An analytic representation of weak mutually unbiased bases

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Submitted for the Degree
Of Doctor of Philosophy

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University of Bradford

2016

Keywords: Finite quantum systems, weak mutually unbiased bases, finite geometry, Theta functions**.**

Abstract

Quantum systems in the d-dimensional Hilbert space are considered. The mutually unbiased bases is a deep problem in this area. The problem of finding all mutually unbiased bases for higher (non-prime) dimension is still open. We derive an alternate approach to mutually unbiased bases by studying a weaker concept which we call weak mutually unbiased bases. We then compare three rather different structures.

The first is weak mutually unbiased bases, for which the absolute value of the overlap of any two vectors in two different bases is $1/\sqrt{k}$ (where $k|d$) or 0. The second is maximal lines through the origin in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space. The third is an analytic representation in the complex plane based on Theta functions, and their zeros. The analytic representation of the weak mutually unbiased bases is defined with the zeros examined.

It is shown that there is a correspondence (triality) that links strongly these three apparently different structures. We give an explicit breakdown of this triality.

Dedication

I would like to dedicate this thesis first to the almighty God whose grace kept me going through the program. Also to my husband and pillar Abayomi Olupitan for all his daily encouragement, support and unmeasurable love showered on me in every area, you are indeed one in a million. To my ever supportive parents Pastor and Deaconess Akinola, for your prayers and love which cannot be quantified. Thanks for believing in me and motivating me towards a PhD. To my siblings and all my family members for the support and encouragement. Finally to my supervisors, Prof A. Vourdas for being a mentor, father figure and motivating supervisor, as well as Dr Ci, for his help and support always.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text. Part of this research has been published in the following work:

Journals

Olupitan, T., Lei, C., and Vourdas, A. (2016). *An analytic function approach to weak mutually unbiased bases.* Annals. Phys. 371(2016):1-19

Proceedings

Olupitan, T., Lei, C., and Vourdas, A. (2016) *A triality between weak mutually unbiased bases, zeros of their analytic representations, and finite geometries.* 31st International Colloquium on Group Theoretical Methods in Physics (to appear)

Acknowledgements

I would like to acknowledge the University of Bradford 2013 PhD studentship/grant provided by the Faculty of Engineering and Informatics, Department of Electrical Engineering and computer science, to sponsor my PhD program.

List of notations

Notation	Meaning
$\mathbb{Z}(d)$	Integers modulo d.
$H(d)$	Hilbert space d
$\Pi(d)$	Finite geometry $\mathbb{Z}(d) \times \mathbb{Z}(d)$
Γ	Cell $[0, d) \times [0, d)$ in the complex plane
X	Position operator
P	Momentum operator
x	Position variable
p	Momentum variable
$D(z)$	Displacement operator
$\mathcal{P}$	Parity Operator
$\mathcal{W}(X, P)$	Wigner Function
$\tilde{\mathcal{W}}(\rho; x, p)$	Weyl function
$\|\mathbb{X}; a\rangle$	position state
$\|\mathbb{P}; a\rangle$	momentum state
$S(\eta, \zeta, \rho, \vartheta)$	Symplectic transformation
$\|\mathbb{X}(\rho, \vartheta)\rangle$	Mutually unbiased bases for prime 'd'

$M(a,b)$	Maximal line through the origin
$\mathfrak{M}(\vartheta_1,\vartheta_2)$	Factorized maximal line
$\|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$	Weak mutually unbiased bases for a given a
$\mathfrak{W}(\vartheta_1,\vartheta_2)$	Weak mutually unbiased bases
$\Theta(u,\tau)$	Theta function
$\varphi(\vartheta_1,\vartheta_2;\overline{a}_1,\overline{a}_2,R)$	Line of zeros for the d zeros of the analytic representation of weak mutually unbiased bases
$\mathfrak{K}(\vartheta_1,\vartheta_2;\overline{a}_1,\overline{a}_2)$	Set of d zeros representing the weak mutually unbiased bases
$\mathfrak{N}((\vartheta_1,\vartheta_2)$	Set of parallel lines of zeros in Γ

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Chapter 1

Introduction

1.1 Introduction

There has been lot of work done on various areas of quantum system $H(d)$ with variables in $\mathbb{Z}(d)$ and various analytic representations have been studied in quantum mechanics. This work combines Quantum Physics with Discrete Mathematics and it has numerous applications in quantum key distribution, quantum cryptography, quantum coding, quantum entanglement and measurements [8, 8, 14, 17, 34, 65, 67]. The open problem of mutually unbiased bases (MUBs) in this area of research has been going on for years [1, 3, 19, 32, 35, 49, 51, 61, 64, 71]. This is a set of bases for which the absolute overlap of any two vectors in different bases is $1/\sqrt{(}d)$. The only known result is for prime d dimensions. For prime d the maximum number of mutually unbiased bases in d-dimensional system is $d + 1$. When d is no longer prime, $\mathbb{Z}(d)$ becomes a ring and the study of mutually unbiased bases becomes a more difficult problem. However, the existence of a complete set of mutually unbiased bases in d-dimensional systems where d is non prime still remains open.

Mutually unbiased bases in higher dimensional systems has numerous applications in quantum information sciences. These applications include enhancing cryptographic security, it corresponds to the optimal choice of measurements to be performed in order to obtain a full reconstruction of density. Systems in higher dimensional Hilbert space can store more information per carrier. Protocols using higher MUBs also result in higher generation rate of secure keys bits. For a given disturbance the eavesdropper's information decreases with higher dimension.

This led to a recent work which introduced a weaker concept called weak mutually unbiased bases (WMUBs). This is defined as a set of bases, for which absolute value of the overlap of any two vectors in two different bases is $1/\sqrt(k)$, where k is a divisor of d or k is zero. These bases are considered in a d-dimensional Hilbert space $H(d)$ where d is non-prime which are factorized into smaller subspaces i.e, $H(d) = H(d_1) \times H(d_2)$ where $d = d_1 \times d_2$. This factorization is based on the Chinese remainder theorem and a mapping introduced by Good [21]. These weak mutually unbiased bases are expressed as a tensor product of mutually unbiased bases. There are $\psi(d)$ WMUBS ($\psi(d)$ is Dedekind ψ-function).

Finite geometries which are geometries with finite number of points and lines that obey certain axioms have been studied extensively in literature. Most of the work is focused on near-linear geometry where two lines have at most one point in common. However when d is composite, $\mathbb{Z}(d) \times \mathbb{Z}(d)$ geometry is based on rings and does not obey this axiom.

There has been a lot of work in the study of analytic representations for quantum systems with variables in $\mathbb{Z}(d)$ [2] . The most popular analytic representation is the Bargmann representation in the complex plane for the harmonic oscillator [3], which uses the resolution of identity of coherent states [20, 40]. Overtime, the Theta function has used to study the analytic representation of these systems as given in [31, 45]. Theta functions are seen as Gaussian 'bundle' on discrete circle. Gaussians are very useful in the study of quantum systems due to the fact that they can be easily normalized. It has been shown that the $'d'$ zeros of an analytic function representing a quantum state, define the state uniquely. In this thesis, we make use of this to extend the work on mutually unbiased bases, thereby introducing an alternate approach to the study of mutually unbiased bases in higher dimensions.

This work study the weak mutually unbiased bases in $H(d)$. We make use of concept of factorization and symplectic transformation to construct weak mutually unbiased bases. We also study the phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ as a finite geometry $\Pi(d)$. We then consider the problem of analytic representation of the weak mutually unbiased bases in the cell $\Gamma = [0, d) \times [0, d)$. The zeros of the vectors were examined to establish a deep connection between the three different structures stated above.
In this thesis we use this language of analytic functions for the study weak mutually unbiased bases (WMUBs), using their zeros.

1.2 Aims and Objective

In this thesis we show the following novel results

- Each of the d vectors in a WMUB has d zeros on a straight line.
- A WMUB which consist of various vectors (d vectors to be precise) have zeros on parallel lines. Each WMUB is distinguished by the slope of the lines of zeros, which is different for each WMUB.
- In each WMUB, the d^2 zeros form a regular lattice in the cell Γ, which doesn't change irrespective of the transformation which defines the WMUBs.

Based on these results we establish that there is a triality between

- WMUBs
- Lines through the origin in the finite geometry $\Pi(d)$ of the phase space

- Sets of parallel lines of zeros of the vectors in WMUBs in the cell Γ

Thereby providing an alternate approach to the study of MUBs through the study of zeros analytic functions that describes them.

1.3 Structure of this thesis

The thesis is organized as follows; chapter one which is the brief introduction focuses on background knowledge, motivation for the research and description of the thesis structure.

In chapter two, we present the fundamentals of quantum system on $\mathbb{R}$. We define the position and momentum and then the displacement operator and parity operator, coherent state, Winger function and Weyl function. We close this by giving Winger and Weyl function for quantum systems on $\mathbb{R}$.

In chapter three, we start by a review of some important mathematical tools used, such as the Theta function. We give its definition and basic properties. ee present the fundamentals of finite quantum system on $\mathbb{Z}(d) \times \mathbb{Z}(d)$, such the linear operators, symplectic transformation. Wigner and Weyl functions in phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ are considered. We study factorization of quantum systems and finally we present a review of mutually unbiased bases.

