2aa^\dagger)\Big)\exp\Big(Ba^\dagger\Big)\exp(-B^*a) \\ &= \exp\Big(-\frac{1}{2}|B|^2(aa^\dagger - a^\dagger a)\Big)\exp\Big(Ba^\dagger\Big)\exp(-B^*a) \\ &= \exp\Big(-\frac{1}{2}|B|^2\Big)[a, a^\dagger]\exp\Big(Ba^\dagger\Big)\exp(-B^*a). \end{aligned} \tag{3.78}$$

By using Eq (3.42) we get:

$$\mathfrak{D}(B) = \exp\Big(-\frac{1}{2}|B|^2\Big)\exp\Big(Ba^\dagger\Big)\exp(-B^*a). \tag{3.79}$$

Let's use $B = \frac{x+ip}{\sqrt{2}}$

$$\begin{aligned}\mathfrak{D}(x,p) &= \mathfrak{D}(B = \frac{x+ip}{\sqrt{2}}) \\ &= \exp\Big(\frac{(x+ip)}{\sqrt{2}}.\frac{(\hat{x}-i\hat{p})}{\sqrt{2}} - \frac{(x-ip)}{\sqrt{2}}\frac{(\hat{x}+i\hat{p})}{\sqrt{2}}\Big) \\ &= \exp\Big(\frac{1}{2}(x\hat{x} + ip\hat{x} - ix\hat{p} + p\hat{p} - x\hat{x} + ip\hat{x} - ix\hat{p} - p\hat{p})\Big) \\ &= \exp(ip\hat{x} - ix\hat{p}), \end{aligned} \tag{3.80}$$

then:

$$\mathfrak{D}(x,p) = \exp(ip\hat{x} - ix\hat{p}). \tag{3.81}$$

Also, it can be written as:

$$\mathfrak{D}(x,p) = \exp(ip\hat{x})\exp(-ix\hat{p})\exp\left(-\frac{1}{2}ixp\right). \tag{3.82}$$

By using Eq (3.74), we get:

$$\begin{aligned}
\mathfrak{D}(x,p) = \exp(ip\hat{x} - ix\hat{p}) &= \exp\left(-\frac{1}{2}[ip\hat{x}, -ix\hat{p}]\right)\exp(ip\hat{x})\exp(-ix\hat{p}) \\
&= \exp\left(-\frac{1}{2}(ip\hat{x})(-ix\hat{p}) - (-ix\hat{p})(ip\hat{x})\right)\exp(ip\hat{x})exp(-ix\hat{p}) \\
&= \exp\left(-\frac{1}{2}xp(\hat{x}\hat{p} - \hat{p}\hat{x})\right)\exp(ip\hat{x})\exp(-ix\hat{p}) \\
&= \exp\left(-\frac{1}{2}xp[\hat{x},\hat{p}]\right)\exp(ip\hat{x})\exp(-ix\hat{p}).
\end{aligned} \tag{3.83}$$

We obtain the following equation by using Eq (3.43):

$$\mathfrak{D}(x,p) = \exp(ip\hat{x})\exp(-ix\hat{p})\exp\left(-\frac{1}{2}ixp\right). \tag{3.84}$$

Properties of the displacement operator.

We have some necessary properties of the displacement operator to explain [35].

1.

$$\mathfrak{D}(B)\mathfrak{D}^{\dagger}(B) = \mathfrak{D}^{\dagger}(B)\mathfrak{D}(B) = 1. \tag{3.85}$$

2.

$$\mathfrak{D}(B)^\dagger = \mathfrak{D}(B)^* = \mathfrak{D}(-B) = \mathfrak{D}(B)^{-1}. \tag{3.86}$$

3.

$$\mathfrak{D}(B)^\dagger a \mathfrak{D}(B) = a + B. \tag{3.87}$$

We can prove this property as follows:

Proof.

$$\mathfrak{D}(B)^\dagger a \mathfrak{D}(B) = \exp\Big(-B^* a - B a^\dagger\Big) a \exp\Big(B a^\dagger - B^* a\Big), \tag{3.88}$$

By using the Baker-Campbell-Hausdorff (BCH) Formulae (Eq (3.71)), we obtain:

$$\begin{aligned}
\mathfrak{D}(B)^\dagger a \mathfrak{D}(B) =& a + [-B^* a - B a^\dagger, a] + \frac{1}{2!}[-B^* a - B a^\dagger, [-B^* a - B a^\dagger, a]] \\
&+ \frac{1}{3!}[-B^* a - B a^\dagger, [-B^* a - B a^\dagger, [-B^* a - B a^\dagger, a]]] + ..., \\
&\text{after that use Jacobi Identity:} \\
=& a + [-B^* a - B a^\dagger, a] + \frac{1}{2!}((-B^* a - B a^\dagger)[-B^* a - B a^\dagger, a] \\
&- [-B^* a - B a^\dagger, a](-B^* a - B a^\dagger)) + ...
\end{aligned} \tag{3.89}$$

We can note the higher order commutators equal to zero, then:

$$\begin{aligned}
\mathfrak{D}(B)^\dagger a \mathfrak{D}(B) &= a + [-B^* a - B a^\dagger, a] \\
&= a + (-B^* a - B a^\dagger) a - a(-B^* a - B a^\dagger) \\
&= a - B^* aa - B a^\dagger a + a B^* a + a B a^\dagger \\
&= a - B^* aa + a B^* a + a B a^\dagger - B a^\dagger a \\
&= a + B^* (aa - aa) + B(a a^\dagger - a^\dagger a) \\
&= a + B^* [a, a] + B[a, a^\dagger] = a + B. \qquad (3.90)
\end{aligned}$$

□

We can also obtain another property from this property, as follows:

4.

$$\begin{aligned}
&\mathfrak{D}(B)\mathfrak{D}(B)^\dagger a \mathfrak{D}(B) = \mathfrak{D}(B)(a + B) \\
&\Rightarrow a \mathfrak{D}(B) = \mathfrak{D}(B)(a + B) \qquad (3.91)
\end{aligned}$$

The similar way can be proved this property:

5.

$$\mathfrak{D}(B) a \mathfrak{D}(B)^\dagger = a - B. \qquad (3.92)$$

Also, in the same way in which we proved number **3**, we can prove the following properties:

6.

$$\mathfrak{D}(B)^\dagger a^\dagger \mathfrak{D}(B) = (a^\dagger + B^*). \tag{3.93}$$

We can get another relation from Eq (3.93)as follows:

$$\begin{aligned} &\mathfrak{D}(B)^\dagger a^\dagger \mathfrak{D}(B)\mathfrak{D}(B)^\dagger = (a^\dagger + B^*)\mathfrak{D}(B)^\dagger \\ &\Rightarrow \mathfrak{D}(B)^\dagger a^\dagger = (a^\dagger + B^*)\mathfrak{D}(B)^\dagger. \end{aligned} \tag{3.94}$$

7.

$$\mathfrak{D}(B) a^\dagger \mathfrak{D}(B)^\dagger = (a^\dagger - B^*). \tag{3.95}$$

8.

$$\mathfrak{D}(A+B) = \mathfrak{D}(A)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}(AB^* - B^*A)\Big) = \exp\big(-iIm(AB^*)\big). \tag{3.96}$$

We can prove this equation by using Eq(3.75):

Proof.

$$
\begin{aligned}
\mathfrak{D}(A+B) =& \exp\Big(Aa^\dagger - A^*a + Ba^\dagger - B^*a\Big) \\
=& \exp\Big(Aa^\dagger - A^*a\Big)\exp\Big(Ba^\dagger - B^*a\Big) \\
& \exp\Big(-\frac{1}{2}[(Aa^\dagger - A^*a),(Ba^\dagger - B^*a)]\Big) \\
=& \mathfrak{D}(A)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}\Big((Aa^\dagger - A^*a)(Ba^\dagger - B^*a) \\
& - (Ba^\dagger - B^*a)(Aa^\dagger - A^*a)\Big)\Big) \\
=& \mathfrak{D}(A)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}(Aa^\dagger Ba^\dagger - Aa^\dagger B^*a - A^*aBa^\dagger + A^*aB^*a)\Big) \\
& \exp\Big(-\frac{1}{2}(-Ba^\dagger Aa^\dagger + Ba^\dagger A^*a + B^*aAa^\dagger - B^*aA^*a)\Big) \\
=& \mathfrak{D}(A)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}(AB(a^\dagger a^\dagger - a^\dagger a^\dagger) + A^*B^*(aa - aa))\Big) \\
& \exp\Big(-\frac{1}{2}(AB^*(aa^\dagger - a^\dagger a) + A^*B(a^\dagger a - aa^\dagger))\Big) \\
=& \mathfrak{D}(B)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}(AB[a^\dagger, a^\dagger] + A^*B^*[a,a] + AB^*[a,a^\dagger] \\
& + A^*B[a^\dagger, a])\Big) \\
=& \mathfrak{D}(A)\mathfrak{D}(B)\exp\Big(-\frac{1}{2}(AB^* - A^*B)\Big) \\
=& \mathfrak{D}(A)\mathfrak{D}(B)\exp\big(-iIm(AB^*)\big)
\end{aligned}
\tag{3.97}
$$

□

Also, we have similar property for this one and the same way to prove it, which is:

$$\mathfrak{D}^{\dagger}(A+B)=\mathfrak{D}^{\dagger}(A)\mathfrak{D}^{\dagger}(B)\exp\Big(\frac{1}{2}(AB^{*}-B^{*}A)\Big)=\exp\big(iIm(AB^{*})\big).$$

3.3.5 Coherent states and their properties.

Coherent states have another name which are Glauber states. They are defined as eigenstates of creation operator a as follows:, $B\in\mathbb{C}$ which is an eigenvalues [35, 34, 31].

$$a|B\rangle = B|B\rangle, \tag{3.98}$$

Expand the following equation onto Eq(3.98):

$$|B\rangle=\sum_{n=0}^{\infty}c_{n}\,|n\rangle\,. \tag{3.99}$$

as following:

$$\begin{aligned}
&\Rightarrow a\sum_{n=0}^{\infty}c_{n}|n\rangle = B\sum_{n=0}^{\infty}c_{n}|n\rangle \Rightarrow \sum_{n=0}^{\infty}c_{n}a|n\rangle = B\sum_{n=0}^{\infty}c_{n}|n\rangle \\
&\Rightarrow \sum_{n=0}^{\infty}c_{n}\sqrt{n}|n-1\rangle = B\sum_{n=0}^{\infty}c_{n}|n\rangle,
\end{aligned} \tag{3.100}$$

multiply from left to right by $\langle m|$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n \sqrt{n} \langle m|n-1\rangle = B \sum_{n=0}^{\infty} c_n \langle m|n\rangle \\
&\sum_{m+1=0}^{\infty} c_{m+1}\sqrt{m+1}\langle m|m\rangle = B \sum_{m=0}^{\infty} c_m \langle m|m\rangle \\
&c_{m+1}\sqrt{m+1} = Bc_m, \qquad (3.101)
\end{aligned}$$

then:

$$\begin{aligned}
c_{m+1} &= \frac{Bc_m}{\sqrt{m+1}} \\
\Rightarrow c_{m+1} &= \frac{B\, c_{m-1}}{\sqrt{m}}. \qquad (3.102)
\end{aligned}$$

Let $c_0 = N(B)$ and try to calculate the first four terms.

$$\begin{aligned}
c_1 &= \frac{B}{\sqrt{1}} c_0 = \frac{B}{\sqrt{1}} N(B) \\
c_2 &= \frac{B}{\sqrt{2}} c_1 = \frac{B^2}{\sqrt{2.1}} N(B) \\
c_3 &= \frac{B}{\sqrt{3}} c_2 = \frac{B^3}{\sqrt{3.2.1}} N(B) \\
c_4 &= \frac{B}{\sqrt{3}} c_3 = \frac{B^4}{\sqrt{4.3.2.1}} N(B). \qquad (3.103)
\end{aligned}$$

We can get this easily:

$$c_n = \frac{B^n}{\sqrt{n!}} N(B), \qquad (3.104)$$

where c_n called a normalization constant.

$$|B\rangle = N(B)\sum_{n=0}^{\infty}\frac{B^n}{\sqrt{n!}}|n\rangle, \tag{3.105}$$

from the definition of normalization we have:

$$\begin{aligned}&\langle B|B\rangle = 1\\ &N(B)N^*(B)\sum_{n,m=0}^{\infty}\frac{(B^*)^m B^n}{\sqrt{n!m!}}\langle m|n\rangle = 1\end{aligned} \tag{3.106}$$

If $m = n$, then:

$$|N(B)|^2\sum_{n=0}^{\infty}\frac{|B|^{2n}}{n!} = 1, \tag{3.107}$$

which gives:

$$|N(B)|^2\exp\big(|B|^2\big) = 1 \Rightarrow N(B) = \exp\left(\frac{-|B|^2}{2}\right). \tag{3.108}$$

From Eq (3.108) into (3.105), we obtain the coherent states equations as following:

$$|B\rangle = \exp\left(\frac{-|B|^2}{2}\right)\sum_{n=0}^{\infty}\frac{B^n}{\sqrt{n!}}|n\rangle, \tag{3.109}$$

and:

$$\langle B| = \langle n|\exp\left(\frac{-|B|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(B^*)^n}{\sqrt{n!}}. \tag{3.110}$$

The vacuum state

It is called the normalized vacuum state or ground state, and can be defined as [31]:

$$|0\rangle = (1, 0, 0, 0...), \tag{3.111}$$

The following are some relations related to a vacuum state.

a.

$$\begin{aligned} a|0\rangle &= 0, & \langle 0|a^\dagger &= 0 \\ \langle 0|a &= 1, & a^\dagger|0\rangle &= 1. \end{aligned} \tag{3.112}$$

b.

$$\langle 0|a^\dagger a|0\rangle = 0, \qquad \langle 0|aa^\dagger|0\rangle = 1. \tag{3.113}$$

c.

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{3.114}$$

Properties of coherent states.

We have some important properties of coherent states, which are [34, 31]:

1.

$$|B\rangle = \mathfrak{D}(B)|0\rangle, \tag{3.115}$$

and

$$\langle B| = \langle 0|\mathfrak{D}^\dagger(B). \tag{3.116}$$

Proof. Using Eq (3.76), we get:

$$\begin{aligned}
\mathfrak{D}(B)|0\rangle =& \exp\left(-\frac{1}{2}|B|^2\right)\exp\left(Ba^\dagger\right)\exp(-B^*a)|0\rangle \\
=& \exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(Ba^\dagger)^n}{n!}\sum_{n=0}^{\infty}\frac{(B^*a)^n}{n!}|0\rangle \\
=& \exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(Ba^\dagger)^n}{n!}\left(1 + B^*a + \frac{(B^*a)^2}{2!} + \frac{(B^*a)^3}{3!} + ...\right)|0\rangle \\
=& \exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(Ba^\dagger)^n}{n!}\left(|0\rangle + B^*a|0\rangle + \frac{(B^*a)^2|0\rangle}{2!}\right. \\
&\left. + \frac{(B^*a)^3|0\rangle}{3!} + ...\right).
\end{aligned} \tag{3.117}$$

We have $a|0\rangle = 0$, then:

$$\mathfrak{D}(B)|0\rangle = \exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(Ba^\dagger)^n}{n!}|0\rangle + 0, \tag{3.118}$$

from Eq (3.114), we obtain:

$$\mathfrak{D}(B)|0\rangle = \exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(B)^n}{\sqrt{n!}}|n\rangle = |B\rangle. \tag{3.119}$$

□

We have proved Eq (3.116) by the same method.

2.

$$a|B\rangle = B|B\rangle, \tag{3.120}$$

and we can prove this relation by using Eq (3.115) and Eq (3.91) as:

$$\begin{aligned} a|B\rangle &= a\mathfrak{D}(B)|0\rangle = \mathfrak{D}(B)(a+B)|0\rangle \\ &= \mathfrak{D}(B)a|0\rangle + B\mathfrak{D}(B)|0\rangle = B|B\rangle. \end{aligned} \tag{3.121}$$

Also, we have:

$$\langle B|a|B\rangle = \langle B|B|B\rangle = B\langle B|B\rangle = B. \tag{3.122}$$

3.

$$\langle B|a^\dagger a|B\rangle = B^*\langle B|a|B\rangle = BB^*\langle B|B\rangle = B^*B = |B|^2. \tag{3.123}$$

4.

$$\langle B|a^\dagger = B^*\langle B|. \tag{3.124}$$

This one is proved from Eq (3.116) and Eq (3.94) as follows:

$$\begin{aligned}\langle B|a^\dagger &= \langle 0|\mathfrak{D}^\dagger(B)a^\dagger = \langle 0|(a^\dagger + B^*)\mathfrak{D}^\dagger(B) \\ &= \langle 0|a^\dagger\mathfrak{D}^\dagger(B) + \langle 0|\, B^*\mathfrak{D}^\dagger(B). \end{aligned} \tag{3.125}$$

Where $\langle 0|a^\dagger = 0$, then:

$$\langle B|a^\dagger = B^*\langle 0|\mathfrak{D}^\dagger(B) = B^*\langle B|. \tag{3.126}$$

Also, we have:

$$\langle B|a^\dagger|B\rangle = \langle B|B^*|B\rangle = B^*\langle B|B\rangle = B^*. \tag{3.127}$$

5.

$$\langle B|\hat{x}|B\rangle = \frac{1}{\sqrt{2}}(B + B^*) = \sqrt{2}B_{Real}. \tag{3.128}$$

$$\langle B|\hat{x}|B\rangle = \langle B|\frac{a + a^\dagger}{\sqrt{2}}|B\rangle = \frac{1}{\sqrt{2}}\langle B|a|B\rangle + \frac{1}{\sqrt{2}}\langle B|a^\dagger|B\rangle. \tag{3.129}$$

By using Eqs (3.122) and (3.127), we get:

$$\begin{aligned}\langle B|\hat{x}|B\rangle &= \frac{1}{\sqrt{2}}\langle B|B|B\rangle + \frac{1}{\sqrt{2}}\langle B|B^*|B\rangle \\ &= \frac{B}{\sqrt{2}}\langle B|B\rangle + \frac{B^*}{\sqrt{2}}\langle B|B\rangle \\ &= \frac{1}{\sqrt{2}}(B + B^*) = \sqrt{2}B_{Real}. \end{aligned} \tag{3.130}$$

6. Also, we can prove this easily in the same way.

$$\langle B|\hat{p}|B\rangle = \frac{1}{i\sqrt{2}}(B - B^*) = \sqrt{2}B_{Image}. \tag{3.131}$$

7.

$$\langle B|\hat{x}^2|B\rangle = \frac{1}{2}((B + B^*)^2 + 1) = \frac{1}{\sqrt{2}}\langle B|\hat{x}|B\rangle^2 + \frac{1}{2}. \tag{3.132}$$

Proof.

$$\begin{aligned}
\langle B|\hat{x}^2|B\rangle &= \frac{1}{2}\langle B|(a + a^\dagger)^2|B\rangle = \frac{1}{2}\langle B|(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger|B\rangle \\
&= \frac{1}{2}\langle B|aa|B\rangle + \frac{1}{2}\langle B|aa^\dagger + a^\dagger a|B\rangle + \frac{1}{2}\langle B|a^\dagger a^\dagger|B\rangle \\
&= \frac{1}{2}\langle B|aB|B\rangle + \frac{1}{2}\langle B|aa^\dagger - a^\dagger a + a^\dagger a + a^\dagger a|B\rangle + \frac{1}{2}\langle B|B^* a^\dagger|B\rangle \\
&= \frac{B}{2}\langle B|a|B\rangle\rangle + \frac{B^*}{2}\langle B|a^\dagger B\rangle + \frac{1}{2}\langle B|[a, a^\dagger] + 2a^\dagger a|B\rangle \\
&= \frac{BB}{2} + \frac{B^* B^*}{2} + \frac{1}{2}\langle B|1 + 2a^\dagger a|B\rangle \\
&= \frac{B^2}{2} + \frac{B^{*2}}{2} + \frac{1}{2}\langle B|1|B\rangle + \frac{1}{2}\langle B|2a^\dagger a|B\rangle \\
&= \frac{B^2}{2} + \frac{B^{*2}}{2} + \frac{1}{2}\langle B|B\rangle + \frac{1}{2}2|B|^2 = \frac{B^2}{2} + \frac{B^{*2}}{2} + \frac{1}{2} + BB^* \\
&= \frac{1}{2}(B^2 + B^{*2} + 2BB^* + 1) \\
&= \frac{1}{2}((B + B^*)^2) + \frac{1}{2} = \frac{1}{\sqrt{2}}\langle B|\hat{x}|B\rangle^2 + \frac{1}{2}.
\end{aligned} \tag{3.133}$$

□

8.

$$\langle B|\hat{p}^2|B\rangle = -\frac{1}{2}((B - B^*)^2 - 1) = \langle B|\hat{p}|B\rangle^2 - \frac{1}{2}. \tag{3.134}$$

9.

$$\Delta\hat{x} = \sqrt{\langle B|\hat{x}^2|B\rangle - \langle B|\hat{x}|B\rangle^2}. \tag{3.135}$$

$$\Delta\hat{x} = \sqrt{\frac{1}{2}((B + B^*)^2) + \frac{1}{2} - (\frac{1}{\sqrt{2}}(B + B^*))^2} = \frac{1}{\sqrt{2}}. \tag{3.136}$$

10.

$$\Delta\hat{p} = \sqrt{\langle B|\hat{p}^2|B\rangle - \langle B|\hat{p}|B\rangle^2}. \tag{3.137}$$

$$\Delta\hat{p} = \sqrt{\text{ffl} - \frac{1}{2}((B - B^*)^2 - 1) - (\frac{1}{i\sqrt{2}}(B - B^*))^2} = \frac{1}{\sqrt{2}}. \tag{3.138}$$

Then:

$$\Delta\hat{x}\Delta\hat{p} = \frac{1}{2} \tag{3.139}$$

11. Non-orthogonality of a coherent state.

$$\begin{aligned}
\langle A|B\rangle &= \langle m|\exp\left(-\frac{1}{2}|A|^2\right)\sum_{n=0}^{\infty}\frac{(A^*)^m}{\sqrt{m!}}\exp\left(-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(B)^n}{\sqrt{n!}}|n\rangle \\
&= \exp\left(-\frac{1}{2}|A|^2-\frac{1}{2}|B|^2\right)\sum_{\substack{n=0\\m=0}}^{\infty}\frac{(A^*)^m B^n}{\sqrt{n!}\sqrt{m!}}\langle m|n\rangle \\
&= \exp\left(-\frac{1}{2}|A|^2-\frac{1}{2}|B|^2\right)\sum_{n=0}^{\infty}\frac{(A^*B)^n}{n!}\langle n|n\rangle \\
&= \exp\left(-\frac{1}{2}|A|^2-\frac{1}{2}|B|^2\right)\exp(A^*B). \qquad (3.140)
\end{aligned}$$

Then:

$$\langle A|B\rangle = \exp\left(-\frac{1}{2}|A|^2-\frac{1}{2}|B|^2+A^*B\right), \qquad (3.141)$$

which is called a scalar product of two coherent states, and similarly:

$$\langle B|A\rangle = \exp\left(-\frac{1}{2}|A|^2-\frac{1}{2}|B|^2+B^*A\right). \qquad (3.142)$$

$$|\langle A|B\rangle|^2 = \langle A|B\rangle\langle B|A\rangle = \exp\big(-|A|^2-|B|^2+A^*B+B^*A\big). \qquad (3.143)$$

Remark

$$\begin{aligned}
\exp\big(-|A-B|^2\big) &= \exp\big(-(A-B)(A^*-B^*)\big) \\
&= \exp\big(-|A|^2-|B|^2+A^*B+B^*A\big).
\end{aligned}$$

Then:

$$|\langle A|B\rangle|^2 = \exp\big(-|A-B|^2\big) \neq 0. \tag{3.144}$$

i.e $\exp\big(-|A-B|\big) \approx 0$ if $|A-B|$ is large, that means coherent states are not orthogonal when $A \neq B$.