Chapter four, discusses the lines in $\mathbb{Z}(d) \times \mathbb{Z}(d)$. We introduce the concept of line factorization and we present some properties of lines in $\mathbb{Z}(d) \times \mathbb{Z}(d)$. Maximal lines through the origin are also introduced.

In chapter five we present the concept of weak mutually unbiased bases (WMUBS) and their construction. Finally we discuss the duality between weak mutually unbiased bases (WMUBS) in $H(d)$ and maximal lines in $\Pi(d)$.

Chapter six examines an analytic representation of weak mutually unbiased bases quantum systems. An analytic representation of quantum systems with variables in $\mathbb{Z}(d) \times \mathbb{Z}(d)$ is examined as well as some of its properties. Also, the zeros of analytic functions are considered. We present as our work in this chapter the analytic representation of the state, the d-zeros of the analytic representation of the vector and the d-squared zeros in the cell Γ.

Chapter seven starts with presenting the lines of zeros of the WMUBS and we give as work, the corresponding slope of the lines which represent the WMUBs. We further established our novel result of the triality between WMUBs, lines through the origin in the finite geometry $\Pi(d)$ of the phase space and sets of parallel lines of zeros of the vectors in WMUBs in the cell Γ .

Finally in chapter eight, we present the discussion and conclusion of our work.

Chapter 2

Quantum systems on $\mathbb{R}$

2.1 Introduction

In this chapter, we discuss the basic properties of quantum mechanics used to describe states of quantum particles on infinite Hilbert space.

We first give a definition of Fourier transform.

2.2 Fourier Transform

Let $p(t)$ be a complex function with respect to time t over and interval $-\infty \leq t \leq \infty$, the Fourier transform $P(f)$ is given by

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-2\pi i f t} dt \tag{2.1}$$

with its inverse given as

$$p(t) = \int_{-\infty}^{\infty} P(f) e^{2\pi i f t} df. \tag{2.2}$$

2.3 Dirac Notation

The Dirac notation was introduced by Dirac to give a representation of quantum states along with their properties.
We represent the quantum state or a wave function by a ket vector as $|\psi\rangle$ and for its conjugate we use a bra vector $\langle\psi|$

The inner product is defined as the multiplication of the 'bra' and 'ket' written as

$$\langle\phi|\psi\rangle = \int \phi^*\psi dx. \tag{2.3}$$

We define our wavefunction $\psi(x) \equiv \langle x|\psi\rangle$.

2.4 Position and momentum operators in infinite quantum systems

Let x and p be the position and momentum quantities respectively. The position operator X and momentum operator P satisfy the condition (where the Planck's constant $\hbar = 1$)

$$X\psi(x) = x\psi(x), \quad P\psi(x) = -i\frac{\partial}{\partial x}\psi(x) \tag{2.4}$$

and they also satisfy the canonical commutation relation

$$[X, P] = i \tag{2.5}$$

Their eigen values are real values. The eigen states form a basis for representation of a quantum system.

$$X|x\rangle = x|x\rangle \tag{2.6}$$

$$P|p\rangle = p|p\rangle, \quad x, p \in \mathbb{R} \tag{2.7}$$

$|x\rangle$ and $|p\rangle$ denotes the position and momentum states respectively.

The Dirac-delta function given below is used for normalization on the account that the eigenvalues x are not discrete.

$$\langle x|x'\rangle = \delta(x - x') \tag{2.8}$$

Their eigen states form a complete set such that

$$\int |x\rangle\langle x|dx = \mathbf{1} \tag{2.9}$$

$$\int |p\rangle\langle p|dp = \mathbf{1} \tag{2.10}$$

Therefore we can write arbitrary kets $|\psi\rangle$ as

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \int dx|x\rangle\langle x|\psi = \int dx\psi(x)|x\rangle, \tag{2.11}$$

using the completeness property in 2.9.

Also in terms of the momentum basis, we have the Dirac-delta function given as

$$\langle p|p'\rangle = \delta(p - p') \tag{2.12}$$

any arbitary state $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \int dx|p\rangle\langle p|\psi\rangle = \int dp\phi(p)|p\rangle, \tag{2.13}$$

using the completeness property in 2.9 and $\phi(p) \equiv \langle p|\psi\rangle$ is the wave function in the momentum basis.

The position representation and the momentum representation of the state $|\psi\rangle$ are related by the Fourier transform

$$\psi(x) = (2\pi)^{-1/2} \int_{\infty}^{\infty} \exp(ixp)\phi(p)dp, \tag{2.14}$$

$$\phi(p) = (2\pi)^{-1/2} \int_{\infty}^{\infty} \exp(-ixp)\psi(x)dp \tag{2.15}$$

The state vectors $|x\rangle$ and $|p\rangle$ are related to each other by Fourier transform

$$|x\rangle = (2\pi)^{-1/2} \int_{\infty}^{\infty} \exp(-ixp)|p\rangle dp, \tag{2.16}$$

$$|p\rangle = (2\pi)^{-1/2} \int_{\infty}^{\infty} \exp(ixp)|x\rangle dx. \tag{2.17}$$

2.5 Density operator

We define the probability density (weight) of a particle x decribed by a state ψ as

$$|\psi(x)|^2 = \langle x|\psi\rangle\langle\psi|x\rangle \tag{2.18}$$

$$= \langle x|\rho|x\rangle \tag{2.19}$$

where ρ is called the density operator

$$\rho \equiv |\psi\rangle\langle\psi|. \tag{2.20}$$

The density operator allows us to describe a quantum state without using the state vector. This is useful when representing a quantum state as a mixed state and is given as

$$\rho = 1/2(|\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|). \tag{2.21}$$

Since any operator can be represented as matrix, the expectation value of a given operator A is defined as the trace of the product of the density matrix and the operator

$$\langle A\rangle = Tr[A\rho] \tag{2.22}$$

2.6 Parity and Displacement operator

We define the ladder operators

$$b = \frac{X + iP}{\sqrt{2}} \qquad b^\dagger = \frac{X - iP}{\sqrt{2}}. \tag{2.23}$$

2.6.1 Displacement operator

The displacement operator which moves a particle in a localized state by a magnitude z is given by

$$D(z) = \exp(zb^\dagger - z^* b) \tag{2.24}$$

where z is a complex number and can be written in terms of position and momentum quantities x and p as

$$z = (x + ip)/\sqrt{2} \tag{2.25}$$

Using the ladder operators in Eq(2.24), the displacement operator is expressed in terms of position and momentum operators as follows

$$D(x,p) = exp(ipX - ixP) \tag{2.26}$$

$exp(ipX)$ and $exp(-ixP)$ denote unitary translation operator in position space and momentum space respectively, and these operators do not commute.

2.6.2 Parity operator

We define the parity operator around the origin as

$$\mathcal{P}_0 \equiv \int_{-\infty}^{\infty} |-x\rangle\langle x| dx = \int_{-\infty}^{\infty} |-p\rangle\langle p| dp \tag{2.27}$$

Acting the parity operator on position and momentum operator and their eigenstates produces their inverses

$$\mathcal{P}_0 X \mathcal{P}_0^\dagger = -X \tag{2.28}$$

$$\mathcal{P}_0 P \mathcal{P}_0^\dagger = -P \tag{2.29}$$

$$\mathcal{P}_0 |x\rangle = |-x\rangle \tag{2.30}$$

$$\mathcal{P}_0 |p\rangle = |-p\rangle \tag{2.31}$$

It also acts on the displacement operator to give its inverse

$$\mathcal{P}_0 D(a_0, b_0) \mathcal{P}_0^\dagger = D(-a_0, -b_0) \tag{2.32}$$

The displaced parity operator is symbolized by $\mathcal{P}(z)$ and expressed as

$$\mathcal{P}(z) = D(z)\mathcal{P}_0[D(z)]^\dagger \tag{2.33}$$

2.7 Coherent states

We start be defininig the vacuum state $|0\rangle$ as $b|0\rangle = 0$.

Coherent states are defined as the eigen state of the destruction operator b.

$$b|z\rangle = z|z\rangle \tag{2.34}$$

where b has been defined in Eq(2.23).

Coherent states are obtained by acting the displacement operator on the vacuum state,

$$D(z)\,|0\rangle = |z\rangle\,. \tag{2.35}$$

They can also be expressed as

$$|z\rangle = e^{-\frac{1}{2}|z|^2} e^{zb^\dagger}\,|0\rangle = e^{zb^\dagger - z^* b}\,|0\rangle = D(z)\,|0\rangle\,. \tag{2.36}$$

$D(z)$ was earlier defined in Eq.(2.35) as the displacement operator.

The position representation of the coherent state is a Gaussian function

$$\langle x|z\rangle = \pi^{-1/4}\exp\left(-\frac{x^2}{2} + \sqrt{2}zx - zz_R\right). \tag{2.37}$$

The momentum representation of the coherent state is

$$\langle p|z\rangle = \pi^{-1/4}\exp\left(-\frac{p^2}{2} - \sqrt{2}zip + ziz_I\right), \tag{2.38}$$

where $z = z_R + iz_I$ in both cases.