Gamma function.

The Gamma function is defined by the following integral [36]:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \qquad \forall\, t > 0,\ x > 0. \tag{3.145}$$

And if we put $t = t+1$, then:

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}dx, \tag{3.146}$$

We integrate this equation by using parts of the integral as:

$$\begin{aligned} u &= x^t, & dv &= e^{-x}dx \\ du &= tx^{t-1}dx, & v &= -e^{-x}. \end{aligned} \tag{3.147}$$

$$\begin{aligned} \int udv &= -x^t e^{-x}|_0^\infty + \int_0^\infty tx^{t-1}e^{-x}dx \\ \Gamma(t+1) &= 0 + t\int_0^\infty x^{t-1}e^{-x}dx = t\Gamma(t). \end{aligned} \tag{3.148}$$

If we integrate this equation by the same method, we will obtain:

$$\Gamma(t) = (t-1)\int_0^\infty x^{t-2}e^{-x}dx = (t-1)\Gamma(t-1). \tag{3.149}$$

We can note that:

$$\Gamma(1) = 1, \Gamma(2) = 1\Gamma(1) = 1!, \Gamma(3) = 2\Gamma(2) = 2!, \Gamma(4) = 3\Gamma(3) = 3!, \dots . \tag{3.150}$$

Then at the end we obtain:

$$\Gamma(t+1) = t! \ , \qquad\qquad t > 0. \tag{3.151}$$

This equation explains the relation between the Gamma function and the factorial of any positive real number.

12. Completeness of coherent states.

The relation for completeness of the coherent states is written as:

$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B| d^2B = 1. \tag{3.152}$$

Proof.

$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B| d^2B = \frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m|\int_0^\infty \exp\left(-|B|^2\right)B^n B^{*m} d^2B. \tag{3.153}$$

Use polar coordinates to solve the integral on the right side, where $B =$

$|B|\exp(i\theta) = r\exp(i\theta)$ and $d^2B = rdrd\theta$.

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B =& \frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m|\int_0^\infty \exp\big(-r^2\big)(r\exp(i\theta))^n \\ &(r\exp(-i\theta))^m rdrd\theta \\ =& \frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m| \\ &\int_0^\infty \exp\big(-r^2\big)r^{n+m}rdr\int_0^\theta \exp\big(-i(n-m)\theta\big)d\theta.\end{aligned} \tag{3.154}$$

We know that:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iwt)dw, \tag{3.155}$$

then:

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B =& \frac{1}{\pi}\sum_{n,m=0}^{\infty}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m|2\pi\delta_{n,m} \\ &\int_0^\infty \exp\big(-r^2\big)r^{n+m}rdr.\end{aligned} \tag{3.156}$$

If $n = m$, then $\delta_{n,m} = 1$.

$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B = \sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle\langle n|\ 2\int_0^\infty \exp\big(-r^2\big)r^{2n}rdr. \tag{3.157}$$

Let $r^2 = t \Rightarrow 2rdr = dt$.

$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B = \sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle\langle n|\int_0^\infty \exp(-t)t^n dt, \tag{3.158}$$

Use Eq (3.146) into Eq (3.158):

$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B = \sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle\langle n|\Gamma(n+1) = \sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle\langle n|n!$$
$$\frac{1}{\pi}\int_0^\infty |B\rangle\langle B|d^2B = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1. \tag{3.159}$$

3.4 Operators in quantum system

3.4.1 Hermitian operator

A Hermitian or (self-adjoint) operator is an operator that satisfies [34]:

$$\int (H\psi)^*\psi dx = \int \psi^* H\psi dx, \tag{3.160}$$

equivalently,

$$\langle\psi|H|\psi\rangle = \langle H\psi|\psi\rangle. \tag{3.161}$$

The following equation is called a Hermitian conjugate (adjoint):

$$\int (H^\dagger\psi)^*\psi dx = \int \psi^* H\psi dx, \tag{3.162}$$

equivalently,

$$\langle H\psi|\psi\rangle = \langle H^\dagger\psi|\psi\rangle = \langle H\psi|\psi\rangle. \tag{3.163}$$

Where ψ is a wave function, and can note that the right side of Eqs (3.160), (3.162) are equal, this means:

$$H = H^{\dagger}. \tag{3.164}$$

The expectation value of H is:

$$\langle H \rangle = \langle \psi | H | \psi \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \langle \psi | H | \psi \rangle. \tag{3.165}$$

Now we can write the Hermitian operator via another way:

$$\langle H\psi | \psi \rangle = \langle \psi | H | \psi \rangle = \langle H^{\dagger}\psi | \psi \rangle. \tag{3.166}$$

Properties of Hermitian operators

1. Every eigenvalue of Hermitian operator are real.

Proof.

$$H|B\rangle = B|B\rangle \qquad\qquad \langle B|H^{\dagger} = B^{*}\langle B| \tag{3.167}$$

From Eq (3.164), we get:

$$\langle B|H = B^{*}\langle B|, \tag{3.168}$$

then:

$$\langle B|H|B\rangle = B\langle B|B\rangle, \quad \langle B|H|B\rangle = B^*\langle B|B\rangle$$
$$\Rightarrow B\langle B|B\rangle = B^*\langle B|B\rangle \Rightarrow B = B^*. \tag{3.169}$$

Where $\langle B|B\rangle = 0$ if $B = 0$, then all eigenvalues are real. □

2. Every eigenvector of a Hermitian operator is orthogonal i.e ($\langle\psi_a|\psi_b\rangle = 0 \, for \; a \neq b$).

Proof. Let $|\psi_a\rangle, |\psi_b\rangle$, and a, b are eigenvalues, where $a \neq b$

Remark. The definition of eigenvector is $A\psi_a = a\psi_a$ or $A|\psi_a\rangle = a|\psi_a\rangle$, where A is an operator.

$$\langle H\psi_a|\psi_b\rangle = \langle a\psi_a|\psi_b\rangle = a\langle\psi_a|\psi_b\rangle. \tag{3.170}$$

$$\langle H\psi_a|\psi_b\rangle = \langle\psi_a|H\psi_b\rangle = \langle\psi_a|b\psi_b\rangle = b\langle\psi_a|\psi_b\rangle. \tag{3.171}$$

Now subtract Eq(3.170)and Eq(3.171), we obtain:

$$(a - b)\langle\psi_a|\psi_b\rangle = 0. \tag{3.172}$$

We suppose that $a \neq b$, this means $(a - b) \neq 0$, then $\langle\psi_a|\psi_b\rangle = 0$.

3.4.2 Parity operator

If we apply a parity operator on $\hat{x}$, $\hat{p}$ and the state $\varphi(x)$ gives the following:

$$U_0\hat{x} = -\hat{x}U_0 = -\hat{x} \quad U_0\hat{p} = -\hat{p}U_0 = -\hat{p} \quad U_0\varphi(x) = \varphi(-x)U_0 = \varphi(-x). \tag{3.173}$$

Or equally:

$$U_0\hat{x}U_0^\dagger = -\hat{x} \qquad U_0\hat{p}U_0^\dagger = -\hat{p} \qquad U_0\varphi(x)U_0^\dagger = \varphi(-x). \tag{3.174}$$

Then the parity operator is defined by creation and annihilation as follows [35, 37]:

$$U_0 = \exp\Big(i\pi a^\dagger a\Big) = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|. \tag{3.175}$$

Since $U_0 = \exp\big(i\pi a^\dagger a\big)$, and let $n = a^\dagger a$ then:

$$\begin{aligned} U_0 &= \exp(i\pi n) \\ U_0|n\rangle &= \exp(i\pi n)|n\rangle \Rightarrow \langle n|U_0|n\rangle = \langle n|\exp(i\pi n)|n\rangle. \end{aligned} \tag{3.176}$$

Here, use De Moivre's theorem which is:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta), \tag{3.177}$$

then this yields to $\exp(i\pi n) = (\exp(i\pi))^n = (-1)^n$, after that take the summation of both sides then obtain:

$$U_0 \sum_{n=0}^{\infty} |n\rangle\langle n| = \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|. \tag{3.178}$$

Note that the identity operator is $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$, then:

$$U_0 = \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|. \tag{3.179}$$

Or an easy way is:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \tag{3.180}$$

multiply both sides by $e^{i\pi a^\dagger a}$, and let $n = a^\dagger a$.

$$\begin{aligned}\exp\Big(i\pi a^\dagger a\Big) &= \sum_{n=0}^{\infty} \exp\Big(i\pi a^\dagger a\Big)|n\rangle\langle n| = \sum_{n=0}^{\infty} \exp(i\pi n)|n\rangle\langle n| \\ &= \sum_{n=0}^{\infty} \exp\big((i\pi)^n\big)|n\rangle\langle n| = \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|. \end{aligned} \tag{3.181}$$

This is called an infinite summation of a parity operator [37].

We have the following properties of parity operator [35, 37] :

1.

$$U_0 a^\dagger U_0^\dagger = -a^\dagger. \tag{3.182}$$

2.

$$U_0 a U_0^\dagger = -a. \tag{3.183}$$

We can prove Eq (3.182)usig Eq (3.71) as follows:

$$\begin{aligned} U_0 a^\dagger U_0^\dagger &= \exp\Big(i\pi a^\dagger a\Big) a^\dagger \exp\Big(i\pi a^\dagger a\Big) \\ &= a^\dagger + i\pi[a^\dagger a, a^\dagger] + \frac{i\pi}{2!}[a^\dagger a, [aa^\dagger, a^\dagger]] + \ldots . \end{aligned} \tag{3.184}$$

Remark. $[a^\dagger a, a^\dagger] = a^\dagger$ and $[a^\dagger a, [aa^\dagger, a^\dagger]] = a^\dagger [a, a^\dagger] = a^\dagger$.

Then:

$$\begin{aligned} \exp\Big(i\pi a^\dagger a\Big) a^\dagger \exp\Big(i\pi a^\dagger a\Big) =& a^\dagger (1 + i\pi + \frac{i\pi}{2!} + \ldots) \\ =& a^\dagger \exp(i\pi) = a^\dagger (\cos(\pi) + i \sin(\pi)) \\ =& - a^\dagger. \end{aligned} \tag{3.185}$$

As a similar way can prove Eq (3.183).

3. If we sum Eq (3.182) and Eq (3.183) we get:

$$U_0 (a^\dagger + a) U_0^\dagger = -(a^\dagger + a). \tag{3.186}$$

4. If we subtract Eq (3.182) from Eq (3.1843) we get:

$$U_0 (a^\dagger - a) U_0^\dagger = -(a^\dagger - a). \tag{3.187}$$

5. Momentum and position with parity operators.

$$U_0\,|\hat{x}\rangle = |-\hat{x}\rangle \qquad\qquad U_0\,|-\hat{x}\rangle = |\hat{x}\rangle. \tag{3.188}$$

We have here some relation:

(a)

$$U_0^2|\hat{x}\rangle = U_0U_0|\hat{x}\rangle = U_0|-\hat{x}\rangle = |\hat{x}\rangle \Rightarrow U_0^2 = I = 1. \tag{3.189}$$

(b)

$$\langle\hat{x}|U_0^\dagger = (U_0|\hat{x}\rangle)^\dagger = |-\hat{x}\rangle^\dagger = \langle-\hat{x}| = \langle\hat{x}|U_0. \tag{3.190}$$

(c)

$$U_0\,|\hat{p}\rangle = |-\hat{p}\rangle \qquad\qquad U_0\,|-\hat{p}\rangle = |\hat{p}\rangle\,. \tag{3.191}$$

6.

$$U_0 = U_0^\dagger, \tag{3.192}$$

we can prove this easily as:

$$\begin{aligned}&\langle U_0\rangle = \langle\psi_1(x)|U_0|\psi_2(x)\rangle = \langle\psi_1(x)|\psi_2(-x)\rangle\\&\langle U_0^\dagger\rangle^\dagger = \langle\psi_2(x)|U_0|\psi_1(x)\rangle^\dagger = \langle\psi_2(x)|\psi_1(-x)\rangle^\dagger = \langle\psi_1(x)|\psi_2(-x)\rangle.\end{aligned} \tag{3.193}$$

7.

$$U_0^2 = 1, \tag{3.194}$$

we can prove this by Eq (3.179) as:

$$\begin{aligned} U_0 &= \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n| \Rightarrow U_0U_0 = \sum_{n=0}^{\infty} U_0(-1)^n|n\rangle\langle n| \\ U_0^2 &= \sum_{n=0}^{\infty}(-1)^n \exp(i\pi n)|n\rangle\langle n| = \sum_{n=0}^{\infty}(-1)^n(-1)^n|n\rangle\langle n| \\ &= \sum_{n=0}^{\infty}(-1)^{2n}|n\rangle\langle n| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1. \end{aligned} \tag{3.195}$$

3.4.3 Displaced parity operator

Displaced parity operator $U(B)$ can be defined as [35]:

$$U(B) = \mathfrak{D}(B)U_0\mathfrak{D}^\dagger(B) = \exp\Big(i\pi(a^\dagger - B^*I)(a - BI)\Big). \tag{3.196}$$

Proof. : We use this relation:

$$\exp\Big(i\pi a^\dagger a\Big) = 1 + i\pi aa^\dagger + \frac{i\pi aa^\dagger}{2!} + ... \tag{3.197}$$

$$\begin{aligned} \mathfrak{D}(B)\exp\Big(i\pi a^\dagger a\Big)\mathfrak{D}^\dagger(B) &= \mathfrak{D}(B)1D^\dagger(B) + i\pi\mathfrak{D}(B)aa^\dagger\mathfrak{D}^\dagger(B) \\ &+ \frac{i\pi\mathfrak{D}(B)aa^\dagger\mathfrak{D}^\dagger(B)}{2!} + \dots . \end{aligned} \tag{3.198}$$

Multiply $\mathfrak{D}^\dagger(B)$ from the right side of Eq (3.198) and $\mathfrak{D}(B)$ from the left side of it.

$$\mathfrak{D}(B)\exp\Big(i\pi a^\dagger a\Big)\mathfrak{D}^\dagger(B) = 1 + i\pi\mathfrak{D}(B)a\mathfrak{D}^\dagger(B)\mathfrak{D}(B)a^\dagger\mathfrak{D}^\dagger(B) + \frac{i\pi\mathfrak{D}(B)a\mathfrak{D}^\dagger(B)\mathfrak{D}(B)a^\dagger\mathfrak{D}^\dagger(B)}{2!} + \dots . \tag{3.199}$$

From Eq (3.932) and Eq (3.95) we obtain:

$$\begin{aligned}\mathfrak{D}(B)\exp\Big(i\pi a^\dagger a\Big)\mathfrak{D}^\dagger(B) &= 1 + i\pi(a^\dagger - B^*I)(a - BI) \\ &+ \frac{i\pi(a^\dagger - B^*I)(a - BI)}{2!} + \dots \\ &= \exp\Big(i\pi(a^\dagger - B^*I)(a - BI)\Big).\end{aligned} \tag{3.200}$$

Also we have the following relations:

$$\begin{aligned}U(B) = \mathfrak{D}(B)U_0\mathfrak{D}^\dagger(B) &\Rightarrow U(x,p) = \mathfrak{D}(x,p)U_0D^\dagger(x,p) \\ &= \mathfrak{D}(x,p)\mathfrak{D}(x,p)U_0 = \mathfrak{D}(2x,2p)U_0 = U_0\mathfrak{D}(-2x,-2p).\end{aligned} \tag{3.201}$$

$$U^2(B) = 1, \qquad\qquad U(B) = U^\dagger(B). \tag{3.202}$$

We can prove this relation $U^2(B) = 1$, easily by using Eq (3.196) as:

$$\begin{aligned}U(B) = \mathfrak{D}(B)U_0\mathfrak{D}^\dagger(B) &\Rightarrow U(B)U(B) = \mathfrak{D}(B)U_0\mathfrak{D}^\dagger(B)U(B) \\ &\Rightarrow U^2(B) = \mathfrak{D}(B)U_0D^\dagger(B)\mathfrak{D}(B)U_0\mathfrak{D}^\dagger(B) \\ &= \mathfrak{D}(B)U_0^2D^\dagger(B) = I = 1.\end{aligned} \tag{3.203}$$

And by Eq(3.196) we can prove easily this relation $U(B) = U^{\dagger}(B)$ as:

$$U(B) = \mathfrak{D}(B)U_0\mathfrak{D}^{\dagger}(B) = \mathfrak{D}(B)U_0^{\dagger}\mathfrak{D}^{\dagger}(B) = U^{\dagger}(B). \tag{3.204}$$

3.5 Bargmann representation

There are many representations which use analytic functions. The Bargmann representation is the one of these representations. Here, we introduce the Bargmann analytic representation in the complex plane using coherent state [38, 39].

Let $|s\rangle$ be an arbitrary normalized state:

$$|s\rangle = \sum_{n=0}^{\infty} s_n |n\rangle, \qquad \sum_{n=0}^{\infty} |s_n|^2 = 1, \tag{3.205}$$

and we use the following expressions:

$$|s^*\rangle = \sum_{n=0}^{\infty} s_n^* |n\rangle, \qquad \sum_{n=0}^{\infty} |s_n|^2 = 1. \tag{3.206}$$

The state $|s\rangle$ is represented by the following analytic function in the Bargmann representation [40, 39, 3]:

$$s(z) = \exp\left(\frac{1}{2}|z|^2\right) \langle z^*|s\rangle = \exp\left(\frac{1}{2}|z|^2\right) \langle s^*|z\rangle = \sum_{n=0}^{\infty} s_n \frac{z^n}{\sqrt{n!}}. \tag{3.207}$$

We are going now to prove the equality of Eq (3.207), so the first proof of the following equation is as follows:

$$\exp\left(\frac{1}{2}|z|^2\right)\langle z^*|s\rangle = \sum_{n=0}^{\infty} s_n \frac{z^n}{\sqrt{n!}}. \tag{3.208}$$

Proof.

By using the definition of a coherent state, which is:

$$\begin{aligned}
\langle z^*| &= \langle n| \exp\left(\frac{-|z|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}} \\
\exp\left(\frac{|z|^2}{2}\right)\langle z^*| &= \langle n| \sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}} \\
\exp\left(\frac{|z|^2}{2}\right)\langle z^*|s\rangle &= \langle n|s\rangle \sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}}.
\end{aligned} \tag{3.209}$$

Use Eq (3.205):

$$\begin{aligned}
\exp\left(\frac{|z|^2}{2}\right)\langle z^*|s\rangle &= \sum_{n,m=0}^{\infty} \langle n|m\rangle\, s_m \frac{(z)^n}{\sqrt{n!}} \\
\exp\left(\frac{|z|^2}{2}\right)\langle z^*|s\rangle &= \sum_{n,m=0}^{\infty} \delta_{nm} s_m \frac{(z)^n}{\sqrt{n!}}.
\end{aligned} \tag{3.210}$$

If $n = m$, then from the definition of Kronecker delta function, we obtain:

$$\exp\left(\frac{|z|^2}{2}\right)\langle z^*|s\rangle = \sum_{n=0}^{\infty} s_n \frac{(z)^n}{\sqrt{n!}}. \tag{3.211}$$

The second proof is similar to the first:

$$
\begin{aligned}
\exp\left(\frac{1}{2}|z|^2\right)\langle s^*|z\rangle &= \sum_{n=0}^{\infty} s_n \frac{z^n}{\sqrt{n!}} \\
|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right) &\sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}}\,|n\rangle \\
\exp\left(\frac{|z|^2}{2}\right)|z\rangle &= \sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}}\,|n\rangle \\
\exp\left(\frac{|z|^2}{2}\right)\langle s^*|z\rangle &= \sum_{n=0}^{\infty} \frac{(z)^n}{\sqrt{n!}}\,\langle s^*|n\rangle \\
\exp\left(\frac{|z|^2}{2}\right)\langle s^*|z\rangle &= \sum_{m,n=0}^{\infty} s_m \frac{(z)^n}{\sqrt{n!}}\,\langle m|n\rangle \\
\exp\left(\frac{|z|^2}{2}\right)\langle s^*|z\rangle &= \sum_{n,m=0}^{\infty} s_n \frac{(z)^n}{\sqrt{n!}}.
\end{aligned}
\tag{3.212}
$$

3.5.1 The scalar product in Bargmann representation

We start to explain the scalar product between ket and bra which is defined as [3, 39]:

$$
\langle\psi|\varphi\rangle = \int_{\infty}^{\infty} \psi^*(x)\varphi(x)dx = \sum_{n=0}^{\infty} \psi_n^*(x)\varphi_n(x). \tag{3.213}
$$

We can define scalar products from the different ways using the following equations:

$$
\begin{aligned}
|s\rangle &= s(a^\dagger)\,|0\rangle\,, \qquad \langle t| = (t(a^\dagger)\,|0\rangle)^\dagger = \langle 0|\,t^*(a), \\
\langle t^*| &= \langle 0|\,t(a).
\end{aligned}
\tag{3.214}
$$

$$s(a^\dagger) = \sum_{n=0}^{\infty} s_n (a^\dagger)^n \Rightarrow s(a^\dagger) |0\rangle = \sum_{n=0}^{\infty} s_n (a^\dagger)^n |0\rangle$$
$$|s\rangle = \sum_{n=0}^{\infty} s_n \sqrt{n!} |n\rangle . \tag{3.215}$$

$$\langle t^*| = \sum_{n=0}^{\infty} t_n \sqrt{n!} \langle n| = \sum_{n=0}^{\infty} G_n \langle n| = \sum_{n=0}^{\infty} G_n \frac{z^{*n}}{\sqrt{n!}}. \tag{3.216}$$

Where s, t are arbitrary states. As follows:

$$\langle 0| t(a)s(a^\dagger) |0\rangle = \langle t^*|s\rangle = \sum_{n=0}^{\infty} t_n \sqrt{n!} \langle n| \sum_{m=0}^{\infty} s_m \sqrt{m!} |m\rangle$$
$$\langle t^*|s\rangle = \sum_{n,m=0}^{\infty} \sqrt{n!}\sqrt{m!} t_n s_m \langle n|m\rangle$$
$$= \sum_{n,m=0}^{\infty} \sqrt{n!}\sqrt{m!} t_n s_m \delta_{nm} = \sum_{n=0}^{\infty} n! t_n s_n. \tag{3.217}$$

From the scalar product in real space, we have the position $\psi(x) = \langle x|\psi\rangle$ and momentum $\psi(p) = \langle p|\psi\rangle$ and from identity, we have $\langle\psi|\varphi\rangle^* = \langle\varphi|\psi\rangle$. However, in vector space for three-dimensions the volume of the scalar product is limited by the volume of the vectors. So we have the following relation, which is called the Schwartz inequality:

$$\langle\psi|\varphi\rangle \leq \sqrt{\langle\psi|\psi\rangle \langle\varphi|\varphi\rangle}, \tag{3.218}$$

where $a.b \leq |a||b|$.