The expectation value of the position and momentum operator can be obtained from Eq(2.37) and Eq(2.38) as follows

$$\langle X\rangle = \langle z|X|z\rangle \ = \sqrt{2}z_R, \tag{2.39}$$

$$\langle P\rangle = \langle z|P|z\rangle \ = \sqrt{2}z_I. \tag{2.40}$$

The variances of X and P are both equal to the value 1/2

$$(\Delta x)^2 = (\Delta p)^2 = \frac{1}{2} \tag{2.41}$$

We give the inner product of two coherent states $|z_1\rangle$ and $|z_2\rangle$ as

$$\langle z_2|z_1\rangle = \exp\left(-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + z_1 z_2^*\right) \tag{2.42}$$

The completeness relation for coherent states is

$$\frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\, d^2z = \mathbf{1} \tag{2.43}$$

where $\mathbb{C}$ is a complex plane. This interesting property of the coherent state makes it very useful in quantum mechanics.

2.8 Wigner and Weyl function

The Wigner function is defined as a quasi probability distribution function in the phase space. It was discovered by E.Wigner in 1932 [53, 70] due to the fact that uncertainty relation forbids the possibility of having a standard probability distribution. It can be derived through the trace of the parity operator.

The Wigner function [23, 44] is given by

$$\mathcal{W}(x,p) = Tr[\rho\mathcal{P}(z)] \tag{2.44}$$

where ρ is the density operator.

The Wigner function can be expressed in both representations as follows

$$\mathcal{W}(\rho;x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ip\nu)\left\langle x-\frac{\nu}{2}|\rho|x+\frac{\nu}{2}\right\rangle d\nu \tag{2.45}$$

$$\mathcal{W}(\rho;x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ix\mu)\left\langle p-\frac{\mu}{2}|\rho|p+\frac{\mu}{2}\right\rangle dp \tag{2.46}$$

where ν and μ are the position increase and momentum increase respectively.
We can also define the Wigner function for an arbitrary operator U by replacing the density operator ρ in Eq(2.44).

$$\mathcal{W}(U;x,p) = Tr[U\mathcal{P}(z)] \tag{2.47}$$

Thus, with the Wigner function of a particle, the expectation value with respect to position and momentum can be derived.

The Weyl function is a correlation function with displacements in both position and momentum. It is a general case of correlation functions since it shows the correlation by displacing the state in both position and momentum. It is given with respect to the density operator,

$$\tilde{\mathcal{W}}(\rho;\nu,\mu) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ix\mu)\left\langle x-\frac{\nu}{2}|\rho|x+\frac{\nu}{2}\right\rangle dx \tag{2.48}$$

$$\tilde{\mathcal{W}}(\rho;\nu,\mu) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ip\nu)\left\langle p-\frac{\mu}{2}|\rho|p+\frac{\mu}{2}\right\rangle dp \tag{2.49}$$

and is expressed in terms if the displacement operator as

$$\tilde{\mathcal{W}}(\rho;\nu,\mu) = Tr[\rho D(\nu,\mu)] \tag{2.50}$$

The Weyl function is derived from the Wigner function by a two-dimensional Fourier transform

$$\mathcal{W}(\rho;x,p) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\mathcal{W}}(\nu,\mu)\exp[-i(p\nu - i\mu x]dxdp. \tag{2.51}$$

2.9 Summary

In this chapter, we reviewed the phase space formalism for quantum systems on the infinite Hilbert space. The position and momentum operators were introduced along with their important properties. We studied the displacement and parity operators along with the coherent state. Finally we introduce the important Wigner and Weyl functions and the their formalisms.

Chapter 3

Quantum systems with finite Hilbert space

3.1 Introduction

In previous chapter, we introduced quantum systems where the values of the position and momentum are described on $\mathbb{R}$, that is position and momentum phase space is $\mathbb{R} \times \mathbb{R}$. This chapter reviews the analogous formalism where we are in the finite quantum system on the d dimensional Hilbert space which we represent as $H(d)$. Study of finite systems was started by Weyl [69] and Schwinger[55], and later some other authors [12, 21, 61, 65, 71] contributed to the research and applications. In the d-dimensional space both position and momentum take values in $\mathbb{Z}(d)$, that is, the set of integers modulo d, hence our phase space is the toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$.

Definition

Vectors will be denoted by Dirac ket notation. A (finite-dimensional) Hilbert space is a vector space H over the complex number field $\mathbb{C}$ which also comes with an inner-product, i.e. a map

$$\langle -|-\rangle : H \times H \to \mathbb{C} \tag{3.1}$$

satisfying $\langle \gamma|\gamma\rangle \in \mathbb{R}^{+}$ and $\langle \gamma|\gamma\rangle = 0 \leftrightarrow \gamma = 0$.

An important axiom which worthy of note due its application in our work is the composite systems.

Axiom: If the Hilbert space of system A is H_A and the Hilbert space of system B is H_B, then the Hilbert space of the composite systems AB is the tensor product of $H_A \otimes H_B$.

For a state $|\gamma_A\rangle$ and state $|\gamma_B\rangle$, their composite system is $|\gamma_A\rangle \otimes |\gamma_B\rangle$.

3.2 Fourier Transform, Position and Momentum operators

Consider a quantum system with a d-dimensional space $H(d)$, and an orthonormal basis of position states which is denoted as $|\mathbb{X}; a\rangle$ where a belongs to $\mathbb{Z}(d)$

The states $|\mathbb{X}; a\rangle$ satisfies the relation:

1. $\langle \mathbb{X}; a|\mathbb{X}; b\rangle = \delta(a, b)$
2. $\sum_a |\mathbb{X}; a\rangle\langle \mathbb{X}; a| = \mathbf{I}$

$\delta(a, b)$ is called the Kronecker delta, which satisfies the condition

$$\delta(a,b) = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases} \tag{3.2}$$

Relation (2) helps us to express any state $|f\rangle$ as linear combination of vector $|\mathbb{X};a\rangle$.

$$\sum_a |\mathbb{X};a\rangle\langle\mathbb{X};a|f\rangle = |f\rangle \tag{3.3}$$

These identities are proved using the following identity

$$\frac{1}{d}\sum_{a=0}^{d-1}\omega[a(k-l)] = \delta(k,l) \tag{3.4}$$

where $\omega(a) = \exp[\frac{i2\pi a}{d}]$, $\quad a \in \mathbb{Z}(d)$

3.3 Fourier Transform

In a d-dimensional quantum system, the position and momentum states $|\mathbb{X};a\rangle$, $|\mathbb{P};a\rangle$ are two orthonormal bases in $H(d)$, where $a \in \mathbb{Z}(d)$ and are related to each other through Fourier transform.
We define the Fourier operator as

$$F = d^{1/2}\sum_{a,b}\omega(ab)|\mathbb{X};a\rangle\langle\mathbb{X};b|, \tag{3.5}$$

$$\omega(\alpha) = \exp(i\frac{2\pi\alpha}{d}). \tag{3.6}$$

In general for d- dimensional system we write

$$F = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)(N-1)}. \end{pmatrix} \tag{3.7}$$

The Fourier transform is unitary operator then

$$FF^{\dagger} = F^{\dagger}F = \mathbf{I}. \tag{3.8}$$

Operating the Fourier operator twice gives the original data in reverse order, .so operating it four times gives back the original data, then

$$F^4 = \mathbf{I}. \tag{3.9}$$

Using Fourier transform the 'momentum states' are defined as:

$$|\mathbb{P}; a\rangle = F|\mathbb{X}; a\rangle = d^{-1/2} \sum_b \omega(ab)|\mathbb{X}; b\rangle. \tag{3.10}$$

We write any arbitrary state $|s\rangle$ in H as:

$$|s\rangle = \sum_b \lambda_a |\mathbb{X}; b\rangle = \sum_a \mu_a |\mathbb{P}; a\rangle, \tag{3.11}$$

where

$$\lambda_b = d^{-1/2} \sum_a \mu_a \omega(ab) \tag{3.12}$$

and λ_b, μ_a are 'wave functions' for the state $|s\rangle$ in the position and momentum representations, correspondingly.

3.3.1 Position and momentum operators

The position and momentum operators are defined as

$$X = \sum_{a=0}^{d-1} a \left|\mathbb{X}; a\right\rangle \left\langle\mathbb{X}; b\right| , \tag{3.13}$$

$$P = \sum_{a=0}^{d-1} a \left|\mathbb{P}; a\right\rangle \left\langle\mathbb{P}; b\right| . \tag{3.14}$$

The position and momentum operator as related to each other through Fourier transform.

$$P = FXF^{\dagger} \tag{3.15}$$

$$X = -FPF^{\dagger} \tag{3.16}$$

3.3.2 Displacement Operator

Since position and momentum in the finite quantum system are integers modulo d, then the position-momentum phase space is the toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$. In this phase-space the displacement operators are defined as

$$\begin{aligned} \mathcal{Z} &= \exp\left[i\frac{2\pi}{d}x\right], \\ \mathcal{X} &= \exp\left[-i\frac{2\pi}{d}p\right] \end{aligned} \tag{3.17}$$

They are unitary operators that perform displacement along momentum, and position axes respectively in the phase space, such that:

$$\mathcal{Z}^{\alpha}|\mathbb{P}, a\rangle = |\mathbb{P}; a + \alpha\rangle, \qquad \mathcal{Z}^{\alpha}|\mathbb{X}; a\rangle = \omega(\alpha a)|\mathbb{X}; a\rangle \tag{3.18}$$

$$\mathcal{X}^{\beta}|\mathbb{P};a\rangle = \omega(-a\beta)|\mathbb{P};a\rangle, \qquad \mathcal{X}^{\beta}|\mathbb{X};a\rangle = |\mathbb{X};a+\beta\rangle \tag{3.19}$$

$$\mathcal{X}^{d} = \mathcal{Z}^{d} = \mathbf{1}; \qquad \mathcal{X}\mathcal{Z} = \mathcal{Z}\mathcal{X}\omega(-1) \tag{3.20}$$

based on Eqs.(3.17), $\mathcal{X}$ and $\mathcal{Z}$, obey the following relation;

$$\mathcal{X}^{\beta}\mathcal{Z}^{\alpha} = \mathcal{Z}^{\alpha}\mathcal{X}^{\beta}\omega(-\alpha\beta) \tag{3.21}$$

where α, β are integers in $\mathbb{Z}(d)$.

In the special case $d = 2$ the $\langle\mathbb{X};a|\mathcal{X}|\mathbb{X};b\rangle$ and $\langle\mathbb{X};a|Z|\mathbb{X};b\rangle$ becomes the pauli matrices σ_x and σ_z correspondingly.

The general displacement operator can be defined as:

$$D(\alpha,\beta) = \mathcal{Z}^{\alpha}\mathcal{X}^{\beta}\omega(-2^{-1}\alpha\beta), \tag{3.22}$$

$$[D(\alpha,\beta)]^{\dagger} = D(-\alpha,-\beta) \tag{3.23}$$

where $2^{-1} \in \mathbb{Z}(d)$ exists only if d is odd.

The $D(\alpha,\beta)$ are unitary operators and are associated with Heisenberg-Weyl group in the context of finite quantum systems. The multiplication of two displacement operators result in the following relation:

$$D(\alpha_1,\beta_1)D(\alpha_2,\beta_2) = D(\alpha_1+\alpha_2,\beta_1+\beta_2)\omega(2^{-1}(\alpha_1\beta_2-\alpha_2\beta_1) \tag{3.24}$$

Also

$$D(\alpha,\beta)X[D(\alpha,\beta)]^{\dagger} = X - \beta\mathbf{1} \tag{3.25}$$

$$D(\alpha,\beta)P[D(\alpha,\beta)]^{\dagger} = P - \alpha\mathbf{1} \tag{3.26}$$

$$D(\alpha,\beta)|\mathbb{X};a\rangle = \omega(-2^{-1}\alpha\beta + \alpha a)|\mathbb{X};a+\beta\rangle \tag{3.27}$$

$$D(\alpha,\beta)|\mathbb{P};a\rangle = \omega(-2^{-1}\alpha\beta - \beta a)|\mathbb{P};a+\alpha\rangle \tag{3.28}$$

Acting Fourier operator on the displacement operators yields

$$F\mathcal{X}F^{\dagger} = \mathcal{Z}, \tag{3.29}$$

$$F\mathcal{Z}F^{\dagger} = \mathcal{X}^{-1} \tag{3.30}$$

$$FD(\alpha,\beta)F^{\dagger} = D(\beta,-\alpha) \tag{3.31}$$

Displacement operator has the following marginal properties [65]

$$\frac{1}{d}\sum_{\beta} D(\alpha,\beta) = |\mathbb{P};2^{-1}\alpha\rangle\langle\mathbb{P};-2^{-1}\alpha| \tag{3.32}$$
$$\frac{1}{d}\sum_{\alpha} D(\alpha,\beta) = |\mathbb{X};2^{-1}\beta\rangle\langle\mathbb{X};-2^{-1}\beta|$$

3.3.3 Parity Operator

Parity operator around the origin, $\mathcal{P}_0$ is defined as

$$\mathcal{P}_0 = F^2, \qquad [\mathcal{P}_0]^2 = \mathbf{1} \tag{3.33}$$

It has $1, -1$ as its eigenvalues. Acting the operator $\mathcal{P}_0$ on position and momentum states $|\mathbb{X};a\rangle$ and $|\mathbb{P};a\rangle$ gives

$$\begin{aligned} \mathcal{P}_0|\mathbb{X};a\rangle &= |\mathbb{X};-a\rangle, & \mathcal{P}_0|\mathbb{P};a\rangle &= |\mathbb{P};-a\rangle. \\ \text{Also, } \mathcal{P}_0 X[\mathcal{P}_0]^\dagger &= -X, & \mathcal{P}_0 P[\mathcal{P}_0]^\dagger &= -P \\ \text{and } \mathcal{P}_0 \mathcal{Z}[\mathcal{P}_0]^\dagger &= \mathcal{Z}^\dagger, & \mathcal{P}_0 \mathcal{X}[\mathcal{P}_0]^\dagger &= \mathcal{X}^\dagger \end{aligned} \tag{3.34}$$

The parity operator about a point (β,α) is called a displaced parity operator. It is defined as;

$$\begin{aligned} \mathcal{P}(\beta,\alpha) &= D(\beta,\alpha)\mathcal{P}_0[D(\beta,\alpha)]^\dagger \\ &= [D(2\beta,2\alpha)]\mathcal{P}_0 \\ &= \mathcal{P}_0[D(2\beta,2\alpha)]^\dagger \end{aligned} \tag{3.35}$$

and

$$[\mathcal{P}(\beta,\alpha)]^2 = \mathbf{1} \tag{3.36}$$

Parity operator has the following marginal properties

$$\begin{aligned} \frac{1}{d}\sum_\beta \mathcal{P}(\alpha,\beta) &= |\mathbb{P};\alpha\rangle\langle\mathbb{P};\alpha| \\ \frac{1}{d}\sum_\beta \mathcal{P}(\alpha,\beta) &= |\mathbb{X};\beta\rangle\langle\mathbb{X};\beta| \end{aligned} \tag{3.37}$$

3.4 Symplectic Transformation

In this section, we explain the concept of symplectic transformation in the phase space $\mathbb{Z}(d)\times\mathbb{Z}(d)$ of a finite quantum system. The symplectic transformation $S(\eta,\zeta,\rho,\vartheta)$

is a unitary transformation with parameters $\eta, \zeta, \rho, \vartheta \in \mathbb{Z}(d)$ such that

$$\eta\vartheta - \zeta\rho = 1(mod \quad d) \tag{3.38}$$

This transformation relies on the idea of multiplicative inverses in $\mathbb{Z}(d)$ due to the fact that only three parameters are independent. For example, if we choose ζ to the dependent variable, the multiplicative inverse of ρ must exist, since we will have $\zeta = \rho^{-1}(\eta\vartheta - 1)$.

The symplectic transformation of the displacement operator $\mathcal{X}$ and $\mathcal{Z}$ are expressed as follows;

$$\mathcal{X}' = S(\eta, \zeta, \rho, \vartheta)\mathcal{X}[S(\eta, \zeta, \rho, \vartheta]^{\dagger} \ = \mathcal{X}^{\eta}\mathcal{Z}^{\zeta}\omega(2^{-1}\eta\zeta) \ = D(\zeta, \eta), \tag{3.39}$$

$$\mathcal{Z}' = S(\eta, \zeta, \rho, \vartheta)\mathcal{Z}[S(\eta, \zeta, \rho, \vartheta]^{\dagger} \ = \mathcal{X}^{\rho}\mathcal{Z}^{\vartheta}\omega(2^{-1}\rho\vartheta) \ = D(\vartheta, \rho) \tag{3.40}$$

We use the constraint in Eq(3.38) in above expression to show that follows Eqs(3.20), which confirms they are displacement operators.

3.5 Wigner and Weyl functions

In previous chapter, we gave the Wigner and Weyl functions for infinite dimensions as well as their properties. We now give the corresponding formalism for finite dimensional systems.

3.5.1 Wigner function

Let A an arbitrary operator with matrix elements A_X and A_P are defined as follows

$$A_{\mathbb{X}} = \langle \mathbb{X}; a| A |\mathbb{X}; b\rangle , \qquad (3.41)$$
$$A_{\mathbb{P}} = \langle \mathbb{P}; a| A |\mathbb{P}; b\rangle$$

We write the Wigner function corresponding to the operator as

$$\mathcal{W}_A(\alpha, \beta) = Tr[A\mathcal{P}(\alpha, \beta)] \qquad (3.42)$$

where $\alpha, \beta \in \mathbb{Z}(d)$.

From Eq(3.42), we note that the Wigner function is defined in terms of the parity operator.

We can also express it as the Fourier transform of the matrix elements of the operator A as follows

$$\mathcal{W}_A(\alpha, \beta) = \omega(2\alpha\beta) \sum_a \omega(-2\alpha a) A_{\mathbb{X}}(a, 2\beta - a) \qquad (3.43)$$
$$\mathcal{W}_A(\alpha, \beta) = \omega(-2\alpha\beta) \sum_a \omega(-2\beta a) A_{\mathbb{P}}(a, 2\alpha - a)$$

For odd-d the marginal properties is given as [65]

$$\frac{1}{d} \sum_\alpha \mathcal{W}_A(\alpha, \beta) = A_{\mathbb{X}}(\beta, \beta), \qquad (3.44)$$
$$\frac{1}{d} \sum_\beta \mathcal{W}_A(\alpha, \beta) = A_{\mathbb{P}}(\alpha, \alpha),$$
$$\frac{1}{d} \sum_{\alpha,\beta} \mathcal{W}_A(\alpha, \beta) = Tr(A)$$

3.5.2 Weyl functions

The Wigner and Weyl function are closely related. The Weyl function for the operator A is given by

$$\tilde{\mathcal{W}}_A(\alpha,\beta) = Tr[aD(\alpha,\beta)] \tag{3.45}$$

we note that this is expressed in terms of the displacement operator.