We use the completeness relation to prove the definition of the scalar product in the Bargmann representation, which is:

$$\int_c |z\rangle \langle z| \frac{d^2z}{\pi} = 1, \tag{3.219}$$

multiply by $\langle t|s\rangle$:

$$\int_c \langle t|z\rangle \langle z|s\rangle \frac{d^2z}{\pi} = \langle t|s\rangle , \tag{3.220}$$

Now by applying Eq (3.214) and the properties of coherent state $(a|z\rangle = z|z\rangle , \langle z| a^\dagger = z^* \langle z|)$, on the following equation we get:

$$\langle z|s\rangle = \langle z| s(a^\dagger) |0\rangle = s(z^*) \langle z|0\rangle , \tag{3.221}$$

and by using the scalar product of two coherent states $|z\rangle$ and $\langle s|$ which is:

$$\begin{aligned} \langle z|s\rangle &= \exp\left(-\frac{1}{2}|z|^2 - \frac{1}{2}|s|^2 + z^* s\right) \\ \langle z|0\rangle &= \exp\left(-\frac{1}{2}z - 0 - 0\right) \Rightarrow \langle z|0\rangle = \exp\left(-\frac{1}{2}|z|^2\right). \end{aligned} \tag{3.222}$$

then we get:

$$\langle z|s\rangle = s(z) = s(z^*) \exp\left(-\frac{1}{2}|z|^2\right), \tag{3.223}$$

and similarly:

$$\begin{aligned} \langle t^*|z\rangle &= t^*(z) = t(z^*) = \langle 0| t(a) |z\rangle \\ &= t(z) \langle 0|z\rangle = t(z) \exp\left(-\frac{1}{2}|z|^2\right). \end{aligned} \tag{3.224}$$

Then by applying Eqs (3.2243), (3.224) on Eq(3.220), we have:

$$\begin{aligned}\langle t|s\rangle &= \int_c t(z)\exp\Big(-\frac{1}{2}|z|^2\Big)s(z^*)\exp\Big(-\frac{1}{2}|z|^2\Big)\\ &= \int_c \exp\Big(-|z|^2\Big)s(z^*)t(z)\frac{d^2z}{\pi}\\ \langle t|s\rangle &= \int_c \exp\Big(-|z|^2\Big)(s(z))^*t(z)\frac{d^2z}{\pi},\end{aligned} \tag{3.225}$$

From Eq (3.207) we obtain:

$$\begin{aligned}\langle t|s\rangle &= \frac{1}{\pi}\int_c \exp\Big(-|z|^2\Big)(\sum_{n=0}^{\infty} s_n\frac{z^n}{\sqrt{n!}})^*(\sum_{m=0}^{\infty} t_m\frac{z^m}{\sqrt{m!}})d^2z\\ &= \sum_{n,m=0}^{\infty}\frac{s_n^*t_m}{\sqrt{n!m!}}\frac{1}{\pi}\int_c \exp\Big(-|z|^2\Big)z^{*n}z^m d^2z.\end{aligned} \tag{3.226}$$

Then use polar coordination, which is: $z = r\exp(i\theta)$, $d^2z = rdrd\theta$ and $|z| = r$ then:

$$\begin{aligned}\langle t|s\rangle &= \sum_{n,m=0}^{\infty}\frac{s_n^*t_m}{\sqrt{n!m!}}\frac{1}{\pi}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi}\exp\big(-r^2\big)(r\exp(-i\theta))^n(r\exp(i\theta))^m rdrd\theta\\ &= \sum_{n,m=0}^{\infty}\frac{s_n^*t_m}{\sqrt{n!m!}}\int_{r=0}^{\infty}\exp\big(-r^2\big)(r)^{n+m+1}dr\frac{1}{\pi}\int_{\theta=0}^{2\pi}\exp\big(i\theta(m-n)\big)d\theta,\end{aligned} \tag{3.227}$$

from the definition of delta function ($\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$), we obtain:

$$\langle t|s\rangle = 2\sum_{n=0}^{\infty}\frac{s_n^*t_n}{n!}\int_{r=0}^{\infty}\exp\big(-r^2\big)(r)^{2n+1}dr. \tag{3.228}$$

We use the result of Gaussian integration, which is $\int_0^{\infty}\exp\big(-ax^2\big)(x)^{2n+1}dx =$

$\frac{n!}{2a^{n+1}}$, so we obtain:

$$\langle t|s\rangle = 2\sum_{n=0}^{\infty}\frac{s_n^* t_n}{n!}\frac{n!}{2} = \sum_{n=0}^{\infty} s_n^* t_n. \tag{3.229}$$

The final result for the scalar product of Bargmann representation is:

$$\langle t|s\rangle = \int_c \exp\Big(-|z|^2\Big)(s(z))^* t(z)\frac{d^2 z}{\pi} = \sum_{n=0}^{\infty} s_n^* t_n \tag{3.230}$$

3.5.2 Operators in Bargmann representation

1. The creation and annihilation operators in the Bargmann representation are defined as:

$$a^\dagger \to z, \qquad\qquad a \to \frac{\partial}{\partial z}. \tag{3.231}$$

We have the following relations:

$$as(a^\dagger)\left|0\right\rangle = \frac{d}{d(a^\dagger)}s(a^\dagger)\left|0\right\rangle. \tag{3.232}$$

Note that if $n \neq 0$,

$$\begin{aligned} a\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle, & a^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle, \\ \left\langle n\right| a^\dagger &= \left\langle n-1\right|\sqrt{n}, & \left\langle n\right| a &= \left\langle n+1\right|\sqrt{n+1}. \end{aligned} \tag{3.233}$$

Also we know that:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle\,, \qquad\qquad \langle n| = \frac{1}{\sqrt{n!}} \langle 0| (a)^n,$$
$$\langle x|n\rangle = \frac{1}{\sqrt{n!}} \langle x| (a^\dagger)^n |0\rangle\,. \tag{3.234}$$

We know that:

$$|n-1\rangle = \frac{1}{\sqrt{(n-1)!}}(a^\dagger)^{n-1} |0\rangle$$
$$\Rightarrow (a^\dagger)^{n-1} |0\rangle = \sqrt{(n-1)!}\,|n-1\rangle\,. \tag{3.235}$$

We can prove Eq (3.232) by using Eq (3.215) as:

$$as(a^\dagger)|0\rangle = \sum_{n=0}^{\infty} s_n a (a^\dagger)^n |0\rangle\,, \tag{3.236}$$

then from Eqs (3.233), (3.235) and (3.215) we get:

$$as(a^\dagger)|0\rangle = \sum_{n=0}^{\infty} s_n a \sqrt{(n-1)!}\sqrt{n}\,|n\rangle$$
$$= \sum_{n=0}^{\infty} s_n \sqrt{n}\sqrt{n}\sqrt{(n-1)!}\,|n-1\rangle$$
$$as(a^\dagger)|0\rangle = \sum_{n=0}^{\infty} s_n n (a^\dagger)^{n-1} |0\rangle = \frac{d}{d(a^\dagger)} s(a^\dagger)|0\rangle\,. \tag{3.237}$$

And similarly:

$$\begin{aligned}
\langle 0| t(a)a^\dagger &= \sum_{n=0}^{\infty} t_n \langle 0| a^n a^\dagger = \sum_{n=0}^{\infty} t_n \langle 0| a^{n-1} a a^\dagger \\
&= \sum_{n=0}^{\infty} t_n \langle n-1| \sqrt{(n-1)!} a a^\dagger = \sum_{n=0}^{\infty} t_n \langle n| a^\dagger \sqrt{(n-1)!}\sqrt{n} \\
&= \sum_{n=0}^{\infty} t_n \langle n-1| a^\dagger \sqrt{(n-1)!}\sqrt{n}\sqrt{n} = \sum_{n=0}^{\infty} t_n \langle 0| n a^{n-1} \\
\langle 0| t(a)a^\dagger &= \langle 0| \frac{d}{da} t(a). \qquad (3.238)
\end{aligned}$$

We can obtain two relations from Eqs (3.237), (3.238) which are:

$$t(a)s(a^\dagger)|0\rangle = t(\frac{d}{da^\dagger})s(a^\dagger)|0\rangle . \qquad (3.239)$$

$$\langle 0| t(a)s(a^\dagger) = \langle 0| s(\frac{d}{da})t(a). \qquad (3.240)$$

2. The equation of displacement operator on a state $|s\rangle$ represented with the function $s(z)$ is given by:

$$D(B)|s\rangle \to \exp\left(-\frac{1}{2}|z|^2 + Bz\right)s(z - B^*). \qquad (3.241)$$

Proof. From Eq (3.214), and we know that $|s\rangle \to s(z) = \langle z|s\rangle$ and

$\langle z|0\rangle = \exp\Big(-\frac{1}{2}|z|^2\Big)$, then we obtain:

$$\begin{aligned} D(B)\,|s\rangle =& \exp\bigg(-\frac{1}{2}|z|^2\bigg)\exp\Big(Ba^\dagger\Big)\exp(-B^*a)s(z) \\ &\to \exp\bigg(-\frac{1}{2}|z|^2\bigg)\exp(Bz)\exp\bigg(-B^*\frac{\partial}{\partial z}\bigg)s(z) \\ &\to \exp\bigg(-\frac{1}{2}|z|^2 + Bz\bigg)\exp\bigg(-B^*\frac{\partial}{\partial z}\bigg)s(z). \end{aligned} \tag{3.242}$$

We can find this relation $\exp\Big(-B^*\frac{\partial}{\partial z}\Big)s(z)$ by using power series of exponential function which is:

$$\begin{aligned} \exp\bigg(-B^*\frac{\partial}{\partial z}\bigg)s(z) =& (1 - B^*\frac{\partial}{\partial z} - \frac{1}{2!}B^*(\frac{\partial}{\partial z})^2 - \frac{1}{3!}B^*(\frac{\partial}{\partial z})^3 - ...)s(z) \\ =& s(z) - B^*\frac{\partial}{\partial z}s(z) - \frac{1}{2!}B^*(\frac{\partial}{\partial z})^2 s(z) \\ & - \frac{1}{3!}B^*(\frac{\partial}{\partial z})^3 s(z) - ... \\ =& s(z) - B^*s'(z) - \frac{1}{2!}B^*s''(z) \\ & - \frac{1}{3!}B^*s^{'''}(z) - ... . \end{aligned} \tag{3.243}$$

And from the Taylor series expansion with the Lagrange Remainder which is:

$$s(x+h) = s(x) + hs'(x) + \frac{1}{2!}hs''(x) + \frac{1}{3!}hs^{'''}(x) + ... \tag{3.244}$$

Then, we obtain:

$$\exp\bigg(-B^*\frac{\partial}{\partial z}\bigg)s(z) = s(z - B^*), \tag{3.245}$$

from Eq (3.245) in Eq (3.242), we get:

$$D(B)\,|s\rangle \to \exp\left(-\frac{1}{2}|z|^2 + Bz\right)s(z - B^*). \tag{3.246}$$

3.6 Summary

In this chapter, we reviewed Hermite polynomials. We also introduced the concept of the quantum harmonic oscillator, which is considered the commutator operator, as well as creation and annihilation operators.
Furthermore, position and momentum operators and displacement operators were studied. In addition, we introduced the important operators of a quantum system. For example, Hermitian and displaced parity operators.
We concluded this chapter with the Bargmann analytic representation.

Chapter 4

Analytic representations for finite quantum systems

4.1 Introduction

Weyl and Schwinger are original researchers for finite quantum systems [41, 42]. Their work can be reviewed through several references, as afforded in these references [43, 44, 45, 46, 47, 48, 49, 50, 51].

In this chapter, we provide a brief summery about the Jacobi theta function and we use the following definition for the theta function in this chapter and the final chapter.

$$\Theta_3(v;\tau) = \sum_{n=-\infty}^{\infty} \exp\left(i\pi\tau n^2\right) \exp(2niv). \tag{4.1}$$

Also, we consider a finite quantum system with D-dimensional Hilbert space ($\mathcal{H}$). This can be described using orthonormal basis of position and momentum states. These states take values in the ring $\mathbb{Z}(D)$, also they can be denoted by $|X_m\rangle$ and $\langle P_m|$ respectively. And P, X are not variables, but indicate the position and momentum states. We introduce a Fourier transform in the third section, and we study displacement operators in the fourth section.

Then, we consider a very important representation which is analytic representation. We conclude this chapter using the analytic function ($F(z)$) Zeros.

4.2 Jacobi theta functions

We have four kinds of Jacobi theta functions as given by [52, 53]:

$$\Theta_1(v;\tau) = \sum_{n=-\infty}^{\infty} (-1)^{\frac{n-1}{2}} \exp\Big(i\pi\tau(\frac{n+1}{2})^2\Big) \exp\big((2n+1)iv\big). \tag{4.2}$$

$$\Theta_2(v;\tau) = \sum_{n=-\infty}^{\infty} \exp\Big(i\pi\tau(\frac{n+1}{2})^2\Big) \exp\big((2n+1)iv\big). \tag{4.3}$$

$$\Theta_3(v;\tau) = \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2\big) \exp(2niv). \tag{4.4}$$

$$\Theta_4(v;\tau) = (-1)^n \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2\big) \exp(2niv). \tag{4.5}$$

In this thesis we consider the third kind of Jacobi theta function:

$$\Theta_3(v;\tau) = \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2\big) \exp(2niv), \tag{4.6}$$

and their important properties, which are:

(a). The Jacobi theta function Θ_3 is periodic [52].

$$\Theta_3(v+\pi;\tau) = \Theta_3(v;\tau+2) = \Theta_3(v;\tau), \tag{4.7}$$

it is easy to prove this:

$$\Theta_3(v+\pi;\tau) = \Theta_3(v;\tau). \tag{4.8}$$

Proof.

$$\begin{aligned}
\Theta_3(v+\pi;\tau) &= \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2\big)\exp\big(2ni(v+\pi)\big)\\
&= \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2\big)\exp(2niv+2ni\pi)\\
&\text{if } n=m-1, \quad \text{then}\\
&\Theta_3(v+\pi;\tau) = \Theta_3(v;\tau).
\end{aligned} \tag{4.9}$$

Also we can prove this easily,

$$\Theta_3(v;\tau+2) = \Theta_3(v;\tau). \tag{4.10}$$

□

Proof.

$$\begin{aligned}
\Theta_3(v;\tau+2) &= \sum_{n=-\infty}^{\infty} \exp\big(i\pi(\tau+2)n^2\big)\exp(2niv)\\
&= \sum_{n=-\infty}^{\infty} \exp\big(i\pi\tau n^2+2i\pi n^2\big)\exp(2niv)\\
&= \Theta_3(v;\tau).
\end{aligned} \tag{4.11}$$

□

(b). The Jacobi theta function Θ_3 is quasi-periodic [52].

$$\Theta_3(v+\tau\pi;\tau)=\Theta_3(v;\tau)\exp\big(-i(\pi\tau+2v)\big). \tag{4.12}$$

Proof.

$$\begin{aligned}\Theta_3(v+\tau\pi;\tau) &= \sum_{n=-\infty}^{\infty}\exp\big(i\pi\tau n^2\big)\exp\big(2ni(v+\tau\pi)\big)\\ &= \sum_{n=-\infty}^{\infty}\exp\big(i\pi\tau n^2\big)\exp(2niv+2ni\pi\tau)\\ &\text{if } n=m-1, \quad \text{then}\\ &\Theta_3(v+\tau\pi;\tau)=\Theta_3(v;\tau)\exp\big(-i(\pi\tau+2v)\big). \end{aligned} \tag{4.13}$$

$$\begin{aligned}\Theta_3(v;\tau)\exp\big(-i(\pi\tau+2v)\big) &= \sum_{n=-\infty}^{\infty}\exp\big(i\pi\tau n^2+2niv\big)\\ &\quad \exp\big(-i(\pi\tau+2v)\big)\\ &= \sum_{n=-\infty}^{\infty}\exp\big(i\pi\tau(n-1)^2+2iv(n-1)\big)\\ &= \sum_{n=-\infty}^{\infty}\exp\big(i\pi\tau(n-1)^2+2iv(n-1)\big)\\ &\quad \exp\big(2i\pi\tau(n-1)\big), \end{aligned} \tag{4.14}$$

put $m = n - 1$ get:

$$\begin{aligned}\Theta_3(v;\tau)\exp\big(-i(\pi\tau+2v)\big) &= \sum_{m=-\infty}^{\infty}\exp\big(i\pi\tau m^2+2ivm\big)\\ &\exp(2i\pi\tau m)\\ &= \sum_{m=-\infty}^{\infty}\exp\big(i\pi\tau m^2+2im(v+\tau\pi)\big)\\ &=\Theta_3(v+\tau\pi;\tau). \end{aligned} \tag{4.15}$$

□

(c).

$$\Theta_3(v,\tau) = (-i\tau)^{-\frac{1}{2}}\exp\left(\frac{v^2}{i\pi\tau}\right)\Theta_3\left(\frac{v}{\tau},\frac{-1}{\tau}\right). \tag{4.16}$$

This property has been proved in [54].

4.3 Fourier transform and position and momentum states

We can see the position states $|X_m\rangle$ satisfy these relations [55, 6, 56]:

$$\langle X_m|X_n\rangle = \delta_{m,n} \qquad \sum_m |X_m\rangle\langle X_m| = \mathbb{I}, \tag{4.17}$$

where $\delta_{m,n}$ is the Kronecker delta which is equal to one if $m = n$, and equal to zero if $m \neq n$. Also we can define the delta function as [55, 6, 56]:

$$\delta_{m,n} = \delta(m,n) = \frac{1}{D}\sum_{k}\exp\left(\frac{2i\pi}{D}k(m-n)\right). \tag{4.18}$$

Finite Fourier transform $\mathbb{F}$ can be defined with orthonormal basis of position states $\{|X_0\rangle, |X_1\rangle, ..., |X_{D-1}\rangle\}$ as follows [55, 6, 56]:

$$\mathbb{F} = \frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\langle X_l|. \tag{4.19}$$

Also, it can have some properties which are:

$$\mathbb{F}\mathbb{F}^{\dagger} = \mathbb{F}^{\dagger}\mathbb{F} = \mathbb{I} \qquad\qquad \mathbb{F}^4 = \mathbb{I}. \tag{4.20}$$

The first relation is easy to prove by using the definition, and the second relation we can prove as:

$$\mathbb{F}^4 = \mathbb{F}^2 \times \mathbb{F}^2. \tag{4.21}$$

Now we find $\mathbb{F}^2$ first as follows:

$$\begin{aligned}
\mathbb{F}^2 &= \frac{1}{\sqrt{D}} \sum_{l,k=0}^{D-1} \exp\left(\frac{2\pi ilk}{D}\right) |X_k\rangle \langle X_l| \frac{1}{\sqrt{D}} \sum_{m,n=0}^{D-1} \exp\left(\frac{2\pi imn}{D}\right) |X_m\rangle \langle X_n| \\
&= \frac{1}{\sqrt{D}} \sum_{m,n=0}^{D-1} \sum_{l,k=0}^{D-1} \exp\left(\frac{2\pi ilk}{D}\right) |X_k\rangle \langle X_l|X_m\rangle \frac{1}{\sqrt{D}} \exp\left(\frac{2\pi imn}{D}\right) \langle X_n| \\
&= \sum_{m,n=0}^{D-1} \frac{1}{\sqrt{D}} \sum_{l,k=0}^{D-1} \exp\left(\frac{2\pi ilk}{D}\right) |X_k\rangle \delta_{l,m} \frac{1}{\sqrt{D}} \exp\left(\frac{2\pi imn}{D}\right) \langle X_n| \\
&= \sum_{m=0}^{D-1} \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} \exp\left(\frac{2\pi imk}{D}\right) |X_k\rangle \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} \exp\left(\frac{2\pi imn}{D}\right) \langle X_n| . \qquad (4.22)
\end{aligned}$$

Now we have:

$$\mathbb{F}^2 = \sum_{n=0}^{D-1} \sum_{k=0}^{D-1} \frac{1}{D} \sum_{m=0}^{D-1} \exp\left(\frac{2\pi im(k+n)}{D}\right) |X_k\rangle \langle X_n| , \qquad (4.23)$$

from Eq (4.18) we have:

$$\mathbb{F}^2 = \sum_{n=0}^{D-1} \sum_{k=0}^{D-1} \delta_{k,-n} |X_k\rangle \langle X_n| , \qquad (4.24)$$

that means $k = -n$, then from Eq (4.28), we get:

$$\begin{aligned}
\mathbb{F}^2 &= \sum_{m=0}^{D-1} \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} \exp\left(\frac{2\pi imk}{D}\right) |X_k\rangle \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} \exp\left(\frac{-2\pi imk}{D}\right) \langle X_{-k}| \\
&= \sum_{m=0}^{D-1} |P_m\rangle \langle P_{-m}| = \mathbf{P}, \qquad (4.25)
\end{aligned}$$

where $\mathbf{P}$ is a new operator, which is

$$\mathbf{P} = \sum_m |P_m\rangle \langle P_{-m}| = \sum_m |X_m\rangle \langle X_{-m}| , \tag{4.26}$$
$$\mathbf{P}^2 = \mathbb{I},$$

then:

$$\mathbb{F}^4 = \mathbb{F}^2 \times \mathbb{F}^2 = \mathbf{P} \times \mathbf{P} = \mathbf{P}^2 = \mathbb{I}. \tag{4.27}$$

We can use finite Fourier transform to consider the orthonormal basis of position and momentum states $\{P_0, P_1, ..., P_D\}$ which satisfy these equations [55, 6] :

1.

$$\left|P_j\right\rangle = \mathbb{F}\left|X_j\right\rangle = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} \exp\left(\frac{2i\pi kj}{D}\right) |X_k\rangle . \tag{4.28}$$

Proof.

$$\begin{aligned} \left|P_j\right\rangle =& \mathbb{F}\left|X_j\right\rangle = \frac{1}{\sqrt{D}} \sum_{l,k=0}^{D-1} \exp\left(\frac{2i\pi lk}{D}\right) |X_k\rangle \left\langle X_l \middle| X_j\right\rangle \\ =& \frac{1}{\sqrt{D}} \sum_{l,k=0}^{D-1} \exp\left(\frac{2i\pi lk}{D}\right) |X_k\rangle \delta_{l,j} \\ =& \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} \exp\left(\frac{2i\pi jk}{D}\right) |X_k\rangle . \end{aligned} \tag{4.29}$$

□

2.

$$\left|X_j\right\rangle = \mathbb{F}^{\dagger}\left|P_j\right\rangle = \frac{1}{\sqrt{D}}\sum_{k=0}^{D-1}\exp\left(\frac{-2i\pi kj}{D}\right)\left|P_k\right\rangle. \tag{4.30}$$

Proof.

$$\begin{aligned}\left|X_j\right\rangle = \mathbb{F}^{\dagger}\left|P_j\right\rangle &= \frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{-2i\pi lk}{D}\right)\left|X_l\right\rangle\left\langle X_k\middle|P_j\right\rangle\\ &=\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{-2i\pi lk}{D}\right)\left|X_l\right\rangle\left\langle X_k\right|\mathbb{F}\left|X_j\right\rangle\\ &=\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{-2i\pi lk}{D}\right)\mathbb{F}\left|X_l\right\rangle\left\langle X_k\middle|X_j\right\rangle\\ &=\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{-2i\pi lk}{D}\right)\left|P_l\right\rangle\delta_{k,j}\\ &=\frac{1}{\sqrt{D}}\sum_{l=0}^{D-1}\exp\left(\frac{-2i\pi lj}{D}\right)\left|P_l\right\rangle. \end{aligned}\tag{4.31}$$

Now we can define the position and momentum operators, which are self-adjoint operators, and are defined as modulo D,where $x, p \ : \mathcal{H} \to \mathcal{H}$, as follows:

$$x = \sum_{m=0}^{D-1} m\left|X_m\right\rangle\left\langle X_m\right| \Rightarrow x\left|X_m\right\rangle = m\left|X_m\right\rangle. \tag{4.32}$$

$$p = \sum_{n=0}^{D-1} n\left|P_n\right\rangle\left\langle P_n\right| \Rightarrow p\left|P_n\right\rangle = n\left|P_n\right\rangle. \tag{4.33}$$

Also, we can obtain some relations using position and momentum operators which are [55, 6]:

1.

$$\mathbb{F}x\mathbb{F}^{\dagger} = p. \tag{4.34}$$

Proof.

$$\begin{aligned}
\mathbb{F}x\mathbb{F}^{\dagger} =& \frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\langle X_l|\sum_{m=0}^{D-1}m\,|X_m\rangle\langle X_m|\,\mathbb{F}^{\dagger}\\
=&\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\langle X_l|\sum_{j,m=0}^{D-1}m\,|X_m\rangle\langle X_m|\,\mathbb{F}^{\dagger}\left|X_j\right\rangle\left\langle X_j\right|\\
=&\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\langle X_l|\sum_{j,m=0}^{D-1}m\mathbb{F}^{\dagger}\langle X_m|X_m\rangle\left|X_j\right\rangle\left\langle X_j\right|\\
=&\sum_{m=0}^{D-1}m\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\langle X_l|X_m\rangle\,\mathbb{F}^{\dagger}\langle X_m|\\
=&\sum_{m=0}^{D-1}m\frac{1}{\sqrt{D}}\sum_{l,k=0}^{D-1}\exp\left(\frac{2\pi ilk}{D}\right)|X_k\rangle\,\delta_{l,m}\\
&\frac{1}{\sqrt{D}}\sum_{n,t=0}^{D-1}\exp\left(\frac{-2\pi int}{D}\right)\langle X_n|X_t\rangle\langle X_m|\\
=&\sum_{m=0}^{D-1}m\frac{1}{\sqrt{D}}\sum_{k=0}^{D-1}\exp\left(\frac{2\pi imk}{D}\right)|X_k\rangle\\
&\frac{1}{\sqrt{D}}\sum_{n,t=0}^{D-1}\exp\left(\frac{-2\pi int}{D}\right)|X_t\rangle\,\delta_{n,m}\\
=&\sum_{m=0}^{D-1}m\,|P_m\rangle\,\frac{1}{\sqrt{D}}\sum_{t=0}^{D-1}\exp\left(\frac{-2\pi itm}{D}\right)|X_t\rangle\\
=&\sum_{m=0}^{D-1}m\,|P_m\rangle\langle P_m| = p.
\end{aligned}\tag{4.35}$$

OR by easiest way as:

$$\begin{aligned}\mathbb{F}x\mathbb{F}^{\dagger} &= \mathbb{F}\sum_{m=0}^{D-1} m\,|X_m\rangle\,\langle X_m|\,\mathbb{F}^{\dagger} \\ &= \sum_{m=0}^{D-1} m\mathbb{F}\,|X_m\rangle\,\langle X_m|\,\mathbb{F}^{\dagger} = \sum_{m=0}^{D-1} m\,|P_m\rangle\,\langle P_m| = p. \end{aligned} \tag{4.36}$$

□

2.

$$\mathbb{F}p\mathbb{F}^{\dagger} = -x. \tag{4.37}$$

In a similar way we can prove this equation.

3.

$$x\,|P_k\rangle = \frac{1}{D}\sum_{l,j=0}^{D-1} l\exp\left(\frac{2\pi i l(k-j)}{D}\right)\left|P_j\right\rangle. \tag{4.38}$$

Proof.

$$
\begin{aligned}
x\mathbb{F}\left|X_k\right\rangle &= x\frac{1}{\sqrt{D}}\sum_{m,j=0}^{D-1}\exp\left(\frac{2\pi imj}{D}\right)\left|X_l\right\rangle\left\langle X_j\middle|X_k\right\rangle \\
&= x\sum_{j=0}^{D-1}\left|P_j\right\rangle\left\langle X_j\middle|X_k\right\rangle = \sum_{j=0}^{D-1}\sum_{l=0}^{D-1} l\left|X_l\right\rangle\left\langle X_l\right|\delta_{j,k}\left|P_j\right\rangle \\
&= \sum_{j=0}^{D-1}\sum_{l=0}^{D-1} l\left|X_l\right\rangle\left\langle X_l\right|\frac{1}{D}\sum_{n}\exp\left(\frac{2\pi in(j-k)}{D}\right)\left|P_j\right\rangle \\
&= \frac{1}{D}\sum_{j,n=0}^{D-1} l\exp\left(\frac{2\pi in(j-k)}{D}\right)\left|P_j\right\rangle \\
&= \frac{1}{D}\sum_{l,j=0}^{D-1} l\exp\left(\frac{2\pi il(k-j)}{D}\right)\left|P_j\right\rangle. \qquad (4.39)
\end{aligned}
$$

□

4.

$$
p\left|X_l\right\rangle = \frac{1}{D}\sum_{k,m=0}^{D-1} k\exp\left(\frac{2\pi ik(m-l)}{D}\right)\left|X_m\right\rangle. \qquad (4.40)
$$

In similar way we can prove the this equation:

4.3.1 Δ_n function

We first define the function Δ_0 as follows:

$$
\Delta_0(x) = \frac{1}{D}\sum_{j=0}^{D-1}\exp\left(\frac{2i\pi}{D}jx\right), \qquad (4.41)
$$

where x real number then:

$$\Delta_0(x+D) = \Delta_0(x), \tag{4.42}$$

we can prove this easily:

$$\begin{aligned}\Delta_0(x+D) &= \frac{1}{D}\sum_{j=0}^{D-1}\exp\left(\frac{2i\pi}{D}j(x+D)\right) = \frac{1}{D}\sum_{j=0}^{D-1}\exp\left(\frac{2i\pi}{D}jx\right)\exp\left(\frac{2i\pi}{D}jD\right)\\ &= \frac{1}{D}\sum_{j=0}^{D-1}\exp\left(\frac{2i\pi}{D}jx\right)\exp(2i\pi j) = \Delta_0(x).\end{aligned} \tag{4.43}$$

and

$$\Delta_0(x) = \frac{\exp\left(\frac{2i\pi}{D}xD\right)-1}{D(\exp\left(\frac{2i\pi}{D}\right)-1)},\ \ if\ \ x \neq 0. \tag{4.44}$$

We can obtain this equation easily by using Eq (4.41), and Geometric series which is:

$$\sum_{n=0}^{m-1} r^n = \frac{1-r^m}{1-r}. \tag{4.45}$$

$$\Delta_0(0) = 1,\ \ if\ \ x = 0. \tag{4.46}$$

Also, we have:

$$\Delta_0(x) = \delta(x,0), \tag{4.47}$$

we can prove this from Eq (4.18), where x integer number ($x \in \mathbb{Z}(D)$), and $\delta(x,0)$ is the Kronecker delta in $\mathbb{Z}(D)$.

The derivatives of $\Delta_0(x)$ is:

$$\Delta_n(x) = \frac{d^n}{dx^n}\Delta_0(x) = \frac{1}{D}\sum_{j=0}^{D-1}(\frac{2i\pi j}{D})^n \exp\left(\frac{2i\pi}{D}jx\right), \tag{4.48}$$

where n are integers modulo D, and Δ_n called the analogues of the delta function [57]. Δ_n are useful in the calculation of matrix elements, we can calculate the commutator as:

$$\langle X_n| [x, p] |X_m\rangle = \frac{1}{2i\pi}(n-m)\Delta_1(n-m). \tag{4.49}$$

4.4 Displacement operator

In finite quantum systems, the position and momentum are integers modulo D and we have two kinds of unitary operators, which are perform displacements along the P and X axis in quantum systems. The unitary operators are [55, 6]:

$$E = \exp\left(-\frac{2i\pi}{D}p\right) \qquad C = \exp\left(\frac{2i\pi}{D}x\right) \qquad E^{\dagger} = \exp\left(\frac{2i\pi}{D}p\right), \tag{4.50}$$

where $E, C, E^{\dagger} : \mathcal{H} \to \mathcal{H}$, and these operators satisfy the following relations:

1.

$$C^{\alpha}|P_m\rangle = |P_{m+\alpha}\rangle \qquad C^{\alpha}|X_m\rangle = \exp\left(\frac{2i\pi}{D}m\alpha\right)|X_m\rangle\,. \tag{4.51}$$

2.

$$E^{\beta}\left|P_m\right\rangle = \exp\left(-\frac{2i\pi}{D}m\beta\right)\left|P_m\right\rangle \qquad E^{\beta}\left|X_m\right\rangle = \left|X_{m+\beta}\right\rangle. \tag{4.52}$$

3.

$$C^d = E^d = \mathbb{I} \qquad E^{\beta}C^{\alpha} = \exp\left(\frac{-2i\pi}{D}\alpha\beta\right)C^{\alpha}E^{\beta}. \tag{4.53}$$

Proof.

$$E^{\beta}C^{\alpha} = \exp\left(\frac{-2i\pi}{D}\alpha\beta\right)C^{\alpha}E^{\beta}, \tag{4.54}$$

by using Eqs (4.51), (4.52) we have:

$$\begin{aligned}
&E^{\beta}\exp\left(\frac{2i\pi}{D}m\alpha\right)\left|X_m\right\rangle = \exp\left(-\frac{2i\pi}{D}\alpha\beta\right)C^{\alpha}\left|X_{m+\beta}\right\rangle \\
&\exp\left(\frac{2i\pi}{D}m\alpha\right)E^{\beta}\left|X_m\right\rangle = \exp\left(\frac{-2i\pi}{D}\alpha\beta\right)C^{\alpha}\left|X_{m+\beta}\right\rangle \\
&\exp\left(\frac{2i\pi}{D}m\alpha\right)\left|X_{m+\beta}\right\rangle = \exp\left(\frac{-2i\pi}{D}\alpha\beta\right)\exp\left(\frac{2i\pi}{D}(m+\beta)\alpha\right)\left|X_{m+\beta}\right\rangle \\
&\exp\left(\frac{2i\pi}{D}m\alpha\right)\left|X_{m+\beta}\right\rangle = \exp\left(\frac{2i\pi}{D}m\alpha\right)\left|X_{m+\beta}\right\rangle \\
&\Rightarrow \left|X_{m+\beta}\right\rangle = \left|X_{m+\beta}\right\rangle.
\end{aligned} \tag{4.55}$$

Then:

$$E^{\beta}C^{\alpha} = \exp\left(\frac{-2i\pi}{D}\alpha\beta\right)C^{\alpha}E^{\beta}. \tag{4.56}$$

□

4.

$$E^{\dagger}\left|X_m\right\rangle = \left|X_{m-1}\right\rangle \qquad E^{\dagger}\left|P_n\right\rangle = \exp\left(\frac{2i\pi}{D}n\right)\left|P_n\right\rangle, \tag{4.57}$$

that means:

$$(E^{\dagger})^{\alpha}\left|X_m\right\rangle = \left|X_{m-\alpha}\right\rangle \qquad (E^{\dagger})^{\alpha}\left|P_n\right\rangle = \exp\left(\frac{2i\pi}{D}n\alpha\right)\left|P_n\right\rangle. \tag{4.58}$$

5. If $\alpha = 1$, then from Eq (4.51), we obtain:

$$C\left|X_m\right\rangle = \exp\left(\frac{2i\pi}{D}m\right)\left|X_m\right\rangle \qquad C\left|P_n\right\rangle = \left|P_{n+1}\right\rangle. \tag{4.59}$$

Also:

$$C^{\beta}\left|X_m\right\rangle = \exp\left(\frac{2i\pi}{D}m\beta\right)\left|X_m\right\rangle \qquad C^{\beta}\left|P_n\right\rangle = \left|P_{n+\beta}\right\rangle. \tag{4.60}$$

6.

$$(E^{\dagger})^{D} = C^{D} = \mathbb{I} \qquad (E^{\dagger})^{\alpha}C^{\beta} = \exp\left(\frac{2i\pi}{D}\alpha\beta\right)C^{\beta}(E^{\dagger})^{\alpha}, \tag{4.61}$$

where α, β are integers in $\mathbb{Z}(D)$.

In the same way that proved Eq (4.53), we can prove this relation by using Eq(4.58)and Eq(4.60).

The general displacement operators with respect to α, β are defined as :

$$\mathfrak{D}(\alpha,\beta), = C^{\alpha}E^{\beta}\exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right) \qquad [\mathfrak{D}(\alpha,\beta)]^{\dagger} = \mathfrak{D}(-\alpha,-\beta), \tag{4.62}$$

also the displacement operators are single value and unitary operators.

The displacement operators have some relations which are:

1. The multiplication relation.

$$\begin{aligned}\mathfrak{D}(\alpha_1,\beta_1)\mathfrak{D}(\alpha_2,\beta_2) =& \mathfrak{D}(\alpha_1+\alpha_2,\beta_1+\beta_2)\\ &\times \exp\left(\frac{i\pi}{D}(\alpha_1\beta_2-\alpha_2\beta_1)\right).\end{aligned} \tag{4.63}$$

Proof.

$$\begin{aligned}\mathfrak{D}(\alpha_1,\beta_1)\mathfrak{D}(\alpha_2,\beta_2) =& C^{\alpha_1}E^{\beta_1}\exp\left(-2\,2^{-1}\frac{i\pi}{D}\alpha_1\beta_1\right)C^{\alpha_2}E^{\beta_2}\\ &\times \exp\left(-2\,2^{-1}\frac{i\pi}{D}\alpha_2\beta_2\right)\\ =& \exp\left(-\frac{i\pi}{D}\alpha_1\beta_1 - \frac{i\pi}{D}\alpha_2\beta_2\right)\\ &\times C^{\alpha_1}E^{\beta_1}C^{\alpha_2}E^{\beta_2},\end{aligned} \tag{4.64}$$

Use Eq (4.53):

$$\begin{aligned}\mathfrak{D}(\alpha_1,\beta_1)\mathfrak{D}(\alpha_2,\beta_2) =& \exp\left(-\frac{i\pi}{D}\alpha_1\beta_1 - \frac{i\pi}{D}\alpha_2\beta_2\right)\\ &\times C^{\alpha_1}C^{\alpha_2}E^{\beta_1}E^{\beta_2}\exp\left(\frac{-2i\pi}{D}\alpha_2\beta_1\right)\\ =& \exp\left(-\frac{i\pi}{D}\alpha_1\beta_1 - \frac{i\pi}{D}\alpha_2\beta_2 - 2\frac{i\pi}{D}\alpha_2\beta_1\right)\\ &\times C^{\alpha_1+\alpha_2}E^{\beta_1+\beta_2}\\ =& C^{\alpha_1+\alpha_2}E^{\beta_1+\beta_2}\exp\left(-2^{-1}2\frac{i\pi}{D}(\alpha_1+\alpha_2)(\beta_1+\beta_2)\right)\\ &\times\exp\left(\frac{i\pi}{D}(\alpha_1\beta_2-\alpha_2\beta_1)\right)\\ =& \mathfrak{D}(\alpha_1+\alpha_2,\beta_1+\beta_2)\exp\left(\frac{i\pi}{D}(\alpha_1\beta_2-\alpha_2\beta_1)\right).\end{aligned} \tag{4.65}$$

□

2.

$$\mathfrak{D}(\alpha,\beta)x[\mathfrak{D}(\alpha,\beta)]^\dagger, = x - \beta\mathbb{I}. \tag{4.66}$$

Proof.

$$\begin{aligned}
\mathfrak{D}(\alpha,\beta)x[\mathfrak{D}(\alpha,\beta)]^\dagger =& C^\alpha E^\beta \exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)\sum_{m=1}^{D-1} m\left|X_m\right\rangle\left\langle X_m\right| \\
& \times C^{-\alpha}E^{-\beta}\exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right) \\
=& \sum_{m=1}^{D-1} m\exp\left(\frac{-2\,2^{-1}2i\pi}{D}\alpha\beta\right) \\
& \times C^\alpha E^\beta\left|X_m\right\rangle\left\langle X_m\right| C^{-\alpha}E^{-\beta} \\
=& \sum_{m=1}^{D-1} m\exp\left(\frac{-2i\pi}{D}\alpha\beta\right) \\
& \times C^\alpha\left|X_m+\beta\right\rangle\left\langle X_m\right|\exp\left(-\frac{2i\pi}{D}\alpha m\right)E^{-\beta} \\
=& \sum_{m=1}^{D-1} m\exp\left(\frac{-2i\pi}{D}\alpha\beta\right)\exp\left(\frac{-2i\pi}{D}\alpha m\right) \\
& \times\exp\left(\frac{2i\pi}{D}\alpha(m+\beta)\right)\left|X_{m+\beta}\right\rangle\left\langle X_{m+\beta}\right| \\
=& \sum_{m=1}^{D-1} m\left|X_{m+\beta}\right\rangle\left\langle X_{m+\beta}\right|.
\end{aligned} \tag{4.67}$$

We know that $\left\langle X_m\right| E^\beta = \left\langle X_{m-\beta}\right|$, and put $n = m+\beta \Rightarrow m = n-\beta$

$$\begin{aligned}
\mathfrak{D}(\alpha,\beta)x[\mathfrak{D}(\alpha,\beta)]^\dagger =& \sum_{n=1}^{D-1}(n-\beta)\left|X_n\right\rangle\left\langle X_n\right| \\
=& \sum_{n=1}^{D-1} n\left|X_n\right\rangle\left\langle X_n\right| - \beta\sum_{n=1}^{D-1}\left|X_n\right\rangle\left\langle X_n\right| \\
=& x - \beta\mathbb{I}.
\end{aligned} \tag{4.68}$$

□

3.

$$\mathfrak{D}(\alpha,\beta)p[\mathfrak{D}(\alpha,\beta)]^{\dagger}, = x - \alpha\mathbb{I}, \tag{4.69}$$

to prove this one uses a similar way as the previous one.