We also define it as

$$\tilde{\mathcal{W}}_A(\alpha,\beta) = \omega(2^{-1}\alpha\beta)\sum_a \omega(\alpha a)A_{\mathbb{X}}(a,\beta+a) \tag{3.46}$$
$$\tilde{\mathcal{W}}_A(\alpha,\beta) = \omega(-2^{-1}\alpha\beta)\sum_a \omega(-\beta a)A_{\mathbb{P}}(a,\alpha+a)$$

For odd-d the marginal properties is given as [65]

$$\frac{1}{d}\sum_{\alpha} \tilde{\mathcal{W}}_A(\alpha,\beta) = A_{\mathbb{X}}(-2^{-1}\beta,-2^{-1}\beta), \tag{3.47}$$
$$\frac{1}{d}\sum_{\beta} \tilde{\mathcal{W}}_A(\alpha,\beta) = A_{\mathbb{P}}(-2^{-1}\alpha,-2^{-1}\alpha),$$
$$\frac{1}{d}\sum_{\alpha,\beta} \mathcal{W}_A(\alpha,\beta) = \mathcal{W}_A(0,0).$$

Weyl function is the fourier transformation of the Wigner function given by

$$\tilde{\mathcal{W}}_A(\alpha,\beta) = \frac{1}{d}\sum_{\kappa,\lambda} \mathcal{W}_A(\kappa,\lambda)\omega(\alpha\lambda-\beta\kappa) \tag{3.48}$$

3.5.3 Symplectic Transformation of Displacement Operators

We express the symplectic transformation of the displacement operator as

$$S(\eta,\zeta,\rho,\vartheta)D(\alpha,\beta)[S(\eta,\zeta,\rho,\vartheta)]^{\dagger} = D(\vartheta\alpha+\zeta\beta,\rho\alpha+\eta\beta) \tag{3.49}$$

The proof is as follows

Proof. From Eq(3.39)

$$S(\eta,\zeta,\rho,\vartheta)D(\alpha,\beta)[S(\eta,\zeta,\rho,\vartheta)]^\dagger = S(\eta,\zeta,\rho,\vartheta)Z^\alpha X^\beta \omega(-2^{-1}\alpha\beta)[S(\eta,\zeta,\rho,\vartheta)]^\dagger \tag{3.50}$$

$$= S(\eta,\zeta,\rho,\vartheta)Z^\alpha[S(\eta,\zeta,\rho,\vartheta)]^\dagger S(\eta,\zeta,\rho,\vartheta)X^\beta[S(\eta,\zeta,\rho,\vartheta)]^\dagger$$

and from Eq(3.38) we have

$$\begin{aligned} S(\eta,\zeta,\rho,\vartheta)Z^\alpha[S(\eta,\zeta,\rho,\vartheta)]^\dagger &= X^{\alpha\vartheta}Z^{\alpha\rho}\omega(2^{-1}\alpha^2\vartheta\rho) \;= D(\alpha\vartheta,\alpha\vartheta) \\ S(\eta,\zeta,\rho,\vartheta)X^\beta[S(\eta,\zeta,\rho,\vartheta)]^\dagger &= X^{\beta\eta}Z^{\beta\zeta}\omega(2^{-1}\beta^2\zeta\eta) \;= D(\beta\zeta,\beta\eta) \end{aligned} \tag{3.51}$$

Substituting Eq(3.51) into Eq(3.50) we have

$$X^{\alpha\vartheta+\beta\eta}Z^{\alpha\rho+\beta\zeta}\omega(2^{-1}\alpha^2\vartheta\rho + (2^{-1}\beta^2\zeta\eta)) = D(\vartheta\alpha+\zeta\beta,\rho\alpha+\eta\beta) \tag{3.52}$$

where $\omega(2^{-1}\alpha^2\vartheta\rho + (2^{-1}\beta^2\zeta\eta))$ represent the phase factor. □

3.5.4 Marginal Properties of displacement operator

Let $|\mathbb{X}(\eta,\zeta,\rho,\vartheta);a\rangle = S(\eta,\zeta,\rho,\vartheta)|\mathbb{X};a\rangle$ and $|\mathbb{P}(\eta,\zeta,\rho,\vartheta);a\rangle = S(\eta,\zeta,\rho,\vartheta)|\mathbb{P};a\rangle$. Taking into account marginal properties of the displacement operator given in Eq(3.32) and acting the symplectic transformation on them using Eq(3.49), we have

$$\frac{1}{d}\sum_\beta D(\vartheta\alpha+\zeta\beta,\rho\alpha+\eta\beta) = |\mathbb{P}(\eta,\zeta,\rho,\vartheta);2^{-1}\alpha\rangle\langle\mathbb{P}(\eta,\zeta,\rho,\vartheta);-2^{-1}\alpha|, \tag{3.53}$$

$$\frac{1}{d}\sum_\alpha D(\vartheta\alpha+\zeta\beta,\rho\alpha+\eta\beta) = |\mathbb{X}(\eta,\zeta,\rho,\vartheta);2^{-1}\beta\rangle\langle\mathbb{X}(\eta,\zeta,\rho,\vartheta);-2^{-1}\beta| \tag{3.54}$$

In a similar way, we derive for the marginal properties of the parity operator given in Eq(3.32) by acting the symplectic ransformation, we get

$$\frac{1}{d}\sum_{\beta} S(\eta,\zeta,\rho,\vartheta)\mathcal{P}(\alpha,\beta)[S(\eta,\zeta,\rho,\vartheta)]^{\dagger} = |\mathbb{P}(\eta,\zeta,\rho,\vartheta);\alpha\rangle\langle\mathbb{P}(\eta,\zeta,\rho,\vartheta);-\alpha|, \tag{3.55}$$

$$\frac{1}{d}\sum_{\alpha} S(\eta,\zeta,\rho,\vartheta)\mathcal{P}(\alpha,\beta)[S(\eta,\zeta,\rho,\vartheta)]^{\dagger} = |\mathbb{X}(\eta,\zeta,\rho,\vartheta);\beta\rangle\langle\mathbb{X}(\eta,\zeta,\rho,\vartheta);-\beta| \tag{3.56}$$

3.6 Factorization of quantum systems

In this section we discuss in detail the bijective mapping introduced by Good [21] and we look at the concept of factorization in the context of finite quantum systems. We also give a review of factorization of symplectic transformations which are used through the course of our main results.

3.6.1 The bijective mappings

Computation of quantum systems with large dimensions have a higher difficulty level due to the fact that calculations become more difficult as the time required increases rapidly. Fast Fourier transform overcame this problem by factorizing the large space into smaller subspaces, performing Fourier transform in each subspace, and finally combining the results to obtain Fourier transform in the large system.

Here we consider the systems with dimension $d = d_1 \times d_2$, where d_1, d_2 are odd prime numbers different from each other. We use the fast Fourier transform scheme which is the two bijective mapping introduced by Good. This method is based on the Chinese remainder theorem, and on the factorization of finite Fourier transforms, we introduce two bijective maps between $\mathbb{Z}(d)$ and $\mathbb{Z}(d_1) \times \mathbb{Z}(d_2)$:

$$a \leftrightarrow (a_1, a_2) \qquad a_i = a(\text{mod } d_n); \qquad a = a_1 s_1 + a_2 s_2 \ (\text{mod } d), \tag{3.57}$$

and

$$a \leftrightarrow (\overline{a}_1, \overline{a}_2); \qquad \overline{a}_n = at_n = a_i t_n (\text{mod } d_i); \qquad ; m = \overline{a}_1 r_1 + \overline{a}_2 r_2 \ (\text{mod } d). \qquad (3.58)$$

Here r_n, t_n, s_n are the constants

$$r_1 = \frac{d}{d_1} = d_2; \qquad r_2 = \frac{d}{d_2} = d_1; \qquad t_i r_i = 1\ (mod\ d_i); \qquad s_n = t_n r_n \in \mathbb{Z}(d). \qquad (3.59)$$

We note the following relations

$$s_1 s_2 = 0\ (\text{mod } d); \qquad s_1^2 = s_1\ (\text{mod } d) \qquad s_2^2 = s_2\ (\text{mod } d) \qquad s_1 + s_2 = 1\ (\text{mod } d)$$
$$d_2 s_1 = d_2\ (\text{mod } d); \qquad d_1 s_2 = d_1\ (\text{mod } d); \qquad d_1 s_1 = d_2 s_2 = 0\ (\text{mod } d). \qquad (3.60)$$

Also for the map of Eq.(3.57)

$$a + k \leftrightarrow (a_1 + k_1, a_2 + k_2); \quad ak \leftrightarrow (a_1 k_1, a_2 k_2), \qquad (3.61)$$

and for the map of Eq.(3.58)

$$a + k \leftrightarrow (\overline{a}_1 + \overline{k}_1, \overline{a}_2 + \overline{k}_2); \quad ak \leftrightarrow (\overline{a_1} k_1, \overline{a_2} k_2) \qquad (3.62)$$

3.6.2 Example

We now give an example based on the maps explained above. Let $d = 35$, then $d_1 = 5$, $d_2 = 7$. Based on the definition above we have $r_1 = 7, r_2 = 5$, $t_1 = 3, t_2 = 3$ $s_1 = 21, s_2 = 15$.