4.

$$\mathfrak{D}(\alpha,\beta)\left|X_m\right\rangle = \exp\left(\frac{2\pi i}{D}(2^{-1}\alpha\beta + \alpha m)\right)\left|X_{m+\beta}\right\rangle. \tag{4.70}$$

Proof.

$$\begin{aligned}
\mathfrak{D}(\alpha,\beta)\left|X_m\right\rangle &= C^{\alpha}E^{\beta}\exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)\left|X_m\right\rangle \\
&= \exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)C^{\alpha}E^{\beta}\left|X_m\right\rangle \\
&= \exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)C^{\alpha}\left|X_{m+\beta}\right\rangle \\
&= \exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)\exp\left(\frac{2i\pi}{D}\alpha(m+\beta)\right)\left|X_{m+\beta}\right\rangle \\
&= \exp\left(\frac{-2^{-1}2i\pi}{D}\alpha\beta\right)\exp\left(\frac{2i\pi}{D}\alpha m\right)\exp\left(\frac{2i\pi}{D}\alpha\beta\right)\left|X_{m+\beta}\right\rangle,
\end{aligned} \tag{4.71}$$

then:

$$\begin{aligned}
\mathfrak{D}(\alpha,\beta)\left|X_m\right\rangle &= \exp\left(\frac{i\pi}{D}\alpha\beta\right)\exp\left(\frac{2i\pi}{D}\alpha m\right)\left|X_{m+\beta}\right\rangle \\
&= \exp\left(\frac{2\pi i}{D}(2^{-1}\alpha\beta + \alpha m)\right)\left|X_{m+\beta}\right\rangle.
\end{aligned} \tag{4.72}$$

□

5.

$$\mathfrak{D}(\alpha,\beta)\left|P_m\right\rangle = \exp\left(\frac{2\pi i}{D}(-2^{-1}\alpha\beta - \beta m)\right)\left|P_{m+\alpha}\right\rangle. \tag{4.73}$$

Also, we can prove this in the same way we proved the previous relation.

6.

$$\mathbb{F}E\mathbb{F}^{\dagger} = C. \tag{4.74}$$

Proof.

$$\begin{aligned}\mathbb{F}E\mathbb{F}^{\dagger} =& \mathbb{F}\exp\left(-\frac{2i\pi}{D}p\right)\mathbb{F}^{\dagger}\\ =& \mathbb{F}\exp\left(-\frac{2i\pi}{D}\sum_n n\left|P_n\right\rangle\left\langle P_n\right|\right)\mathbb{F}^{\dagger}\\ =& \exp\left(-\frac{2i\pi}{D}\sum_n n\mathbb{F}\left|P_n\right\rangle\left\langle P_n\right|\mathbb{F}^{\dagger}\right).\end{aligned} \tag{4.75}$$

We know that $\mathbb{F}\left|P_n\right\rangle = \left|X_{-n}\right\rangle$ and $\left\langle P_n\right|\mathbb{F}^{\dagger} = \left\langle X_{-n}\right|$, then:

$$\mathbb{F}E\mathbb{F}^{\dagger} = \exp\left(-\frac{2i\pi}{D}\sum_n n\left|X_{-n}\right\rangle\left\langle X_{-n}\right|\right). \tag{4.76}$$

Now put $n = -m$ then:

$$\begin{aligned}\mathbb{F}E\mathbb{F}^{\dagger} =& \exp\left(-\frac{2i\pi}{D}\sum_{m} -m\left|X_m\right\rangle\left\langle X_m\right|\right)\\ =& \exp\left(\frac{2i\pi}{D}x\right) = C.\end{aligned} \tag{4.77}$$

□

7.

$$\mathbb{F}C\mathbb{F}^{\dagger} = E. \tag{4.78}$$

This relation can be proved in a similar way as the previous one.

8.

$$\mathbb{F}\mathfrak{D}(\alpha,\beta)\mathbb{F}^{\dagger} = \mathfrak{D}(\beta,-\alpha). \tag{4.79}$$

4.5 Analytic representation

We consider an arbitrary (normalized) state [55, 6]:

$$\begin{aligned}\left|f\right\rangle =& \sum_{m=0}^{D-1} f_m\left|X_m\right\rangle = \sum_{m=0}^{D-1}\tilde{f}_m\left|P_m\right\rangle; \qquad \sum_{m=0}^{D-1}\left|f_m\right|^2 = 1,\\ \tilde{f}_m =& D^{-\frac{1}{2}}\sum_{n=0}^{D-1}\exp\left(-\frac{2i\pi nm}{D}\right)f_n.\end{aligned} \tag{4.80}$$

We have used these nations:

$$|f^*\rangle = \sum_{m=0}^{D-1} f_m^* |X_m\rangle ; \qquad \langle f| = \sum_{m=0}^{D-1} f_m^* \langle X_m| ;$$
$$\langle f^*| = \sum_{m=0}^{D-1} f_m \langle X_m| . \tag{4.81}$$

The analytic representation of $|f\rangle$ is defined as:

$$F(z) = \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} \Theta_3 \left(\frac{\pi m}{D} - z\frac{\pi}{D}; \frac{i}{D} \right) f_m, \tag{4.82}$$

where Θ_3 is the theta function as mentioned, which is defined as:

$$\Theta_3(v;\tau) = \sum_{n=-\infty}^{\infty} \exp\left(i\pi\tau n^2\right) \exp(2niv). \tag{4.83}$$

We also have very important properties of the theta function which are:

$$\Theta_3(v;\tau) = (-i\tau)^{-\frac{1}{2}} \exp\left(\frac{v^2}{i\pi\tau} \right) \Theta_3\left(\frac{v}{\tau}; \frac{-1}{\tau} \right), \tag{4.84}$$

this property also has mentioned.

We have the following properties of theta function which are:

$$\Theta_3(v+\tau\pi;\tau) = \exp(-2iv)\exp(-i\tau\pi)\Theta_3(v;\tau), \tag{4.85}$$

and

$$\Theta_3(v+M\tau\pi;\tau) = \exp(-2iMv)\exp\left(-i\tau\pi M^2\right)\Theta_3(v;\tau). \tag{4.86}$$

Also we have periodic relations which we can then prove easily:

$$F(z+D) = F(z), \tag{4.87}$$

and

$$F(z+iD) = F(z)\exp(\pi D - 2i\pi z). \tag{4.88}$$

Where $F(z)$ is defined on $\mathbf{S}$ which is a square area of complex plain and is defined by:

$$\mathbf{S} = [hD, hD+D) \times [kD, kD+D); \quad h, k \in \mathbb{Z}. \tag{4.89}$$

We can prove the first periodic relation as the same way in [7] as follows:

$$\begin{aligned}
F(z+D) &= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} \Theta_3 \left(\frac{\pi m}{D} - (z+D)\frac{\pi}{D}; \frac{i}{D} \right) f_m \qquad (4.90)\\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \sum_{n=-\infty}^{\infty} \exp\left(\frac{i^2\pi}{D} n^2 + 2in(\frac{\pi m}{D} - (z+D)\frac{\pi}{D}) \right)\\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \sum_{n=-\infty}^{\infty} \exp\left(\frac{i^2\pi}{D} n^2 + 2in(\frac{\pi m}{D} - z\frac{\pi}{D}) - 2in\pi \right)\\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \sum_{n=-\infty}^{\infty} \exp\left(\frac{i^2\pi}{D} n^2 + 2in(\frac{\pi m}{D} - z\frac{\pi}{D}) \right) \exp(-2in\pi)\\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \Theta_3 \left(\frac{\pi m}{D} - z\frac{\pi}{D}; \frac{i}{D} \right) = F(z).
\end{aligned}$$

We can prove the second periodic relation by using Eq (4.86) as follows:

$$
\begin{aligned}
F(z+iD) &= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} \Theta_3\left(\frac{\pi m}{D} - (z+iD)\frac{\pi}{D}; \frac{i}{D}\right) f_m \qquad (4.91)\\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \Theta_3\left(\frac{\pi m}{D} - z\frac{\pi}{D} - i\pi; \frac{i}{D}\right), \quad \text{since } M = -D \\
&= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \exp\left(2iD(\frac{\pi m}{D} - z\frac{\pi}{D})\right) \exp\left(-i\frac{i}{D}\pi D^2\right) \Theta_3\left(\frac{\pi m}{D} - z\frac{\pi}{D}; \frac{i}{D}\right),
\end{aligned}
$$

then:

$$
\begin{aligned}
F(z+iD) &= \pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} f_m \Theta_3\left(\frac{\pi m}{D} - z\frac{\pi}{D}; \frac{i}{D}\right) \exp(2i\pi m - 2iz\pi)) \exp(\pi D) \qquad (4.92)\\
&= F(z)\exp(\pi D - 2iz\pi).
\end{aligned}
$$

The coefficients $f_m, \tilde{f}_m$ are given by:

$$
\begin{aligned}
f_m =& \frac{1}{\sqrt{2}} \pi^{-\frac{3}{4}} D^{-\frac{3}{2}} \int_S d\nu(z) \Theta_3\left(\frac{\pi m}{D} - \frac{z\pi}{D}; \frac{i}{D}\right) F(z^*), \qquad (4.93)\\
\tilde{f}_m =& \frac{1}{\sqrt{2}} \pi^{-\frac{3}{4}} D^{-\frac{3}{2}} D^{-\frac{1}{2}} \sum_{n=0}^{D-1} \exp\left(-\frac{2i\pi nm}{D}\right) \int_S d\nu(z) \Theta_3\left(\frac{\pi n}{D} - \frac{z\pi}{D}; \frac{i}{D}\right) F(z^*).
\end{aligned}
$$

The following function represents the momentum state $|P_n\rangle$.

$$
\begin{aligned}
F(z;l) =& \pi^{-\frac{1}{4}} D^{-\frac{1}{2}} \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi ml}{D}\right) \Theta_3\left(\frac{\pi m}{D} - \frac{z\pi}{D}; \frac{i}{D}\right) \qquad (4.94)\\
=& \pi^{-\frac{1}{4}} \exp\left(-\frac{\pi z^2}{D}\right) \Theta_3\left(\frac{\pi l}{D} + \frac{iz\pi}{D}; \frac{i}{D}\right).
\end{aligned}
$$

We prove this equation as follows [21]:

$$
\begin{aligned}
&\pi^{-\frac{1}{4}} D^{-\frac{1}{2}} \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi ml}{D}\right) \Theta_3\left(\frac{\pi m}{D} - \frac{z\pi}{D}; \frac{i}{D}\right) \\
=&\pi^{-\frac{1}{4}} D^{-\frac{1}{2}} \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi ml}{D}\right) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{D} + \frac{2i\pi nm}{D} - \frac{2i\pi nz}{D}\right) \\
=&\pi^{-\frac{1}{4}} D^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{D} - \frac{2i\pi nz}{D}\right) \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi ml}{D} + \frac{2i\pi nm}{D}\right) \\
=&\pi^{-\frac{1}{4}} D^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi n^2}{D} - \frac{2i\pi nz}{D}\right) \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi m}{D}(l+n)\right),
\end{aligned}
\tag{4.95}
$$

let $n = -l + DN$ then the last part of Eq (4.95) is:

$$
\begin{aligned}
\sum_{m=0}^{D-1} \exp\left(\frac{2i\pi m}{D}(l+n)\right) &= \sum_{m=0}^{D-1} \exp\left(\frac{2i\pi m}{D}(l-l+DN)\right) \\
&= \sum_{m=0}^{D-1} \exp(2i\pi mN) = \sum_{m=0}^{D-1} 1 = D,
\end{aligned}
\tag{4.96}
$$

then from Eq (4.96) in Eq (4.95) we get:

$$\pi^{-\frac{1}{4}}D^{-\frac{1}{2}}\sum_{m=0}^{D-1}\exp\left(\frac{2i\pi ml}{D}\right)\Theta_3\left(\frac{\pi m}{D}-\frac{z\pi}{D};\frac{i}{D}\right) \tag{4.97}$$
$$=\pi^{-\frac{1}{4}}\frac{1}{D^{-\frac{1}{2}}}\sum_{n=-l+DN}^{\infty}\exp\left(-\frac{\pi n^2}{D}-\frac{2i\pi nz}{D}\right).D$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\sum_{n=-l+DN}^{\infty}\exp\left(-\frac{\pi n^2}{D}-\frac{2i\pi nz}{D}\right)$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi(-l+DN)^2}{D}-\frac{2i\pi(-l+DN)z}{D}\right)$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{(\pi l^2-2\pi lDN+\pi D^2N^2)}{D}-\frac{(-2i\pi lz+2i\pi DNz)}{D}\right)$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi l^2}{D}+2\pi lN-\pi DN^2+\frac{2i\pi lz}{D}-2i\pi Nz\right)$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi l^2}{D}+\frac{2i\pi lz}{D}\right)\exp\big(2\pi ln-\pi Dn^2-2i\pi nz\big))$$
$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\exp\left(-\frac{\pi l^2}{D}+\frac{2i\pi lz}{D}\right)\Theta_3(-i\pi l-z\pi;iD).$$

Now we use Eq (4.84) as follows:

$$=\pi^{-\frac{1}{4}}D^{\frac{1}{2}}\exp\left(-\frac{\pi l^2}{D}+\frac{2i\pi lz}{D}\right)D^{-\frac{1}{2}}\exp\left(\frac{(-i\pi l-z\pi)^2}{-\pi D}\right)\Theta_3(\frac{-i\pi l-z\pi}{iD};-\frac{1}{iD}) \tag{4.98}$$
$$=\pi^{-\frac{1}{4}}\exp\left(-\frac{\pi l^2}{D}+\frac{2i\pi lz}{D}\right)\exp\left(\frac{\pi l^2}{D}-\frac{2izl\pi}{D}-\frac{\pi z^2}{D}\right)\Theta_3(\frac{\pi l}{D}-\frac{z\pi}{iD};\frac{i}{D})$$
$$=\pi^{-\frac{1}{4}}\exp\left(-\frac{\pi z^2}{D}\right)\Theta_3(\frac{\pi l}{D}+\frac{iz\pi}{D};\frac{i}{D}).$$

The scalar product of finite systems is given by:

$$\langle f_1|f_2\rangle = \frac{\sqrt{2}}{D^{\frac{5}{2}}}\int_S (F_1(z))^* F_2(z)d\nu, \quad d\nu = d^2z\exp\left(-\frac{2\pi}{D}z_I^2\right), \tag{4.99}$$

where $z = z_R + z_I$, $d^2z = d_{z_R}d_{z_I}$, and z_R, z_I are the real and imaginary parts of z respectively.

4.5.1 Displacement operators in analytic representation

We can express the displacement operators E and C in the context of analytic representations. We are written Eqs (4.51) and (4.52) as:

$$\begin{aligned} EF(z) &= F(z - \frac{2\pi}{D}), \\ CF(z) &= F(z + \frac{2\pi}{D})\exp\left(zi\frac{2\pi}{D} - \frac{\pi}{D}\right). \end{aligned} \tag{4.100}$$

Therefore, E, C are given by:

$$\begin{aligned} E &= \exp\left(-\frac{2\pi}{D}\partial_z\right), \\ C &= \exp\left(zi\frac{2\pi}{D} - \frac{\pi}{D}\right)\exp\left(i\frac{2\pi}{D}\partial_z\right). \end{aligned} \tag{4.101}$$

The general displacement operators are written as follows:

$$\mathfrak{D}(\alpha,\beta) = \exp\left(-\frac{1}{\sqrt{2}}\frac{2i\pi}{D}\alpha\beta\right)\exp\left(zi\alpha\frac{2\pi}{D} - \frac{\alpha^2\pi}{D}\right)\exp\left((\alpha i - \beta)\frac{2\pi}{D}\partial_z\right). \tag{4.102}$$

Where β, α are integers in $\mathbb{Z}(D)$.

4.6 The analytic function ($F(z)$) and their zeros

The analytic function $F(z)$ has a number of zeros which is denoted by N and the zeros are denoted by ζ_n, that is $F(\zeta_n) = 0$. We are considered the following integrals [54, 55, 3, 58]:

$$\mathcal{K}_1 = \oint_L \frac{dz}{2\pi i} \frac{\partial_z F(z)}{F(z)} = \mathsf{N}; \quad \mathcal{K}_2 = \oint_L \frac{dz}{2\pi i} \frac{\partial_z F(z)}{F(z)} z = \sum_{n=0}^{D} \zeta_n. \tag{4.103}$$

The first integral $\mathcal{K}_1$ into the contour L is equal to the number of the analytic function $F(z)$ zeros, and the second integral $\mathcal{K}_2$ is equal to the summation of these zeros. The analytic function $F(z)$ has exactly D zeros ζ_n in every cell. Now we have the next relation which can be proved by using periodicity equations Eq (4.87) and Eq (4.88) [54, 55, 3, 58]:

$$\frac{1}{2\pi i} \oint_L dz \frac{\partial_z F(z)}{F(z)} = \sum_{n=0}^{D} \zeta_n = D(R + iT) + \frac{D^2}{2}(1 + i), \tag{4.104}$$

where every cell is labelled by two integers R, T The function $F(z)$ has been defined into a cell $\mathcal{C}$ which is defined as [54, 55, 3, 58]:

$$\mathcal{C} = \big[RD, (R + 1)D\big) \times \big[TD, (T + 1)D\big). \tag{4.105}$$

Where the $D - 1$ of those zeros are independent . The set which contains those $D - 1$ independent zeros, defined uniquely the quantum state. We can also write the analytic function $F(z)$ using a product of the theta function as follows [54] :

$$F(z) = \mathcal{M} \exp\left[-i\frac{2\pi}{D}Tz\right] \prod_{n=1}^{D} \Theta_3 \left[\frac{\pi}{D}(z - \zeta_n) + \frac{\pi(1+i)}{2};\ i\right]. \qquad (4.106)$$

Where $\mathcal{M}$ is the normalization constant which is not dependent on z, and T is an integer number which labels the cell [54, 58].

4.7 Summary

In this chapter, we reviewed the Jacobi theta function which has four types. We considered the Fourier transform and position and momentum states. While in the fourth section, we introduced the topic of displacement operator. The analytic representations were considered in the fifth section.

Finally, we introduced the analytic function zeros.

Chapter 5

Circuits with reversible classical gates

5.1 Introduction

Boolean algebra describes classical gates which are used in classical computation [10, 11]. Reversible gates are a particular case of classical gates [12, 13]. For instance, CNOT gates and Toffoli gates.

In this chapter, we consider reversible circuits with classical gates.

In the second section, we provide a brief summary of classical computation. Also, we consider CNOT gates and binary inputs and outputs for CNOT gates with reversible circuits. In addition, we introduce Toffoli gates and consider implementation of an an arbitrary Boolean expression with CNOT gates and Toffoli gates. Also, we consider reversible circuits with CNOT gate and Toffoli gates. Finally, we provide summery about this chapter.

5.2 Classical computation

A reversible logical gate is a logical circuit which has the same number of inputs and outputs. In other words, a reversible logical gate is a one-to-one correspondence map [12]. Here we can apply this definition to the power set which will be used in this thesis. Let $\mathcal{Y}$ be a logical circuit with r input and t output. This is a map of two power sets which are $(2^B)^n, (2^B)^m$ [23, 22, 25].
i.e

$$\mathcal{Y}(M_1, ..., M_n) = (N_1, ..., N_m); \quad M_j, N_j \in 2^B. \tag{5.1}$$

Where B is a general finite set, $(B = I)$ is the greatest element and ϕ $(\phi = 0)$ is the least element. In the binary system $B = \{1\}$ and $2^B = \{\phi, B\}$.
If $\mathcal{Y}$ is reversible logical circuit, that means there a bijective map from $(2^B)^r$ to $(2^B)^r$, and there another map which is called an inverse logical circuit. This map is denoted here as $(\mathcal{Y})^{-1}$, then we have:

$$\mathcal{Y} \circ (\mathcal{Y})^{-1} = (\mathcal{Y})^{-1} \circ \mathcal{Y} = 1, \tag{5.2}$$

where $\circ$ is a composition operation.

In this chapter, we will consider both the classical CNOT gate and the classical Toffoli gate. Furthermore, quantum CNOT and Toffoli gates will be considered later.

5.2.1 CNOT gate

The CNOT gate, as illustrated in Figure (5.1) is a bijective map from $(2^B)^2$ to $(2^B)^2$ which is defined as [23, 22, 25]:

$$\begin{aligned} \mathcal{Y}_{\text{CNOT}}(M_1, M_2) =& (M_1, M_1 \oplus M_2) \\ &= \left[M_1, (M_1 \wedge \neg M_2) \vee (M_2 \wedge \neg M_1)\right], \end{aligned} \tag{5.3}$$

and we can obtain this relation easily:

$$\mathcal{Y}_{\text{CNOT}}(M_1, M_2) = \left[\mathcal{Y}_{\text{CNOT}}(M_1, M_2)\right]^{-1}. \tag{5.4}$$

This means that the CNOT gate inverse, is a CNOT gate itself.

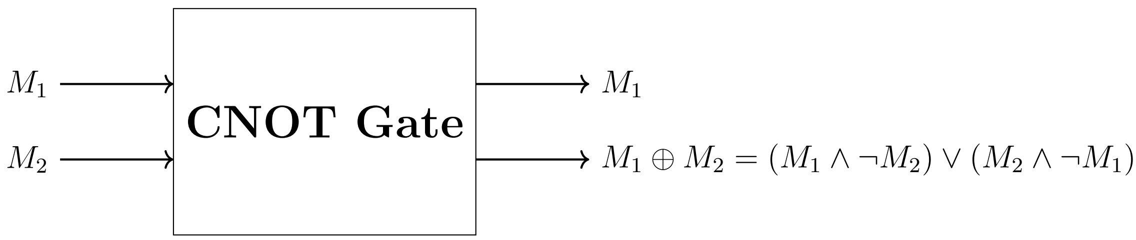

Figure 5.1: The CNOT gate.

Example 5.2.1. Let $B = \{x\}$, where B is a set, then we have its power set as follows:

$$2^B = \{\phi, \{x\}\}. \tag{5.5}$$

Where ϕ is empty set, and $\{x\}$ the biggest set. We consider this set B with binary system (0,1). We introduce the shorthand notation for its subsets as follows:

$$\phi \to 0, \quad \{x\} \to 1. \tag{5.6}$$

In Table 5.1, we provide four inputs and four outputs for a CNOT gate. Moreover, we can write the relation between inputs and outputs for a CNOT gate of a matrix system as follows:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_1, x_2, y_1, y_2 \in 2\mathbb{Z}. \tag{5.7}$$

Where $2\mathbb{Z}$ is the integers numbers set modulo 2. Also we have the following relation:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}. \tag{5.8}$$

Input(M, N)	output $(M, M \oplus N)$
$(0, 0)$	$(0, 0)$
$(0, 1)$	$(0, 1)$
$(1, 0)$	$(1, 1)$
$(1, 1)$	$(1, 0)$

Table 5.1: The CNOT gate of the set $B = \{x\}$ for the binary system.