A number $a = 23$ in $\mathbb{Z}(d)$ is factorized into $a_1 = 3, a_2 = 2$ where $a_1 \in \mathbb{Z}(5)$ and $a_2 \in \mathbb{Z}(7)$. We can also factorize according to dual map into $\overline{a}_1 = 4, \overline{a}_2 = 6$

It is important to give the following useful relation for $\omega_n(b) = \exp(\frac{2\pi b_n}{d_n})$ where $b_n \in \mathbb{Z}(d_n)$,

$$\omega(ab) = \omega_1(a_1\overline{b}_1)\omega_2(a_2\overline{b}_2). \tag{3.63}$$

Eqs.(3.60), (3.61), (3.62), (3.63), are important for the proof of our results in this work.

3.7 Factorization of finite quantum systems

We introduce an isomorphism from $H(d)$ to the product of the Hilbert spaces $H(d_1) \otimes H(d_2)$ as follows [65]. Using the map of Eq.(3.58), the position states is mapped to its corresponding states in $H(d_1) \otimes H(d_2)$ as follows

$$|\mathbb{X}; a\rangle \;\; \leftrightarrow \;\; |\mathbb{X}_1; \overline{a}_1\rangle \otimes |\mathbb{X}_2; \overline{a}_2\rangle\,, \tag{3.64}$$

where $|\mathbb{X}_n; \overline{a}_n\rangle$ are position states in $H(d_n)$. Using Eq.(3.63) we prove that the corresponding map for momentum states, is based on the map of Eq.(3.57), and it is given by

$$|\mathbb{P}; a\rangle \;\; \leftrightarrow \;\; |\mathbb{P}_1; a_1\rangle \otimes |\mathbb{P}_2; a_2\rangle \tag{3.65}$$

where $|\mathbb{X}_n; a_n\rangle$ are momentum states in $H(d_n)$.

We note that due to the Fourier transform between position and momentum states, if the map of Eq.(3.58) is used for position states, then the map of Eq.(3.57) should be used for momentum states.

Also the based on Eq(3.64) and Eq(3.65) the displacement operator in $H(d)$ can be expressed in terms of displacement operators in $H(d_n)$ as follows

$$D(\rho,\sigma) = \prod_{n=1}^{d} D_n(\rho_n,\overline{\sigma}_n) \tag{3.66}$$

where $\rho,\sigma,\rho_n,\overline{\sigma}_n$ are related to Eq.(3.57) and Eq.(3.58)

3.7.1 Factorization of Symplectic Transformations

In this subsection, we illustrate how the symplectic transformation is factorized and state the special cases needed in our work. The $Sp(2,\mathbb{Z}(d))$ is factorized as $Sp(2,\mathbb{Z}(d_1))\times Sp(2,\mathbb{Z}(d_2))$ as follows

$$S(\eta,\zeta|\rho,\vartheta) = S(\eta_1,\zeta_1 r_1|\overline{\rho}_1,\vartheta_1)\otimes S(\eta_2,\zeta_2 r_2|\overline{\rho}_2,\vartheta_2) \tag{3.67}$$

where $\eta_1,\zeta_1 r_1,\overline{\rho}_1,\vartheta_1$ are related to $\eta,\zeta,\rho,\vartheta$ based on Eq.(3.57) and Eq.(3.58).

For our work, we define the following which are special cases

- $$S(0,-\rho^{-1}|\rho,\vartheta) = S(0,-1|1,\vartheta_1)\otimes S(0,-1|1,\vartheta_2) \tag{3.68}$$

  with the parameters ρ,ϑ,ζ are defined as follows

  $$\vartheta = \vartheta_1 s_1 + \vartheta_2 s_2; \quad \rho = d_1 + d_2; \quad \zeta^{-1} = d_2^{-1}s_1 + d_1^{-1}s_2 \pmod d$$

- $$S(\eta,\zeta|\rho,\vartheta) = \mathbf{1}\otimes S(0,-1|1,\vartheta_2) \tag{3.69}$$

The parameters η, ζ, ϑ defined as

$$\eta = s_1; \quad \zeta = -s_2 d_1^{-1} \quad \rho = d_1; \qquad \vartheta = s_1 + \vartheta_2 s_2$$

and

•

$$S(\eta, \zeta|\rho, \vartheta) = S(0, -1|1, \vartheta_1) \otimes \mathbf{1} \tag{3.70}$$

$$\eta = s_2; \quad \zeta = -s_1 d_2^{-1} \quad \rho = p_2; \qquad \vartheta = s_2 + \vartheta_1 s_1.$$

3.7.2 Example

To give an example for the above symplectic transformation, we consider the case that $d = 35$, i.e., $d_1 = 5$ and $d_2 = 7$. Then

$$\begin{aligned} r_1 = 7; \quad t_1 = 3; \quad s_1 = 21 \\ r_2 = 5; \quad t_2 = 3; \quad s_2 = 15 \\ \rho = 12; \quad -\rho^{-1} = -3 \end{aligned} \tag{3.71}$$

So corresponding to Eq(3.68), Eq(3.69), and Eq(3.70), we get

$$\begin{aligned} S(0, -3|12, 21\nu_1 + 15\nu_2) = S(0, -1|1, \nu_1) \otimes S(0, -1|1, \nu_2) \\ S(21, -10|5, 21 + 15\nu_2) = \mathbf{1} \otimes S(0, -1|1, \nu_2) \\ S(15, 7|715 + 21\nu_1) = S(0, -1|1, \nu_1) \otimes \mathbf{1} \end{aligned} \tag{3.72}$$

3.8 Mutually Unbiased Bases

The notion of mutually unbiased bases emerged in the literature of quantum mechanics in 1960 in the works of Schwinger [54]. Mutually unbiased bases have important applications in Quantum Computation, quantum information science. They have diverse applications in Quantum key distribution, quantum cryptography, quantum tomography to mention a few. However these applications rely on the existence of a complete set of such bases. Even though they're being studied since the 1970's the problem of finding a complete set of mutually unbiased bases is only solved for dimensions which are a power of a prime. It remains open for higher dimensions, for non prime d. A comprehensive study of mutually unbiased bases exists in [17].

In this section we give the definition of mutually unbiased bases in prime power dimensions. It is a set of bases, for which the absolute value of the overlap of any two vectors in two different bases is $1/\sqrt{d}$. Mathematically two orthonormal bases $|\mathcal{B}_a; m\rangle$ and $|\mathcal{B}_b; n\rangle$ in the Hilbert space $H(d)$ are mutually unbiased if

$$|\langle\mathcal{B}_a; m|\,|\mathcal{B}_b; n\rangle|^2 = \frac{1}{d} \tag{3.73}$$

It is known that the number K of mutually unbiased bases satisfies the inequality $K \leq d+1$, and that when d is a prime number $K = d+1$.

3.8.1 Mutually unbiased bases using Sp(2, Z(d)) symplectic transformations, with odd prime d

The following special case of symplectic transformations are considered

$$\begin{aligned} X' &= S(0,-\rho^{-1}|\rho,\vartheta)\; X\; [S(0,-\rho^{-1}|\rho;\vartheta)]^\dagger \;= Z^{-\rho^{-1}}; \quad \rho,\vartheta \in \mathbb{Z}(d) \\ Z' &= S(0,-\rho^{-1}|\rho,\vartheta)\; Z\; [S(0,-\rho^{-1}|\rho,\vartheta)]^\dagger \;= X^\rho Z^\vartheta \omega(2^{-1}\rho\vartheta) \end{aligned} \tag{3.74}$$

We note that $S(0,-1|1,0) = \mathcal{F}^{-1}$. These transformations preserve Eq.(3.20).

The symplectic transformation on the position basis gives new bases, which we define as follows

$$|\mathbb{X}(\rho,\vartheta);a\rangle \equiv S(0,-\rho^{-1}|\rho,\vartheta)\,|\mathbb{X};a\rangle\,; \quad \vartheta = 0,...,d-1 \tag{3.75}$$

We note that the corresponding momentum state is actually one of these transformations which is

$$|\mathbb{X}(\rho,0);a\rangle = |\mathbb{P};-\rho^{-1}a\rangle\,. \tag{3.76}$$

We now give a formal representation of the state $|\mathbb{X}(\rho,\vartheta);a\rangle$.

Proposition 3.8.1.

$$|\mathbb{X}(\rho,\vartheta);a\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a,j,\vartheta)]|\mathbb{X};j\rangle; \quad \sigma(a,j,\vartheta) = -ja + 2^{-1}\vartheta j^2 \tag{3.77}$$

Proof. From Eq(3.18) we see that a property of these states is that they are eigenstates of Z. So we need to show that $|\mathbb{X}(\rho,\vartheta);a\rangle$ are eigenstates of $Z' = X^{\rho}Z^{\vartheta}\omega(2^{-1}\vartheta\rho)$.

$$\begin{aligned} Z'\,|\mathbb{X}(\rho,\vartheta);a\rangle &= \frac{1}{\sqrt{d}}\omega(2^{-1}\vartheta\rho)\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a,j,\vartheta)]X^{\rho}Z^{\vartheta}|X;j\rangle \\ &= \frac{1}{\sqrt{d}}\omega(2^{-1}\vartheta\rho)\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a,j,\vartheta)]\omega(\vartheta j)|\mathbb{X};j+\rho\rangle \end{aligned} \tag{3.78}$$

We now change variables $j' = j + \rho$ and we get

$$Z'\,|\mathbb{X}(\rho,\vartheta);a\rangle = \omega(a)\,|\mathbb{X}(\rho,\vartheta);a\rangle \tag{3.79}$$

We next show that $X'\,|\mathbb{X}(\rho,\vartheta);a\rangle = |\mathbb{X}(\rho,\vartheta);a+1\rangle$.

$$\begin{aligned} Z^{-\rho^{-1}}\,|\mathbb{X}(\rho,\vartheta);a\rangle &= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a,j,\vartheta)]Z^{-\rho^{-1}}|\mathbb{X};j\rangle \\ &= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a,j,\vartheta)]\omega(-j\rho^{-1})|\mathbb{X};j\rangle \\ &= \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega[\rho^{-1}\sigma(a+1,j,\vartheta)]|\mathbb{X};j\rangle \\ &= |\mathbb{X}(\rho,\vartheta);a+1\rangle \end{aligned} \tag{3.80}$$

□

The position state $|\mathbb{X};a\rangle$ together with the 'd' symplectic transformations on the position state $|\mathbb{X}(\rho,\vartheta);a\rangle$ gives the '$d+1$' mutually unbiased bases. We can alternatively represent these sets as

$$B(\rho,-1) = \{|\mathbb{X};a\rangle\}; \quad B(\rho,\vartheta) = \{|\mathbb{X}(\rho,\vartheta);a\rangle\}; \quad \vartheta = 0,1,...,d-1. \tag{3.81}$$

We note that ρ is fixed and $B(\rho,0)$ is the basis of momentum states $\{|\mathbb{X}(\rho,0);a\rangle = |\mathbb{P};-\rho^{-1}a\rangle\}$.

They agree with the definition of mutually unbiased bases because for all $\vartheta \neq \vartheta'$

and for all b, a

$$|\langle \mathbb{X}(\rho, \vartheta); b | \mathbb{X}(\rho, \vartheta'); a \rangle| = d^{-1/2}. \tag{3.82}$$

3.9 Summary

In this chapter, we have discussed in detail quantum systems with finite dimensional Hilbert space $H(d)$. We have looked into the basic concepts involved. We introduced the Fourier transform along with the position and momentum states formalism for finite systems. The displacement in phase space was studied through the displacement and parity operator. These were linked to the Weyl and Wigner function. We studied the symplectic transformation along with its properties as a unitary transformation.

We also presented the concept of factorization according to Good along with examples. We used this concept of factorization for factorization of finite quantum systems and factorization of symplectic transformation.

We ended the chapter with a brief review of mutually unbiased bases and we also presented as part of our work, representation of mutually unbiased bases along with its proof.

Chapter 4

Lines through origin in finite geometry $\Pi(d)$

In this chapter, we discuss lines in the finite geometry $\Pi(d)$. We begin by examining the properties of lines in the phase space and explain in details the factorization of lines in $\mathbb{Z}(d) \times \mathbb{Z}(d)$ which is part of our results. Our novel results in this chapter are presented as propositions.

4.1 Lines and sublines in $\Pi(d)$

Literature has extensive research on the phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ as a finite geometry [4, 29, 30] $\Pi(d)$. Most of the studies were based on near-linear geometries, which we define as a geometry where two lines have at most one point in common. The study of the geometry $\Pi(d)$ is entirely based on rings as opposed the near-linear geometry which is based on fields. For prime d, we note that in $\mathbb{Z}(d)$ all its elements have an inverse (except 0), hence a field. For non-prime d, $\mathbb{Z}(d)$ is a ring of integers modulo d. In this section, we show that the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ is a non near linear geometry which violates the axiom of two lines having at most one point in common.

4.2 Definition

The geometry $\Pi(d)$ is defined as $(\mathbf{P}(d), \mathbf{M}(d))$, which is a pair of set of points $\mathbf{P}(d)$ and lines, $\mathbf{L}(d)$.

We define $\mathbf{P}(d)$ as the set of d^2 points (k, l) in $\mathbb{Z}(d) \times \mathbb{Z}(d)$

$$\mathbf{P}(d) = \{(\alpha, \gamma) | \alpha, \gamma \in \mathbb{Z}(d)\} \tag{4.1}$$

and $\mathbf{M}(d)$ is the set of lines. A line through the origin is the set of points

$$\mathbf{M}(k, l) = \{(\beta k, \beta l) | \beta \in \mathbb{Z}(d)\} \tag{4.2}$$

where$(\beta k, \beta l)$ are calculated modulo d. The number of points in $L(k, l)$ is $d/GCD(k, l, d)$ (where $GCD(k, l, d)$ is the greatest common divisor of these integers (k, l, d).

A maximal line is the line $\mathbf{M}(k, l) \in \mathbb{Z}(d) \times \mathbb{Z}(d)$ which has exactly d points, that is the $GCD(k, l, d) = 1$. A maximal line through the origin can also be defined as the set of 'd' points

$$\mathbf{M}(k, l) = \{(\beta k, \beta l) | \beta \in \mathbb{Z}(d)\} \tag{4.3}$$

Provided β has an inverse in $\mathbb{Z}(d)$, then the line $L(k, l)$ is the same line as $\mathbf{L}(\beta k, \beta l)$.

A non-maximal line $\mathbf{M}(k, l) = \{(d_1 k, d_1 l) | d_1 \in \mathbb{Z}(d)\}$ has only d_2 points and also non-maximal line $\mathbf{M}(k, l) = \{(d_2 k, d_2 l) | d_2 \in \mathbb{Z}(d)\}$ has only d_1 points.

The total number of maximal lines in geometry $\Pi(d)$ is $\psi(d)$, where

$$\psi(d) = (p_2 + 1)(p_1 + 1) \tag{4.4}$$

This is called the Dedekind function.

4.2.1 Example

We consider the finite geometry $\Pi(35)$, with $d = 35$. The the line $\mathbf{M}(2,11)$ is a maximal line with the set of points

$$\begin{aligned}\mathbf{M}(2,11) = \{&(0,0)(2,11)(4,22)(6,33)(8,9)(10,20)(12,31)\\&(14,7)(16,18)(18,29)(20,5)(22,16)(24,27)(26,3)\\&(28,14)(30,25)(32,1)(34,12)(1,23)(3,34)(5,10)\\&(7,21)(9,32)(11,8)(13,19)(15,30)(17,6)(19,17)\\&(21,28)(23,4)(25,15)(27,26)(29,2)(31,13)\}\end{aligned} \tag{4.5}$$

as shown in Fig. 4.1

We say $\mathbf{M}(10,20)$ is a sub line of line $\mathbf{M}(2,11)$ and it has only $7 = p_2$ points, hence a non-maximal line.

$$\mathbf{M}(10,20) = \{(0,0)(5,10)(10,20)(15,30)(20,5)(25,15)(30,25)\} \tag{4.6}$$

This is shown in Fig. 4.2.

Also $\mathbf{M}(14,7)$ is a sub line of line $\mathbf{M}(2,11)$ and it has only $5 = p_1$ points.

$$\mathbf{M}(14,7) = \{(0,0)(7,21)(14,7)(21,28)(28,14)\} \tag{4.7}$$

This is also a non-maximal line as shown in Fig. 4.3.

4.3 Symplectic Transformation on points and lines

We earlier defined the symplectic transformation $S(\eta,\zeta,\rho,\vartheta)$, where $\eta,\zeta,\rho,\vartheta \in \mathbb{Z}(d)$ and $\eta\vartheta - \zeta\rho = 1$.

The symplectic transformation on a point $(\alpha,\gamma) \in \mathbb{Z}(d) \times \mathbb{Z}(d)$ is defined as

$$S(\eta,\zeta,\rho,\vartheta)(\alpha,\gamma) = (\alpha,\gamma)\begin{pmatrix}\eta & \zeta \\ \rho & \vartheta\end{pmatrix} = S(\eta\alpha + \zeta\gamma, \rho\alpha + \vartheta\gamma) \tag{4.8}$$

which can also be expressed as

$$S(\eta,\zeta,\rho,\vartheta)(\alpha,\gamma)\begin{pmatrix}\alpha \\ \gamma\end{pmatrix} = \begin{pmatrix}\eta & \zeta \\ \rho & \vartheta\end{pmatrix}^{\dagger}\begin{pmatrix}\alpha \\ \gamma\end{pmatrix} = \begin{pmatrix}\eta\alpha + \zeta\gamma \\ \rho\alpha + \vartheta\gamma.\end{pmatrix} \tag{4.9}$$

The symplectic transformation on points lead to symplectic transformation on lines expressed as

$$S(\eta,\zeta,\rho,\vartheta)\mathbf{M}(\alpha,\gamma) = \mathbf{M}(\eta\beta + \zeta\gamma, \rho\beta + \vartheta\gamma) \tag{4.10}$$

4.4 Factorization of maximal lines

We represent a point (as defined earlier), (α,β) in $\mathbb{Z}(d) \times \mathbb{Z}(d)$ in the factorized form as

$$(\alpha,\beta) = (\overline{\alpha}_1, \beta_1) \times (\overline{\alpha}_2, \beta_2); \quad \overline{\alpha}_n, \beta_n \in \mathbb{Z}(d_n) \tag{4.11}$$

The dual maps of Eq.(3.58) and the map of Eq.(3.57) was used for the first variable and the second variable respectively. These helps us to express the duality between maximal lines through the origin in $\Pi(d)$, and weak mutually unbiased bases in $H(d)$.