Example 5.2.2. Let $B = \{x_1, x_2\}$, where B is a set, then we have its power set as follows:

$$2^B = \{\phi, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}. \tag{5.9}$$

Where ϕ is an empty set and $\{x_1, x_2\}$ the biggest set. We consider this set B with binary system (0,1). We introduce the short hand notation for its subsets as follows:

$$\phi \to 0, \quad \{x_1\} \to 1, \{x_2\} \to 2, \{x_1, x_2\} \to 3. \tag{5.10}$$

In Table 5.2, we provide sixteen inputs and sixteen outputs for a CNOT gate.

Input(M, N)	output $(M, M \oplus N)$
$(0, 0)$	$(0, 0)$
$(0, 1)$	$(0, 1)$
$(0, 2)$	$(0, 2)$
$(0, 3)$	$(0, 3)$
$(1, 0)$	$(1, 1)$
$(1, 1)$	$(1, 0)$
$(1, 2)$	$(1, 3)$
$(1, 3)$	$(1, 2)$
$(2, 0)$	$(2, 2)$
$(2, 1)$	$(2, 3)$
$(2, 2)$	$(2, 0)$
$(2, 3)$	$(2, 1)$
$(3, 0)$	$(3, 3)$
$(3, 1)$	$(3, 2)$
$(3, 2)$	$(3, 1)$
$(3, 3)$	$(3, 0)$

Table 5.2: The CNOT gate of the set $B = \{x_1, x_2\}$ for the binary system.

In this example, we can not represent the relation between the inputs and outputs in a matrix system similar to Eq (5.7).

5.2.2 Binary inputs and outputs for CNOT gate with reversible circuits

In this subsection, we consider the circuits with a CNOT gate and binary system (inputs and outputs). As we explained in previous subsection, we can describe CNOT gates with matrices in Eq (5.7). We explain that these circuits can be described with matrices. We use the multiplication and the direct sum of matrices.

We can explain the circuits with the matrices as follows:

1. The circuits in a sequence with the left inputs are represented with the matrices $(L_n, ..., L_2, L_1)$. These matrices are described with the multiplication of matrices as follows:

$$L = L_n.L_{n-1}.L_{n-2}...L_2.L_1. \tag{5.11}$$

   Where L_n is given in the right part of the circuit and L_{n-1} in the left part of L_n of the circuit and so on.

2. The circuits in parallel are represented with the direct sum of matrices describing different components.

We have two operations which are the multiplication of matrices and the direct sum of matrices to describe these circuits as matrices. As a result, the circuit with n inputs and n outputs can be represented by $n \times n$ matrix. An example is given to explain better these rules.

Example 5.2.3. Figure 5.2 illustrates the circuit considered. This circuit contains three circuits in a sequence as follows:

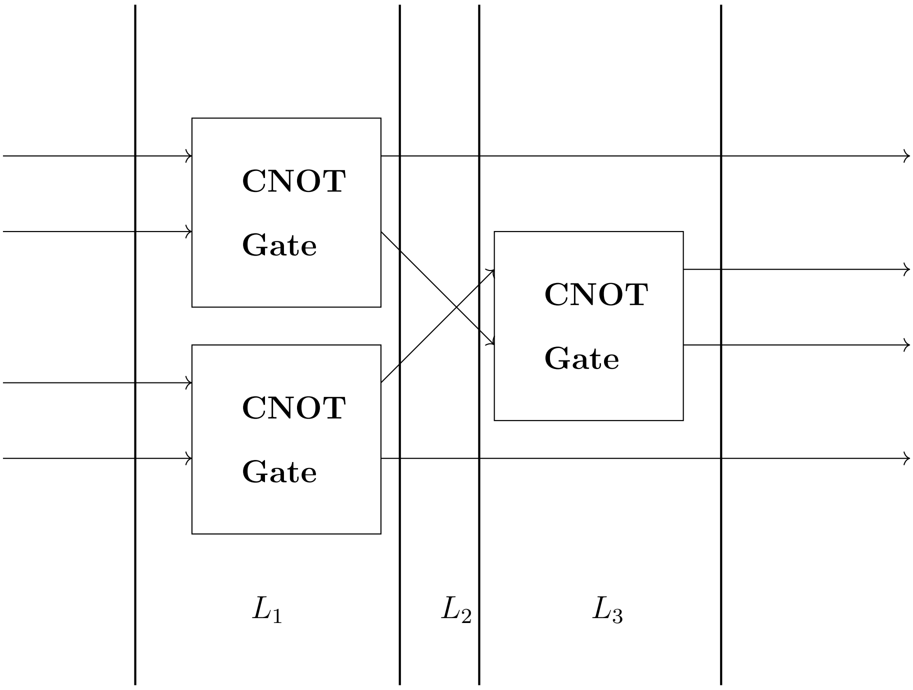

Figure 5.2: The circuit which contains CNOT gates.

We can write the matrices that represent this circuit from the right side of the circuit to the left side of it, as follows:

$$L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.12}$$

This matrix includes the direct sum of the three matrices in three dotted boxes.

$$L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.13}$$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.14}$$

Here we note that the two matrices inside the two dotted boxes of this matrix represent the two CNOT gates. The final matrix (L) which describes the whole circuit is:

$$L = L_3.L_2.L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.15}$$

Where we use here the multiplication of matrices. All the elements of matrix L belong to integers numbers set modulo 2 ($2\mathbb{Z}$).

If we inverse the arrows in Figure 5.2, then the order of the different components circuits will be inverted. The circuit is reversible, we can see that from the following relations:

$$L_1 = (L_1)^{-1}, \quad L_2 = (L_2)^{-1}, \quad L_3 = (L_3)^{-1}, \tag{5.16}$$
$$L.L^{-1} = (L_3.L_2.L_1)(L_1.L_2.L_3)^{-1} = 1.$$

5.2.3 Toffoli gate

The Toffoli gate in Figure 5.3 is a bijective map from $(2^B)^3$ to $(2^B)^3$ which is defined as [23, 22, 25]:

$$\begin{aligned} \mathcal{Y}_{\text{Toffoli}}(M_1, M_2, M_3) =& (M_1, M_2, M_1.M_2 \oplus M_3) \\ =& \Big[M_1, M_2, (M_1 \wedge M_2 \wedge \neg M_3) \vee \left[M_3 \wedge \neg (M_1 \wedge M_2) \right] \Big], \end{aligned} \tag{5.17}$$

and we can get this relation easily:

$$\mathcal{Y}_{\text{Toffoli}}(M_1, M_2, M_3) = \left[\mathcal{Y}_{\text{Toffoli}}(M_1, M_2, M_3) \right]^{-1}. \tag{5.18}$$

This means that the Toffoli gate inverse is a Toffoli gate itself.

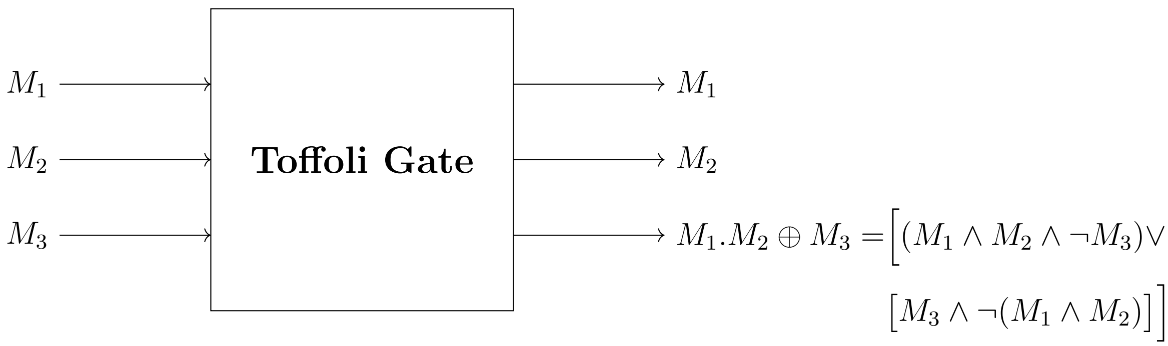

Figure 5.3: The Toffoli gate.

Here we also use the Example 5.2.1 with a Toffoli gate. In Table 5.3, we provide eight inputs and eight outputs which are actually all the inputs and outputs for a Toffoli gate. However, we can not represent the relation between the inputs and outputs in a matrix system similar to Eq (5.7) [25]:

Input(M_1, M_2, M_3)	output $\left[M_1, M_2, (M_1.M_2 \oplus M_3)\right]$
$(0,0,0)$	$(0,0,0)$
$(0,0,1)$	$(0,0,1)$
$(0,1,0)$	$(0,1,0)$
$(0,1,1)$	$(0,1,1)$
$(1,0,0)$	$(1,0,0)$
$(1,0,1)$	$(1,0,1)$
$(1,1,0)$	$(1,1,1)$
$(1,1,1)$	$(1,1,0)$

Table 5.3: The Toffoli gate of the set $B = \{x\}$ for the binary system.

5.2.4 Implementation of an arbitrary Boolean expression with CNOT and Toffoli gates

We can implement the classical OR, AND, NOT gates in terms of CNOT and Toffoli gates as follows:

i. As shown in Figure 5.4, we can implement a NOT gate in terms of a CNOT gate with these inputs M, I as follows:

$$\mathcal{Y}_{\text{CNOT}}(M, I) = (M, I \oplus M) = (M, \neg M). \tag{5.19}$$

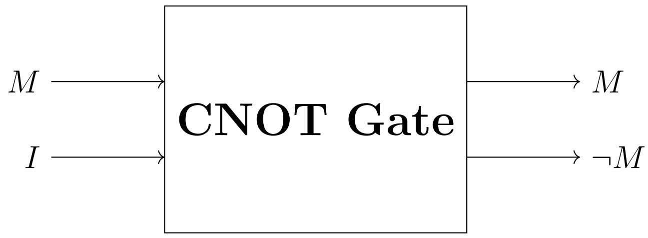

Figure 5.4: The NOT gate in terms of a CNOT gate.

Also, in Figure 5.5, we can implement a NOT gate in terms of a Toffoli gate with these inputs I, I, M as follows:

$$\mathcal{Y}_{\text{Toffoli}}(I, I, M) = (I, I, I.I \oplus M) = (I, I, I \oplus M) = (I, I, \neg M) \quad (5.20)$$

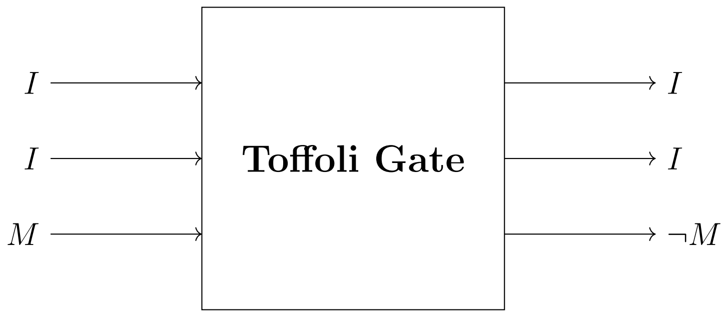

Figure 5.5: The NOT gate in terms of the Toffoli gate.

ii. As shown in Figure 5.6, we can implement an OR gate in terms of both a CNOT and Toffoli gates with these inputs $M, N, M \oplus N$ as follows:

$$\begin{aligned} \mathcal{Y}_{\text{Toffoli}}(M, N, M \oplus N) =& \big[M, N, M.N \oplus (M \oplus N)\big] \\ =& (M, N, M \vee N). \end{aligned} \quad (5.21)$$

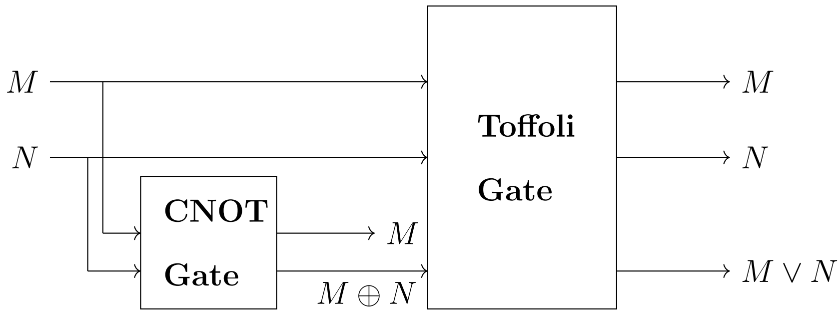

Figure 5.6: The circuit which shows OR gate in terms of the CNOT gate and the Toffoli gate.

As shown in Figure 5.7, we also can implement an OR gate in terms of a Toffoli gate only with these inputs M, I, N as follows:

$$\mathcal{Y}_{\text{Toffoli}}(M, I, N) = (M, I, M.I \oplus (M \oplus N) = (M, I, M \vee N). \qquad (5.22)$$

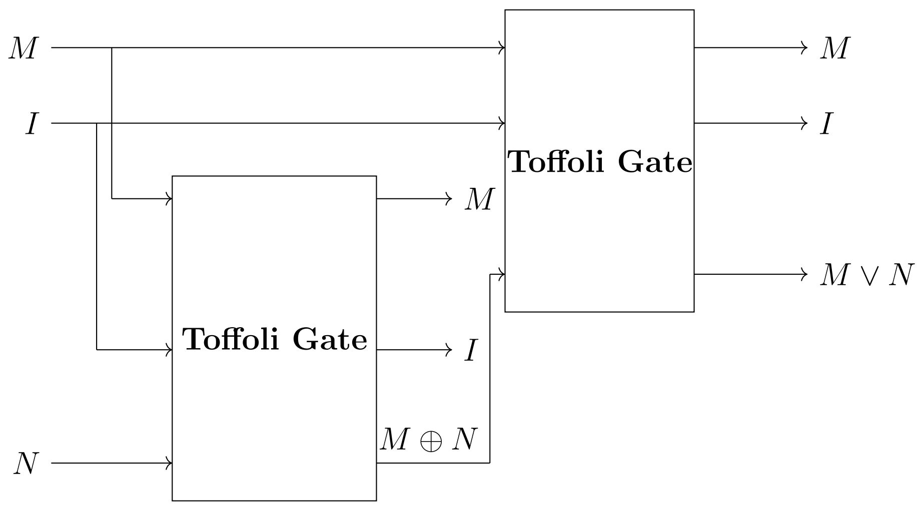

Figure 5.7: The circuit which shows OR gate in terms of the Toffoli gate only.

iii. As shown in Figure 5.8, we can implement an AND gate in terms of a Toffoli gate only with these inputs $M, N, 0$ and the first two outputs can be ignored as follows:

$$\mathcal{Y}_{\text{Toffoli}}(M, N, 0) = (M, N, M.N) = (M, N, M \wedge N). \tag{5.23}$$

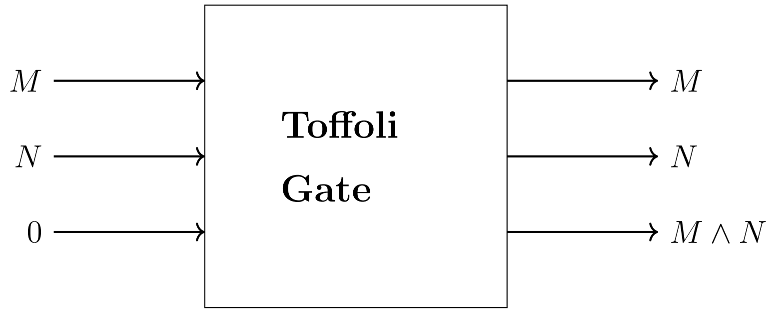

Figure 5.8: The AND gate in trems of the Toffoli gate only.

We can see that if we ignore some of the outputs, the reversibility will be lost. In general, we can express an arbitrary Boolean expression in terms of intersection, union and complementation. They can be implemented in terms of the NOT , AND and OR gates. In addition, according to the above cases, we can also implemented an arbitrary Boolean expression in terms of a Toffoli gates only, and in terms of both a Toffoli and CNOT gates.

Example 5.2.4. The half adder logical circuit is a very important logical circuit. This circuit has two binary inputs which are M_1, M_2 and two outputs which are the sum of M_1, M_2 and carry. The Boolean expression of the sum A and carry B are given as follows [17]:

$$A = (M_1 \wedge \neg M_2) \vee (M_2 \wedge \neg M_1), \quad B = M_1 \wedge M_2. \tag{5.24}$$

Using reversible gates, A can be implemented as the second output of a CNOT gate, as in Eq(5.3), and B as an AND gate. Also B can be implemented in terms of the Toffoli gate, as we studied earlier.

Example 5.2.5. We consider the Boolean expression $(M_1 \wedge M_2) \vee (\neg M_1 \wedge M_3)$. The logical circuit for this Boolean expression is shown in Figure 5.9 in terms of a Toffoli gate and CNOT gate:

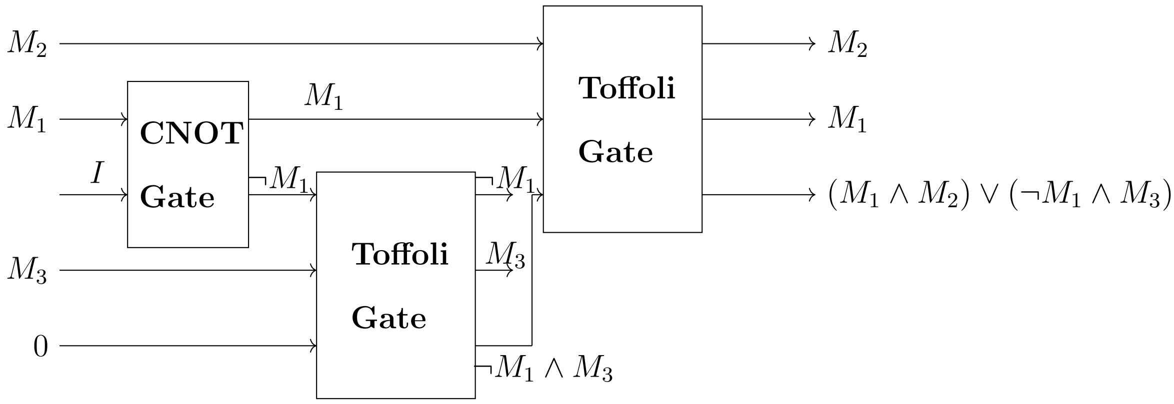

Figure 5.9: Logical ciruit which implements the Boolean expression $(M_1 \wedge M_2) \vee (\neg M_1 \wedge M_3)$ with a CNOT gate and Toffoli gate.

5.2.5 Reversible circuits with CNOT and Toffoli gates

We can describe the circuits with a matrices but is limited with a CNOT gate, and with binary inputs and outputs. Generally, we compute separately each case, for all potential input.

Example 5.2.6. We consider the circuit in Figure 5.10 which involves a Toffoli gate and CNOT gate. This circuit implements the following map:

$$\mathcal{Y}(M_1, M_2, M_3, M_4) = \Big[M_1, (M_1 \oplus M_2), M_3, \big[(M_1 \oplus M_2).M_3\big] \oplus M_4\Big]. \quad (5.25)$$

Where $\mathcal{Y} : (2^B)^4 \longrightarrow (2^B)^4$.

In Figure 5.11, we illustrate the inverse of this circuit which can be written as a function as follows:

$$\mathcal{Y}^{-1}(M_1, M_2, M_3, M_4) = \Big[M_1, (M_1 \oplus M_2), M_3, \big[(M_2.M_3)\big] \oplus M_4\Big], \quad (5.26)$$

and we easily have this relation:

$$\mathcal{Y} \circ \mathcal{Y}^{-1} = 1. \quad (5.27)$$

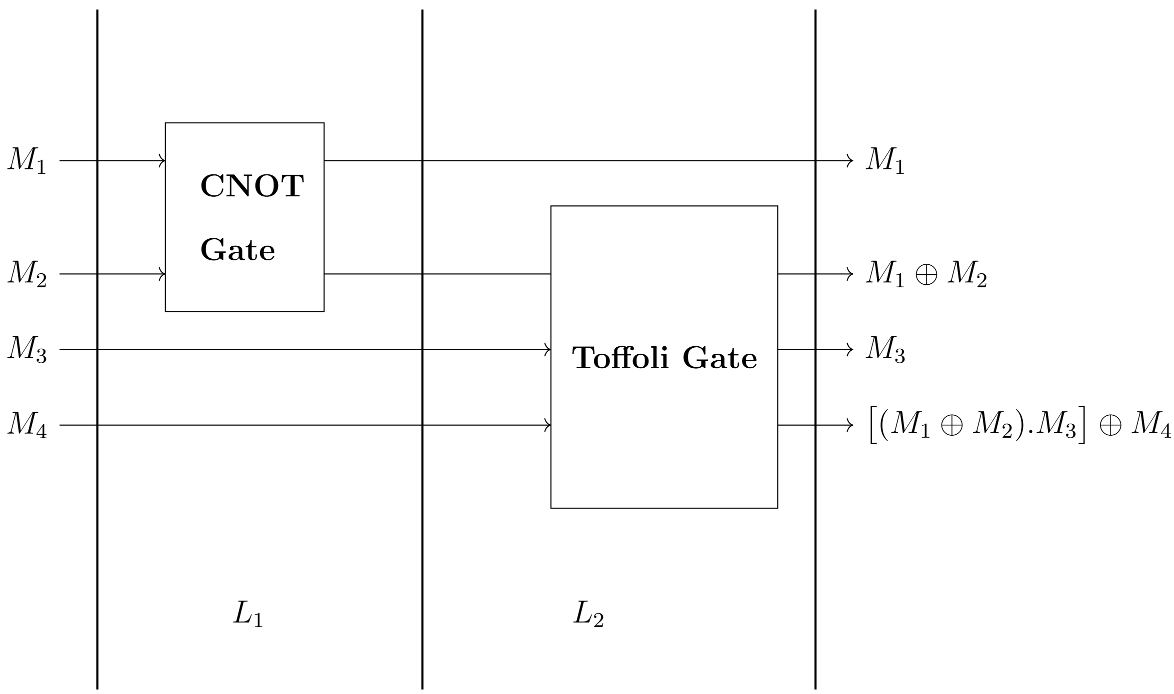

Figure 5.10: The circuit which contains the CNOT gate and the Toffoli gate.

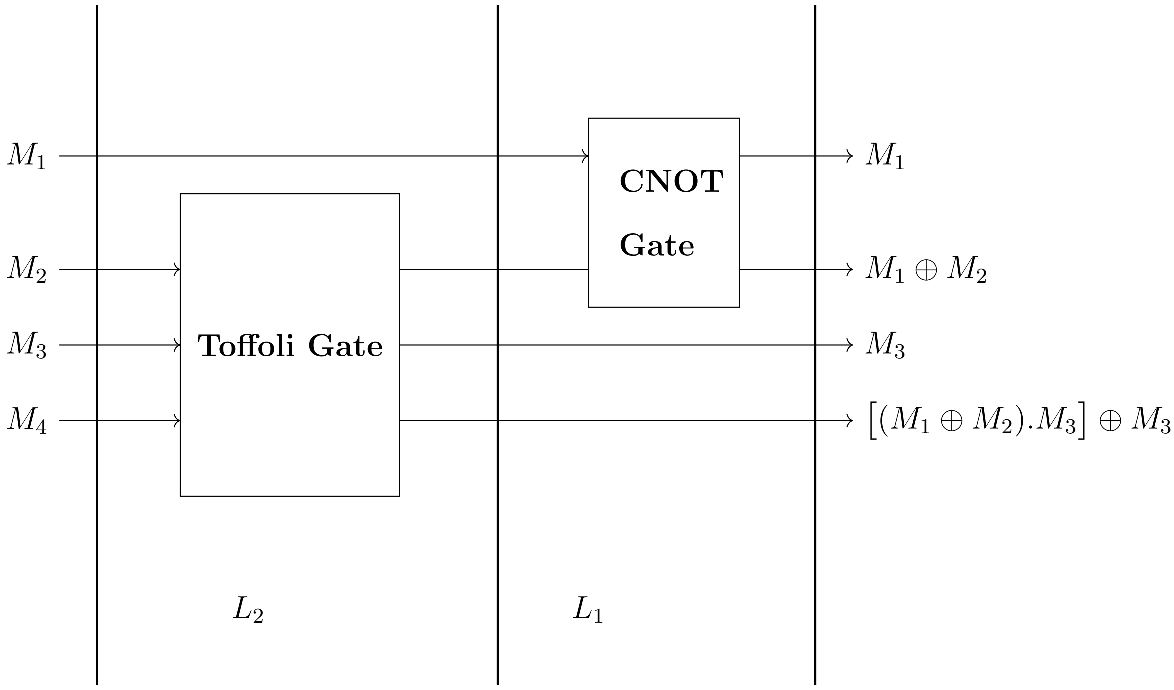

Figure 5.11: The inverse circuit for the circuit in Figure 5.10.

Table 5.4, shows the outputs for all potential inputs of the circuit in Figure 5.10, and here we suppose the binary inputs and outputs.

Input(M_1, M_2, M_3, M_4)	$M_1, M_1 \oplus M_2, M_3,$ output $\big[(M_1 \oplus M_2).M_3\big] \oplus M_3\big]$
$(0,0,0,0)$	$(0,0,0,0)$
$(0,0,0,1)$	$(0,0,0,1)$
$(0,0,1,0)$	$(0,0,1,0)$
$(0,0,1,1)$	$(0,0,1,1)$
$(0,1,0,0)$	$(0,1,0,0)$
$(0,1,0,1)$	$(0,1,0,1)$
$(0,1,1,0)$	$(0,1,1,1)$
$(0,1,1,1)$	$(0,1,1,0)$
$(1,0,0,0)$	$(1,1,0,0)$
$(1,0,0,1)$	$(1,1,0,1)$
$(1,0,1,0)$	$(1,1,1,1)$
$(1,0,1,1)$	$(1,1,1,0)$
$(1,1,0,0)$	$(1,0,0,0)$
$(1,1,0,1)$	$(1,0,0,1)$
$(1,1,1,0)$	$(1,0,1,0)$
$(1,1,1,1)$	$(1,0,1,1)$

Table 5.4: The binary inputs and outputs for the circuit in Figure 5.10.

5.3 Summary

In this chapter, we studied reversible circuits with classical gates. We studied reversible circuits with CNOT and Toffoli gates. The CNOT gates were considered using a number of examples. Also, we studied the reversible circuits with CNOT gates and binary inputs and outputs. The Toffoli gates were considered using some examples.

Finally, implementation of an arbitrary Boolean expression and reversible circuits with CNOT and Toffoli gates were studied.

Chapter 6

Quantum gates : a study using analytic representations and their zeros

6.1 Introduction

We know that quantum logical gates are reversible. In this chapter we consider reversible circuits with quantum gates. In the second section, we consider quantum computation which includes CNOT and Toffoli gates.
Third section consider analytic representations and their zeros. Also, we consider zeros of the inputs and outputs for CNOT and Toffoli gates, as well as we provide numerical examples. Moreover, we compute the zeros on the output approximately. In addition, we consider some quantum circuits.
Finally, we conclude with a summary about what we done in this chapter.

6.2 Quantum computation

6.2.1 CNOT gate

The unitary transformations in the controlled quantum machine is given by the following relation [59, 60]:

$$\begin{aligned} |u\rangle \otimes |v\rangle \longrightarrow & (1 \otimes \mathbb{U})(|u\rangle \otimes |v\rangle) \\ & = \big[\, |u\rangle \otimes (\mathbb{U}\,|v\rangle)\big]; \quad |u\rangle \in \mathcal{H}_1, \quad |v\rangle \in \mathcal{H}_2. \end{aligned} \tag{6.1}$$

Where $|u\rangle, |v\rangle$ are the quantum states in the control and target inputs respectively. The Hilbert spaces of these quantum states at the control and target inputs are $\mathcal{H}_1, \mathcal{H}_2$ respectively. The control output is the same quantum state as in the control input. Here, the unitary transformations ($\mathbb{U}$) have been implemented on the target output by this gate depending on the control input. This unitary transformation ($\mathbb{U}$) could be 1 or a different unitary operator.

Example 6.2.1. We consider the Hilbert spaces ($\mathcal{H}_1, \mathcal{H}_2$), where they are two dimensional. We have the control state $|u\rangle$ is $|u_1\rangle$ or $|u_2\rangle$ or superposition of $|u_1\rangle, |u_2\rangle$. These states are orthogonal in Hilbert space $\mathcal{H}_1$. The states $|u_1\rangle, |u_2\rangle$ are represented as follows:

$$|u_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |u_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.2}$$

Also, we have the target states which are superposition of:

$$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.3}$$

Here, we have three cases:

1. If $|u\rangle = |u_1\rangle$, then the unitary transformation is:

$$\mathbb{U} = \mathbb{U}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1. \tag{6.4}$$

2. If $|u\rangle = |u_2\rangle$, then the unitary transformation is:

$$\mathbb{U} = \mathbb{U}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad (\mathbb{U}_2)^2 = 1. \tag{6.5}$$

3. If control state is the superposition $c_1 |u_1\rangle + c_2 |u_2\rangle; \quad |c_1|^2 + |c_2|^2 = 1$, then we obtain at the output:

$$\big[c_1 |u_1\rangle + c_2 |u_2\rangle\big] \otimes \mathbb{U} |v\rangle \rightarrow \big(c_1 |u_1\rangle \otimes \mathbb{U}_1 |v\rangle\big) + \big(c_2 |u_2\rangle \otimes \mathbb{U}_2 |v\rangle\big). \tag{6.6}$$

If $|u\rangle = c_1 |u_1\rangle + c_2 |u_2\rangle, \quad |v\rangle = c_3 |v_1\rangle + c_4 |v_2\rangle$, then we can rewrite Eq (6.6), using the matrix system as follows:

We have some explanations and we start with the left side of Eq (6.6), as follows:

$$c_1 |u_1\rangle + c_2 |u_2\rangle = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{6.7}$$

$$c_3 |v_1\rangle + c_4 |v_2\rangle = c_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} .$$

Then the full left side of Eq (6.6) is:

$$\begin{aligned} \big[c_1 |u_1\rangle + c_2 |u_2\rangle\big] \otimes \big[c_3 |v_1\rangle + c_4 |v_2\rangle\big] &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \otimes \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} \\ &= \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_3 \\ c_2c_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = |f\rangle . \end{aligned} \tag{6.8}$$

$$\text{Where} f_1 = c_1c_3, \;\; f_2 = c_1c_4, \;\; f_3 = c_2c_3, \;\; f_4 = c_2c_4. \tag{6.9}$$

Now the right side of Eq (6.6)is:

$$c_1 |u_1\rangle \otimes \mathbb{U}_1 |v\rangle = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ 0 \\ 0 \end{bmatrix}. \tag{6.10}$$

$$c_2 |u_2\rangle \otimes \mathbb{U}_2 |v\rangle = \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \otimes \begin{bmatrix} c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c_2c_4 \\ c_2c_3 \end{bmatrix},$$

where

$$\mathbb{U}_1 |v\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} \tag{6.11}$$

$$\mathbb{U}_2 |v\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_3 \end{bmatrix}.$$

Then the full right side of Eq (6.6) is:

$$
\begin{aligned}
\big[c_1 \left|u_1\right\rangle \otimes \mathbb{U}_1 \left|v\right\rangle \big] + \big[c_2 \left|u_2\right\rangle \otimes \mathbb{U}_2 \left|v\right\rangle \big] &= \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_2c_4 \\ c_2c_3 \end{bmatrix} \\
&= \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_4 \\ c_2c_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_3 \end{bmatrix}.
\end{aligned} \tag{6.12}
$$

Now we can rewrite the whole Eq (6.6), using the matrix system as follows:

$$
\left|f\right\rangle = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_4 \\ f_3 \end{bmatrix} = L_{CNOT} \left|f\right\rangle, \tag{6.13}
$$

where

$$
L_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{6.14}
$$

6.2.2 Toffoli gate

The unitary transformation in the controlled quantum machine is given by the following relation [59, 60]:

$$\begin{aligned} |u\rangle \otimes |k\rangle \otimes |v\rangle \longrightarrow & (1 \otimes 1 \otimes \mathbb{U})(|u\rangle \otimes |k\rangle \otimes |v\rangle) \\ = & \big[\, |u\rangle \otimes |k\rangle\, (\mathbb{U}\, |v\rangle)\big]; \\ |u\rangle \in & \mathcal{H}_1, \quad |k\rangle \in \mathcal{H}_2, \quad |v\rangle \in \mathcal{H}_3. \end{aligned} \tag{6.15}$$

Where $|u\rangle\,, |k\rangle$ are the quantum states in the first and second control inputs, respectively, and $|v\rangle$ is the quantum states in the target input. The Hilbert spaces of quantum states at the first and second control inputs and target inputs are $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, respectively. The two control outputs are in the same quantum state as the two control inputs. Here, the unitary transformation ($\mathbb{U}$) has been implemented on the target output by this gate depending on the control input. This unitary transformation ($\mathbb{U}$) could be 1 or a different unitary operator.

Example 6.2.2. We consider the Hilbert spaces ($\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$), where they are two dimensional. We have the first control state $|u\rangle$ is $|u_2\rangle$ or $|u_2\rangle$ as in Eq (6.2). Similar to the second control state $|k\rangle$ is $|k_1\rangle$ or $|k_2\rangle$ as in Eq (6.2). These states are orthogonal in Hilbert space $\mathcal{H}_1, \mathcal{H}_2$, respectively. Also we have the target states v as in Eq (6.3).
We have three cases as in Example 6.2.1 which are:

1. If $|u\rangle = |u_1\rangle$ and $|k\rangle = |k_1\rangle$ or $|u\rangle = |u_1\rangle$ and $|k\rangle = |k_2\rangle$ or $|u\rangle = |u_2\rangle$ and

$|k\rangle = |k_1\rangle$, then the unitary transformation is $\mathbb{U} = \mathbb{U}_1$ as in Eq (6.4).

2. If $|u\rangle = |u_2\rangle$ and $|k\rangle = |k_2\rangle$, then the unitary transformation is $\mathbb{U} = \mathbb{U}_2$ as in Eq (6.5).

3. If the control states are the superposition $|u\rangle = c_1|u_1\rangle + c_2|u_2\rangle\,;\ \ |c_1|^2 + |c_2|^2 = 1$ and $|k\rangle = c_3|k_1\rangle + c_4|k_2\rangle\,;\ \ |c_3|^2 + |c_4|^2 = 1$, then we obtain at the output:

$$
\begin{aligned}
&\big[(c_1|u_1\rangle + c_2|u_2\rangle) \otimes (c_3|k_1\rangle + c_4|k_2\rangle) \otimes |v\rangle\,\big] = \\
&\Big[\big[c_1|u_1\rangle \otimes c_3|k_1\rangle) + (c_1|u_1\rangle \otimes c_4|k_2\rangle\,\big] + \\
&\big[(c_2|u_2\rangle \otimes c_3|k_1\rangle) + (c_2|u_2\rangle \otimes c_4|k_2\rangle)\big]\Big] \otimes |v\rangle = \\
&\big[(c_1c_3|u_1\rangle \otimes |k_1\rangle) + (c_1c_4|u_1\rangle \otimes |k_2\rangle) + \\
&(c_2c_3|u_2\rangle \otimes |k_1\rangle) + (c_2c_4|u_2\rangle \otimes |k_2\rangle)\big] \otimes |v\rangle \to \\
&\Big[\big[(c_1c_3|u_1\rangle \otimes |k_1\rangle) + (c_1c_4|u_1\rangle \otimes |k_2\rangle) + (c_2c_3|u_2\rangle \otimes |k_1\rangle)\big] \otimes \mathbb{U}_1|v\rangle\,\Big] + \\
&\big[(c_2c_4|u_2\rangle \otimes |k_2\rangle) \otimes \mathbb{U}_2|v\rangle\,\big].
\end{aligned}
\tag{6.16}
$$

If $|u\rangle = c_1|u_1\rangle + c_2|u_2\rangle\,,\quad |k\rangle = c_3|k_1\rangle + c_4|k_2\rangle\,,\quad |v\rangle = c_5|v_1\rangle + c_6|v_2\rangle$, then we can rewrite Eq (6.16) in the matrix system in the same way as in Example 6.2.1 as follows:

We have some explanations and we start with the left side of Eq (6.16), as follows:

$$c_1c_3\,|u_1\rangle \otimes |k_1\rangle = \begin{bmatrix} c_1c_3 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1c_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.17}$$

$$c_1c_4\,|u_1\rangle \otimes |k_2\rangle = \begin{bmatrix} c_1c_4 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1c_4 \\ 0 \\ 0 \end{bmatrix}$$

$$c_2c_3\,|u_2\rangle \otimes |k_1\rangle = \begin{bmatrix} 0 \\ c_2c_3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c_2c_3 \\ 0 \end{bmatrix}$$

$$(c_2c_4\,|u_2\rangle \otimes |k_2\rangle = \begin{bmatrix} 0 \\ c_2c_4 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_2c_4 \end{bmatrix}.$$

$$[(c_1c_3 |u_1\rangle \otimes |k_1\rangle) + (c_1c_4 |u_1\rangle \otimes |k_2\rangle) + \tag{6.18}$$
$$(c_2c_3 |u_2\rangle \otimes |k_1\rangle) + (c_2c_4 |u_2\rangle \otimes |k_2\rangle) =$$
$$\begin{bmatrix} c_1c_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_1c_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_2c_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_2c_4 \end{bmatrix} = \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_3 \\ c_2c_4 \end{bmatrix}.$$

Then, the full left side of Eq (6.16) is:

$$\big[(c_1c_3 |u_1\rangle \otimes |k_1\rangle) + (c_1c_4 |u_1\rangle \otimes |k_2\rangle) + \tag{6.19}$$
$$(c_2c_3 |u_2\rangle \otimes |k_1\rangle) + (c_2c_4 |u_2\rangle \otimes |k_2\rangle)\big] \otimes |v\rangle = \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_3 \\ c_2c_4 \end{bmatrix} \otimes \begin{bmatrix} c_5 \\ c_6 \end{bmatrix}$$
$$= \begin{bmatrix} c_1c_3c_5 \\ c_1c_3c_6 \\ c_1c_4c_5 \\ c_1c_4c_6 \\ c_2c_3c_5 \\ c_2c_3c_6 \\ c_2c_4c_5 \\ c_2c_4c_6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = |f\rangle.$$

$$\text{Where } f_1 = c_1c_3c_5,\ f_2 = c_1c_3c_6,\ f_3 = c_1c_4c_5,\ f_4 = c_1c_4c_6 \qquad (6.20)$$

$$f_5 = c_2c_3c_5,\ f_6 = c_2c_3c_6,\ f_7 = c_2c_4c_5,\ f_8 = c_2c_4c_6.$$

Now the right side of Eq (6.16):

$$\begin{aligned}
&\big[(c_1c_3\,|u_1\rangle \otimes |k_1\rangle) + (c_1c_4\,|u_1\rangle \otimes |k_2\rangle) \qquad (6.21)\\
&+(c_2c_3\,|u_2\rangle \otimes |k_1\rangle)\big] \otimes \mathbb{U}_1\,|v\rangle = \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_3 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} c_5 \\ c_6 \end{bmatrix} \\
&= \begin{bmatrix} c_1c_3c_5 \\ c_1c_3c_6 \\ c_1c_4c_5 \\ c_1c_4c_6 \\ c_2c_3c_5 \\ c_2c_3c_6 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

$$(c_2c_4 |u_2\rangle \otimes |k_2\rangle) \otimes \mathbb{U}_2 |v\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_2c_4 \end{bmatrix} \otimes \begin{bmatrix} c_6 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_2c_4c_6 \\ c_2c_4c_5 \end{bmatrix}. \tag{6.22}$$

Then, the full right side of Eq (6.16) is:

$$\Big[\big[(c_1c_3 |u_1\rangle \otimes |k_1\rangle) + (c_1c_4 |u_1\rangle \otimes |k_2\rangle) + (c_2c_3 |u_2\rangle \otimes |k_1\rangle)\big] \otimes \mathbb{U}_1 |v\rangle \Big] +$$

$$\big[(c_2c_4 |u_2\rangle \otimes |k_2\rangle) \otimes \mathbb{U}_2 |v\rangle \big] = \begin{bmatrix} c_1c_3c_5 \\ c_1c_3c_6 \\ c_1c_4c_5 \\ c_1c_4c_6 \\ c_2c_3c_5 \\ c_2c_3c_6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c_2c_4c_6 \\ c_2c_4c_5 \end{bmatrix} = \begin{bmatrix} c_1c_3c_5 \\ c_1c_3c_6 \\ c_1c_4c_5 \\ c_1c_4c_6 \\ c_2c_3c_5 \\ c_2c_3c_6 \\ c_2c_4c_6 \\ c_2c_4c_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_8 \\ f_7 \end{bmatrix} = |f\rangle . \tag{6.23}$$

Now we can rewrite the whole Eq (6.16) using a matrix system as follows:

$$|f\rangle = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} \rightarrow \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_8 \\ f_7 \end{bmatrix} = L_{\text{Toffoli}} \, |f\rangle \, . \tag{6.24}$$

Where

$$L_{\text{Toffoli}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.25}$$

6.3 Analytic representations and their zeros

We used a programme in Matlab. This programme includes the analytic function $F(z_r, z_i)$, where z_r, z_i are real and imaging parts, respectively, of z. And we used different values for $c_1, c_2, c_3, c_4, c_5, c_6$ which are the inputs and outputs of CNOT and Toffoli gates as in Eq (6.7). After that, we plotted the function and cut it by using contour, then, we found the zeros of CNOT and Toffoli gates.

6.3.1 Zeros of the inputs and outputs for CNOT gates and Toffoli gates

In this section, we discuss quantum circuit with input of m qubits which represent with 2^m of corresponding analytic function on a torus, and similar to the outputs. Here, we consider the specific cases with CNOT gate and Toffoli gate where $m = 2,\ m = 3$, respectively.

Example 6.3.1. (Numerical Example)

Here we have two cases as follows:

The first case is for a CNOT gate.

We calculate numerically the zeros of the state $|f\rangle$ and $L_{\text{CNOT}}\,|f\rangle$, for the example

$$c_1 = 0.3,\ \ c_2 = 0.64,\ \ c_3 = 0.4,\ \ c_4 = 0.6. \tag{6.26}$$

The inputs and the outputs of CNOT gate belong to four dimensional Hilbert space ($D = 4$). The results for this choice are provided in Table 6.1.

We have noted that this option corresponds to a factorizable state at the CNOT gate inputs. Generally, any arbitrary state belong to 4-dimensional Hilbert space will be complicated. The zeros obey the constraint in Eq (4.104), and these zeros belong to the cell $(R = -1,\ T = 0)$. The fourth zero has not been used, because it is not independent, that means we just have used the three zeros.

It is seen that, the CNOT gate is a one to one correspondence function from $(B_4)^3 \rightarrow (B_4)^3$.

Input	output
$0.3 + 1.3\,i$	$0.2 + 1.4\,i$
$0.3 + 2.7\,i$	$0.2 + 2.7\,i$
$1.4 + 2\,i$	$0.8 + 2\,i$
$2.1 + 2\,i$	$2.8 + 2\,i$

Table 6.1: The zeros at the inputs and outputs of a CNOT gate, when $c_1 = 0.3,\ c_2 = 0.64,\ c_3 = 0.4,\ c_4 = 0.6$.

Also in Table 6.2, we obtain the different results for CNOT gate inputs and outputs when

$$c_1 = 0.1,\ c_2 = 0.42,\ c_3 = 0.2,\ c_4 = 0.4. \tag{6.27}$$

Input	output
$0.5 + 1.05\,i$	$0.25 + 1.02\,i$
$0.5 + 3\,i$	$0.54 + 2\,i$
$1.17 + 2\,i$	$0.27 + 3\,i$
$1.9 + 2\,i$	$3 + 2\,i$

Table 6.2: The zeros at the inputs and outputs of a CNOT gate, when $c_1 = 0.1,\ c_2 = 0.42,\ c_3 = 0.2,\ c_4 = 0.4$.

The last example considers the inputs and the outputs of the CNOT gate when

$$c_1 = 0.2,\ \ c_2 = 0.9,\ \ c_3 = 0.5,\ \ c_4 = 0.4. \tag{6.28}$$

The results for this choice are shown in Table 6.3.

Input	output
$0.5 + 1.02\,i$	$0.6 + 1.02\,i$
$0.3 + 2\,i$	$0.5 + 2\,i$
$0.5 + 3\,i$	$0.6 + 3\,i$
$2.7 + 2\,i$	$2.3 + 2\,i$

Table 6.3: The zeros at the inputs and outputs of a CNOT gate, when $c_1 = 0.2,\ c_2 = 0.9,\ c_3 = 0.5,\ c_4 = 0.4$.

The second case is for a Toffoli gate.

We calculate numerically the zeros of the state $|f\rangle$ and $L_{\text{Toffoli}}\,|f\rangle$, for the example

$$c_1 = 0.3,\ \ c_2 = 0.54,\ \ c_3 = 0.4,\ \ c_4 = 0.6,\ \ c_5 = 0.5,\ \ c_6 = 0.73. \tag{6.29}$$

The inputs and the outputs of Toffoli gate belong to eight dimensional Hilbert space ($D = 8$). The results for this choice are given in Table 6.4. The zeros obey the constraint in Eq (4.109), and these zeros belong to the cell ($R = -1,\ T = 0$).
It is seen that, the Toffoli gate is a one to one correspondence function from $(B_8)^7 \to (B_8)^7$.

Input	output	output using Eq (6.43)
$0.74 + 1.93\,i$	$0.49 + 1.98\,i$	$0.5274 + 1.9307\,i$
$0.74 + 6.08\,i$	$0.49 + 6.03\,i$	$0.5274 + 6.0793\,i$
$1.81 + 5.07\,i$	$1.76 + 5.12\,i$	$1.8368 + 5.0445\,i$
$1.81 + 2.95\,i$	$1.76 + 2.89\,i$	$1.8368 + 2.9755\,i$
$3.97 + 5.44\,i$	$4.04 + 5.64\,i$	$3.9839 + 5.3646\,i$
$3.97 + 2.58\,i$	$4.04 + 2.39\,i$	$3.9839 + 2.6554\,i$
$5.55 + 5.2\,i$	$4.68 + 4.01\,i$	$5.7219 + 5.5951\,i$
$5.55 + 2.82\,i$	$6.92 + 4\,i$	$5.7219 + 2.4249\,i$

Table 6.4: The zeros at the inputs and outputs of Toffoli gate, when $c_1 = 0.3,\ c_2 = 0.54,\ c_3 = 0.4,\ c_4 = 0.6,\ c_5 = 0.5,\ c_6 = 0.73$.

We obtain different result for the inputs and outputs of Toffoli gate when

$$c_1 = 0.1,\ \ c_2 = 0.42,\ \ c_3 = 0.2,\ \ c_4 = 0.4,\ \ c_5 = 0.2,\ \ c_6 = 0.52, \tag{6.30}$$

which are shown in Table 6.5.

Input	output	output using Eq (6.43)
$1.28 + 1.49\,i$	$0.87 + 1.57\,i$	$0.1097 + 1.494\,i$
$1.28 + 6.5\,i$	$0.87 + 6.45\,i$	$0.1097 + 6.496\,i$
$1.76 + 5.2\,i$	$1.59 + 5.4\,i$	$1.9078 + 5.0599\,i$
$1.76 + 2.8\,i$	$1.6 + 2.59\,i$	$1.9078 + 2.9401\,i$
$3.55 + 5.6\,i$	$3.59 + 5.84\,i$	$3.6263 + 5.185\,i$
$3.55 + 2.4\,i$	$3.59 + 2.16\,i$	$3.6263 + 2.815\,i$
$5.49 + 5.56\,i$	$4.48 + 4\,i$	$6.4363 + 7.7347\,i$
$5.49 + 2.46\,i$	$7.6 + 4\,i$	$6.4363 + 0.2853\,i$

Table 6.5: The zeros at the inputs and outputs of Toffoli gate,when $c_1 = 0.1,\ c_2 = 0.42,\ c_3 = 0.2,\ c_4 = 0.4,\ c_5 = 0.2,\ c_6 = 0.52$.

The last example considers the inputs and the outputs of Toffoli gate when

$$c_1 = 0.4,\ c_2 = 0.76,\ c_3 = 0.5,\ c_4 = 0.7,\ c_5 = 0.6,\ c_6 = 0.85 \tag{6.31}$$

The results for this choice are shown in Table 6.6.

Input	output	output using Eq (6.43)
$0.75 + 1.98\,i$	$0.5 + 2.02\,i$	$0.5496 + 1.9807\,i$
$0.75 + 6.04\,i$	$0.5 + 5.99\,i$	$0.5496 + 6.0393\,i$
$1.9 + 5.1\,i$	$1.86 + 5.16\,i$	$1.9253 + 5.076\,i$
$1.9 + 2.91\,i$	$1.84 + 2.86\,i$	$1.9253 + 2.934\,i$
$3.85 + 5.26\,i$	$3.96 + 5.44\,i$	$3.8631 + 5.189\,i$
$3.85 + 2.86\,i$	$3.96 + 2.58\,i$	$3.8631 + 2.821\,i$
$5.61 + 5.22\,i$	$4.69 + 4\,i$	$5.772 + 5.5923\,i$
$5.61 + 2.79\,i$	$6.85 + 4\,i$	$5.772 + 2.4177\,i$

Table 6.6: The zeros at the inputs and outputs of Toffoli gate, when $c_1 = 0.4,\ c_2 = 0.76,\ c_3 = 0.5,\ c_4 = 0.7,\ c_5 = 0.6,\ c_6 = 0.85$.

6.3.2 Compute the zeros on the output approximately

We have the following matrix:

$$L_{\text{Toffoli}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{6.32}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We can also rewrite this matrix by compact way as follows:

$$L_{\text{Toffoli}} = 1_{8\times 8} + \begin{bmatrix} 0_{6\times 6} & 0_{6\times 2} \\ 0_{2\times 6} & \mathcal{Q} \end{bmatrix}. \tag{6.33}$$

Where

$$\mathcal{Q} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{6.34}$$

Consequently, the output will vary a little from f_m to $f_m + \Delta f_m$, where

$$\Delta f_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -f_7 + f_8 \\ f_7 - f_8 \end{bmatrix}. \tag{6.35}$$

As a result of this, the zeros also will vary from ζ_n to $\zeta_n + \Delta\zeta_n$. Here we used a method similar to the one used in [16] for the approximative computation of $\Delta\zeta_n$, in the case where the output is the same as the input. As in the Toffoli gates, where two of the three outputs stay unchanged, we work here in a cell with $T=0$, then we have from Eqs (4.82),(4.111) the following:

$$\pi^{-\frac{1}{4}}\sum_{m=0}^{D-1}\Theta_3\left(\frac{\pi n}{D}-z\frac{\pi}{D};\frac{i}{D}\right)(f_m+\Delta f_m)=\mathcal{M}\prod_{n=1}^{D}\Theta_3\left[\frac{\pi}{D}(z-\zeta_n-\Delta\zeta_n)+\frac{\pi(1+i)}{2};\ i\right].$$

$$\begin{aligned}&\pi^{-\frac{1}{4}}\sum_{m=0}^{D-1}\Theta_3\left(\frac{\pi n}{D}-z\frac{\pi}{D};\frac{i}{D}\right)f_m\\&+\pi^{-\frac{1}{4}}\sum_{m=0}^{D-1}\Theta_3\left(\frac{\pi n}{D}-z\frac{\pi}{D};\frac{i}{D}\right)\Delta f_m=\mathcal{M}\prod_{n=1}^{D}\Theta_3\left[\frac{\pi}{D}(z-\zeta_n-\Delta\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]\end{aligned} \tag{6.36}$$

As an example we consider the case with $D=2$. Then we apply the Taylor expansion on the right hand side, then we have:

$$\begin{aligned}&\mathcal{M}\left\{\Theta_3\left[\frac{\pi}{D}(z-\zeta_1-\Delta\zeta_1)+\frac{\pi(1+i)}{2};\ i\right].\Theta_3\left[\frac{\pi}{D}(z-\zeta_2-\Delta\zeta_2)+\frac{\pi(1+i)}{2};\ i\right]\right\}+...\\&=\mathcal{M}\left\{\Theta_3\left[\frac{\pi}{D}(z-\zeta_1)+\frac{\pi(1+i)}{2};\ i\right]+\Theta_3'\left[\frac{\pi}{D}(z-\zeta_1)+\frac{\pi(1+i)}{2};\ i\right](-\frac{\pi}{D}\Delta\zeta_1)\right\}\\&\times\mathcal{M}\left\{\Theta_3\left[\frac{\pi}{D}(z-\zeta_2)+\frac{\pi(1+i)}{2};\ i\right]+\Theta_3'\left[\frac{\pi}{D}(z-\zeta_2)+\frac{\pi(1+i)}{2};\ i\right](-\frac{\pi}{D}\Delta\zeta_2)\right\}+...\\&=\mathcal{M}\left\{\Theta_3\left[\frac{\pi}{D}(z-\zeta_1)+\frac{\pi(1+i)}{2};\ i\right].\Theta_3\left[\frac{\pi}{D}(z-\zeta_2)+\frac{\pi(1+i)}{2};\ i\right]\right\}\\&+\mathcal{M}\left\{\Theta_3\left[\frac{\pi}{D}(z-\zeta_1)+\frac{\pi(1+i)}{2};\ i\right].\Theta_3'\left[\frac{\pi}{D}(z-\zeta_2)+\frac{\pi(1+i)}{2};\ i\right](-\frac{\pi}{D}\Delta\zeta_2)\right\}\\&+\mathcal{M}\left\{\Theta_3'\left[\frac{\pi}{D}(z-\zeta_1)+\frac{\pi(1+i)}{2};\ i\right].\Theta_3\left[\frac{\pi}{D}(z-\zeta_2)+\frac{\pi(1+i)}{2};\ i\right](-\frac{\pi}{D}\Delta\zeta_1)\right\}+....\end{aligned} \tag{6.37}$$

In a similar way we get:

$$= \mathcal{M} \prod_{m=1}^{D} \Theta_3 \left[\frac{\pi}{D}(z - \zeta_n) + \frac{\pi(1+i)}{2};\ i \right] -$$
$$\mathcal{M}\frac{\pi}{D} \sum_{n=1}^{D} \Theta_3 \left[\frac{\pi}{D}(z - \zeta_n) + \frac{\pi(1+i)}{2};\ i \right] .\Theta_3' \left[\frac{\pi}{D}(z - \zeta_n) + \frac{\pi(1+i)}{2};\ i \right] \Delta\zeta_n. \tag{6.38}$$

From Eq (4.111), then Eq(6.36) becomes the following:

$$\pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} \Theta_3 \left(\frac{\pi n}{D} - z\frac{\pi}{D}; \frac{i}{D} \right) \Delta f_m = -\mathcal{M}\frac{\pi}{D} \sum_{n=1}^{D} B_n(z) \Theta_3' \left[\frac{\pi}{D}(z - \zeta_n) + \frac{\pi(1+i)}{2};\ i \right] \Delta\zeta_n.$$
$$B_n(z) = \prod_{m \neq n} \Theta_3 \left[\frac{\pi}{D}(z - \zeta_m) + \frac{\pi(1+i)}{2};\ i \right]. \tag{6.39}$$

Now we insert $z = \zeta_n$ on both sides of Eq (6.39), and we suppose that n is fixed, then we obtain:

$$\pi^{-\frac{1}{4}} \sum_{m=0}^{D-1} \Theta_3 \left(\frac{\pi n}{D} - \zeta_n\frac{\pi}{D}; \frac{i}{D} \right) \Delta f_m = -\mathcal{M}\frac{\pi}{D} B_n(z) \Theta_3' \left[\frac{\pi(1+i)}{2};\ i \right] \Delta\zeta_n.$$
$$B_n(\zeta_n) = \prod_{m \neq n} \Theta_3 \left[\frac{\pi}{D}(\zeta_n - \zeta_m) + \frac{\pi(1+i)}{2};\ i \right]. \tag{6.40}$$

Therefore the derivative of ζ_n is :

$$\frac{\partial \zeta_n}{\partial f_m} = \frac{\pi^{-\frac{1}{4}} \Theta_3 \left(\frac{\pi n}{D} - \zeta_n \frac{\pi}{D}; \frac{i}{D} \right)}{-\mathcal{M}\frac{\pi}{D} B_n(z) \Theta_3' \left[\frac{\pi(1+i)}{2};\ i \right]}. \tag{6.41}$$

Where $\Theta_3'\left[\frac{\pi(1+i)}{2}i\right] = 1.9888i, \qquad n, m$ are fixed, and

$$\mathcal{M} = \frac{\pi^{-\frac{1}{4}}\sum_{m=0}^{D-1}\Theta_3\left(\frac{\pi n}{D} - z\frac{\pi}{D};\frac{i}{D}\right)f_m}{\prod_{n=1}^{D}\Theta_3\left[\frac{\pi}{D}(z-\zeta_n) + \frac{\pi(1+i)}{2};\ i\right]}. \tag{6.42}$$

Then, by using a Small Increments Formula we get :

$$\zeta_n + \Delta\zeta_n = \zeta_n + \sum_m \frac{\partial \zeta_n}{\partial f_m}\Delta f_m. \tag{6.43}$$

Because f_m is changed by a small amount (Δf_m), then the corresponding change of ζ_n $(\Delta\zeta_n)$ is $\sum_m \frac{\partial\zeta_n}{\partial f_m}\Delta f_m$.

6.3.3 Some quantum circuits

The quantum circuits methodology is the same as the classical circuit methodology. The r inputs and r outputs of a circuit work with a quantum qubits, here these inputs and outputs are represented with a $2^r \times 2^r$ matrix.

Example 6.3.2. We study the circuit that we considered early in Figure 5.10 in t he classical case. However, we take the following inputs:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}, \begin{bmatrix} c_5 \\ c_6 \end{bmatrix}, \begin{bmatrix} c_7 \\ c_8 \end{bmatrix}. \tag{6.44}$$

Then

$$\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \otimes \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}\right) \otimes \left(\begin{bmatrix} c_5 \\ c_6 \end{bmatrix} \otimes \begin{bmatrix} c_7 \\ c_8 \end{bmatrix}\right) = \begin{bmatrix} c_1c_3 \\ c_1c_4 \\ c_2c_3 \\ c_2c_4 \end{bmatrix} \otimes \begin{bmatrix} c_5c_7 \\ c_5c_8 \\ c_6c_7 \\ c_6c_8 \end{bmatrix} = \begin{bmatrix} c_1c_3c_5c_7 \\ c_1c_3c_5c_8 \\ c_1c_3c_6c_7 \\ c_1c_3c_6c_8 \\ c_1c_4c_5c_7 \\ c_1c_4c_5c_8 \\ c_1c_4c_6c_7 \\ c_1c_4c_6c_8 \\ c_2c_3c_5c_7 \\ c_2c_3c_5c_8 \\ c_2c_3c_6c_7 \\ c_2c_3c_6c_8 \\ c_2c_4c_5c_7 \\ c_2c_4c_5c_8 \\ c_2c_4c_6c_7 \\ c_2c_4c_6c_8 \end{bmatrix}. \tag{6.45}$$

Also from Figure 5.10, we have the following matrices:

$$
L_2 = 1_{2\times 2} \otimes L_{\text{Toffoli}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.46}
$$

$$
= \left[\begin{array}{cccccccc|cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right].
$$

We can write this equation in compact way, as follows:

$$L_2 = 1_{2\times 2} \otimes L_{\text{Toffoli}} = \begin{bmatrix} L_{\text{Toffoli}} & 0_{8\times 8} \\ 0_{8\times 8} & L_{\text{Toffoli}} \end{bmatrix}. \tag{6.47}$$

And

$$L_1 = L_{\text{CNOT}} \otimes 1_{2\times 2} \otimes 1_{2\times 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.48}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$
= \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]. \tag{6.49}
$$

We can also write this equation in a compact way, as follows:

$$
L_1 = L_{\mathrm{CNOT}} \otimes 1_{2\times 2} \otimes 1_{2\times 2} = \begin{bmatrix} 1_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 4} & 1_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 1_{4\times 4} \\ 0_{4\times 4} & 0_{4\times 4} & 1_{4\times 4} & 0_{4\times 4} \end{bmatrix}. \tag{6.50}
$$

Similarly, we can calculate the following matrix:

$$L = L_2 L_1 = \begin{bmatrix} L_{\text{Toffoli}} & 0_{8\times 8} \\ 0_{8\times 8} & K_{\text{Toffoli}} \end{bmatrix}; \quad K_{\text{Toffoli}} = L_{\text{Toffoli}} \begin{bmatrix} 0_{4\times 4} & 1_{4\times 4} \\ 1_{4\times 4} & 0_{4\times 4} \end{bmatrix}. \tag{6.51}$$

The final matrix equation, which describes the circuit in Figure 5.10 with four inputs as in equation (6.44) is:

$$\begin{bmatrix} c_1c_3c_5c_7 \\ c_1c_3c_5c_8 \\ c_1c_3c_6c_7 \\ c_1c_3c_6c_8 \\ c_1c_4c_5c_7 \\ c_1c_4c_5c_8 \\ c_1c_4c_6c_7 \\ c_1c_4c_6c_8 \\ c_2c_3c_5c_7 \\ c_2c_3c_5c_8 \\ c_2c_3c_6c_7 \\ c_2c_3c_6c_8 \\ c_2c_4c_5c_7 \\ c_2c_4c_5c_8 \\ c_2c_4c_6c_7 \\ c_2c_4c_6c_8 \end{bmatrix} \to L \begin{bmatrix} c_1c_3c_5c_7 \\ c_1c_3c_5c_8 \\ c_1c_3c_6c_7 \\ c_1c_3c_6c_8 \\ c_1c_4c_5c_7 \\ c_1c_4c_5c_8 \\ c_1c_4c_6c_7 \\ c_1c_4c_6c_8 \\ c_2c_3c_5c_7 \\ c_2c_3c_5c_8 \\ c_2c_3c_6c_7 \\ c_2c_3c_6c_8 \\ c_2c_4c_5c_7 \\ c_2c_4c_5c_8 \\ c_2c_4c_6c_7 \\ c_2c_4c_6c_8 \end{bmatrix}. \tag{6.52}$$

This example explains the general methodology for studying quantum circuits. Here, in the case when the quantum states have input and output performed by sixteen zeros, we can use the method of analytic representation. The quantum circuit is a one-one corresponding function from $(B_{16})^{15} \to (B_{16})^{15}$.

6.4 Summary

In this chapter, we studied quantum computation. We considered quantum computation with CNOT and Toffoli gates using important examples. Also, analytic representations and their zeros were studied. Then, we studied zeros of the inputs and outputs for CNOT gates and Toffoli gates. After that, we computed the zeros on the output approximately was considered. Also, some quantum circuits were considered. The study of quantum gates using analytic functions and their zeros is one of the novel parts of the thesis.

Chapter 7

Conclusion

There are deep connections between the behaviour of a quantum system and the zeros of the analytic representation of this system. We studied phase space procedures for quantum systems with the Hilbert space of D-dimension. The analytic representations on a torus using Theta function were considered. The quantum states of Eq (4.84) were represented by the analytic function in Eq (4.86) on a torus.

In finite quantum systems, the $D-1$ of D zeros are independent, and the set of these $D-1$ independent zeros, defined uniquely the quantum state of the system. These zeros obey the constraint of Eq (4.114).

The new work in this thesis is described by the following paragraphs.

We studied classical reversible logic circuits which included CNOT gates and Toffoli gates. An arbitrary Boolean expression was performed in terms of the CNOT gates and Toffoli gates. Logical circuits that include CNOT gates only with binary inputs and outputs, can be easily described with the matrices system as in Eq (5.7). However, logical circuits which include Toffoli gates with binary inputs and outputs cannot be described with the matrices system. They use two operations, namely the product of matrices and the direct sum of matrices.

In general, the circuits that include both CNOT gates and Toffoli gates have been studied. As an example, we implemented a NOT gate in terms of a CNOT gate and a Toffoli gate with the chosen inputs as in Eqs (5.19), (5.20), respectively. Also we implemented an OR gate in terms of both a CNOT gate and a Toffoli gate, and in terms of a Toffoli gate only, with the chosen inputs as in Eqs (5.21) and (5.22), respectively. Finally, we implemented an AND gate in terms of a Toffoli gate only, with the chosen inputs as in Eq (5.23).

Reversible circuits with CNOT gates and Toffoli gates were also studied. For example, we implemented a reversible circuit which included a CNOT gate and a Toffoli gate for the function defined in Eq (5.26). Also, we provided a table that included some inputs and outputs of this circuit. Furthermore we implemented the inverse for this circuit, as shown in Figure (5.11).

Quantum mechanics is a time-reversible theory, and only reversible classical logic circuits have quantum analogues. We also considered quantum CNOT gates and quantum Toffoli gates. Unitary transformation can be 1 or different unitary operator. Also, we considered Hilbert spaces when they are two dimensional and three dimensional with CNOT gates and Toffoli gates, respectively. In addition, quantum logic circuits that include quantum CNOT gates and quantum Toffoli gates were discussed.

The representation of quantum states with analytic functions and their zeros has been used widely in quantum physics. In this thesis we used them in the context of quantum gates, particularly CNOT gates and Toffoli gates.

The inputs and the outputs of quantum gates, particularly CNOT gates and Toffoli gates were represented by the zeros on a torus of analytic functions. We calculated numerically the zeros of the quantum states at inputs and outputs of CNOT gates and Toffoli gates. The values which have used at the inputs and the outputs for the CNOT gates have given in Eqs (6.26),(6.27) and (6.28). While the values used at the inputs and the outputs for the Toffoli gates were provided in Eqs (6.29),(6.30) and (6.31).

After that, we computed the zeros on the output approximately. The output varies a little from f_m as in Eq (6.24) to $f_m + \Delta f_m$, and Δf_m as in Eq (6.35). As a result, the zeros will vary from ζ_n to $\zeta_n + \Delta\zeta_n$ as in Eq (6.43). Here the method used is similar the one used in [16].

The quantum circuit methodology is the same for the classical circuit methodology. The r inputs and r outputs of a circuit works with quantum qubits, here these inputs and outputs were represented by a $2^r \times 2^r$ matrix.

Finally, some quantum circuits were considered. As an example, we considered the circuit in Figure 5.10 that we had studied earlier in the classical computation. The matrix equation that describes the circuit in Figure 5.10 with four inputs as in Eq(6.44) is given in Eq(6.51). This example clarify the general methodology for studying quantum circuits. Here, in the case when quantum states have input and output performed by sixteen zeros, we can use the method of analytic representation.

The work may be extended to describe a classical Toffoli gates with matrices. We can also study more quantum gates apart from CNOT gates and Toffoli gates, by using analytic representation. The work may be also extended to quantum gates by using analytic representation for infinite quantum systems.

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