A maximal line $\mathbf{M}(\alpha,\beta)$ in $\mathbb{Z}(d) \times \mathbb{Z}(d)$ can now be factorized as

$$\mathbf{M}(\alpha,\beta) = \mathbf{M}(\overline{\alpha}_1, \beta_1) \times \mathbf{M}(\overline{\alpha}_2, \beta_2); \quad \overline{\alpha}_n, \beta_n \in \mathbb{Z}(d_n) \tag{4.12}$$

We establish this factorization in the following proposition.

Proposition 4.4.1.

(1) The line $\mathbf{M}(\overline{\alpha}_n, \beta_n) = \mathbf{M}(1, (\overline{\alpha}_n)^{-1}\beta_n)$*, if* $\overline{\alpha}_n \neq 0 \pmod{d_n}$*and the inverse* $(\overline{\alpha}_n)^{-1}$ *exists in* $\mathbb{Z}(d_n)$*. Also* $\mathbf{M}(\alpha, \beta) = \mathbf{M}(1, \alpha^{-1}\beta)$.

Equivalently Eq.(4.12) can be written as

$$\mathbf{M}(1, \rho^{-1}\vartheta) = \mathbf{M}(1, \vartheta_1) \times \mathbf{M}(1, \vartheta_2) \equiv \mathfrak{M}(\vartheta_1, \vartheta_2)$$
$$\vartheta = \vartheta_1 s_1 + \vartheta_2 s_2; \qquad \vartheta_n = (\overline{\alpha}_n)^{-1}\beta_n \in \mathbb{Z}(d_n); \qquad \rho^{-1}\vartheta = \alpha^{-1}\beta \in \mathbb{Z}(d)$$
$$\rho = d_1 + d_2 \tag{4.13}$$

(2) If $\overline{\alpha}_1 = d_1 = 0 \pmod{d_1}$ *then* $\vartheta_1 = -1$ *by definition and*

$$\mathbf{M}(d_1, s_1 + s_2\vartheta_2) = \mathbf{M}(0, 1) \times \mathbf{M}(1, \vartheta_2) \equiv \mathfrak{M}(-1, \vartheta_2)$$
$$\vartheta_2 = (\overline{\alpha}_2)^{-1}\beta_2 \tag{4.14}$$

For the case that $\alpha_2 = d_2 = 0 \pmod{d_2}$ *and by definition* $\vartheta_2 = -1$ *, we have a similar result to (2)*

$$\mathbf{M}(d_2, s_2 + s_1\vartheta_1) = \mathbf{M}(1, \vartheta_1) \times \mathbf{M}(0, 1) \equiv \mathfrak{M}(\vartheta_1, -1)$$
$$\vartheta_1 = (\overline{\alpha}_1)^{-1}\beta_1 \tag{4.15}$$

(3) If $\alpha_1 = 0 \pmod{d_1}$ *and* $\alpha_2 = 0 \pmod{d_2}$ *then* $\vartheta_1 = \vartheta_2 = -1$ *by definition and*

$$L(0, 1) = L(0, 1) \times \mathbf{M}(0, 1) \equiv \mathfrak{M}(-1, -1). \tag{4.16}$$

Proof. All we need to do in these different cases is to show that the sets of points in the two sides are identical.

(1) We express the R.H.S as

$$\mathbf{M}(1,\vartheta_1)\times\mathbf{M}(1,\vartheta_2)=\mathcal{S}(0,-1|1,\vartheta_1)\mathbf{M}(0,1)\times\mathcal{S}(0,-1|1,\vartheta_2)L(0,1) \quad (4.17)$$

With the fact that $\mathbf{M}(0,1)\times\mathbf{M}(0,1)$ is the line $\mathbf{M}(0,1)$ in $\Pi(d)$ and using Eq.(3.68) which defines the coressponding symplectic transformation, we have

$$\mathcal{S}(0,-\rho^{-1}|\rho,\vartheta)L(0,1)=\mathbf{M}(1,\rho^{-1}\vartheta) \quad (4.18)$$

which is the L.H.S.

The parameters are given in Eq.(3.68) and the symplectic transformation on the line defined in Eq.(4.10).

(2) For this case we use Eq.(3.69) to derive

$$L(0,1)\times\mathcal{S}(0,-1|1,\vartheta_2)\mathbf{M}(0,1)=\mathcal{S}(\eta,\zeta|\rho,\vartheta)\mathbf{M}(0,1)=\mathbf{M}(d_1,s_1+\vartheta_2 s_2)$$
$$\eta=s_1;\quad \zeta=-s_2d_1^{-1};\quad \rho=d_1;\quad \vartheta=s_1+\vartheta_2 s_2 \quad (4.19)$$

(3) This proves follows from definition for the case $\vartheta_1=\vartheta_2=-1$.

□

Therefore in $\mathfrak{M}(\vartheta_1,\vartheta_2)$ the $\vartheta_n=-1,0,...,d_n-1$. There are $\psi(d)=(d_1+1)(d_2+1)$ such lines through the origin , where $\psi(d)$ is the Dedekind ψ-function.

Table A gives a detailed example of this factorization for $\Pi(35)$ ($d_1=5$ and $d_2=7$)

.

We hereby confirm that the finite geometry is non-near-linear geometry. The common points between two lines are described in the following proposition:

Proposition 4.4.2. *Two maximal lines* $\mathbf{M}(1, \rho^{-1}\vartheta) = \mathfrak{M}(\vartheta_1, \vartheta_2)$ *and* $\mathbf{M}(1, \rho^{-1}\vartheta') = \mathfrak{M}(\vartheta_1', \vartheta_2')$ *through the origin, have in common* $p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')$ *points. The different possible values of* $p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')$ *is given in below.*

$$\begin{aligned} &p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') = 1 \text{ if } \vartheta_1 \neq \vartheta_1' \text{ and } \vartheta_2 \neq \vartheta_2' \\ &p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') = d_1 \text{ if } \vartheta_1 = \vartheta_1' \text{ and } \vartheta_2 \neq \vartheta_2' \\ &p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2') = d_2 \text{ if } \vartheta_1 \neq \vartheta_1' \text{ and } \vartheta_2 = \vartheta_2' \end{aligned} \tag{4.20}$$

Proof. The common points in the two lines should satisfy the relation

$$(\zeta, \zeta\rho^{-1}\vartheta) = (\zeta, \zeta\rho^{-1}\vartheta') \;\; \rightarrow \;\; \zeta[(\vartheta_1 - \vartheta_1')s_1 + (\vartheta_2 - \vartheta_2')s_2] = 0. \tag{4.21}$$

This needs to be proved for the three different cases given in Eq(200)

First for the case $\vartheta_1 \neq \vartheta_1'$ and $\vartheta_2 \neq \vartheta_2'$.

This implies that $(\vartheta_1 - \vartheta_1')s_1 + (\vartheta_2 - \vartheta_2')s_2$ is always different from zero, because the map of Eq.(3.57) is bijective (and $0 \leftrightarrow (0, 0)$). Therefore in this case $\zeta = 0$.

We next consider the case $\vartheta_1 = \vartheta_1'$ and $\vartheta_2 \neq \vartheta_2'$.

We use the relation $d_2 s_2 = 0$ (Eq.(3.60)), this implies that any ζ which is multiple of d_2 will give $\zeta[(\vartheta_2 - \vartheta_2')s_2] = 0$.

Therefore there are d_1 values of ζ which lead to common points.

Similarly for $\vartheta_1 \neq \vartheta_1'$ and $\vartheta_2 = \vartheta_2'$ we use relation $d_1 s_1 = 0$ (Eq.(3.60)).

We need any ζ which is multiple of d_1 to give $\zeta[(\vartheta_1 - \vartheta_1')s_1] = 0$.

Therefore there are d_2 values of ζ which lead to common points. □

4.4.1 Example

We give an example of two lines through the origin in $\Pi(35)$, which have seven points in common. The lines $\mathbf{M}(1,33) = \mathfrak{M}(1,4)$ and $\mathbf{M}(1,19) = \mathfrak{M}(3,4)$ have in common the seven points

$$(0,0),(5,25),(10,15),(15,5),(20,30),(25,30),(30,10) \tag{4.22}$$

and they are shown in Fig. 4.4. This further confirms that our geometry is a non-near-linear geometry.

Analogous example in terms of bases in $H(35)$ and two lines of zeros in $\mathfrak{Z}(35)$ is given later.

4.5 Summary

We have studied the geometry $\Pi(d)$ of maximal lines in $\mathbb{Z}(d) \times \mathbb{Z}(d)$. The properties of lines were examined together with the symplectic transformation on points and lines. The concept 'sublines' was introduced with examples. We also used the concept of factorization on maximal lines in the geometry. We established that the finite geometry $\mathbb{Z}(d) \times \mathbb{Z}(d)$ is a non near linear geometry, which two lines can have more than one point in common. We gave a detailed example to support this.

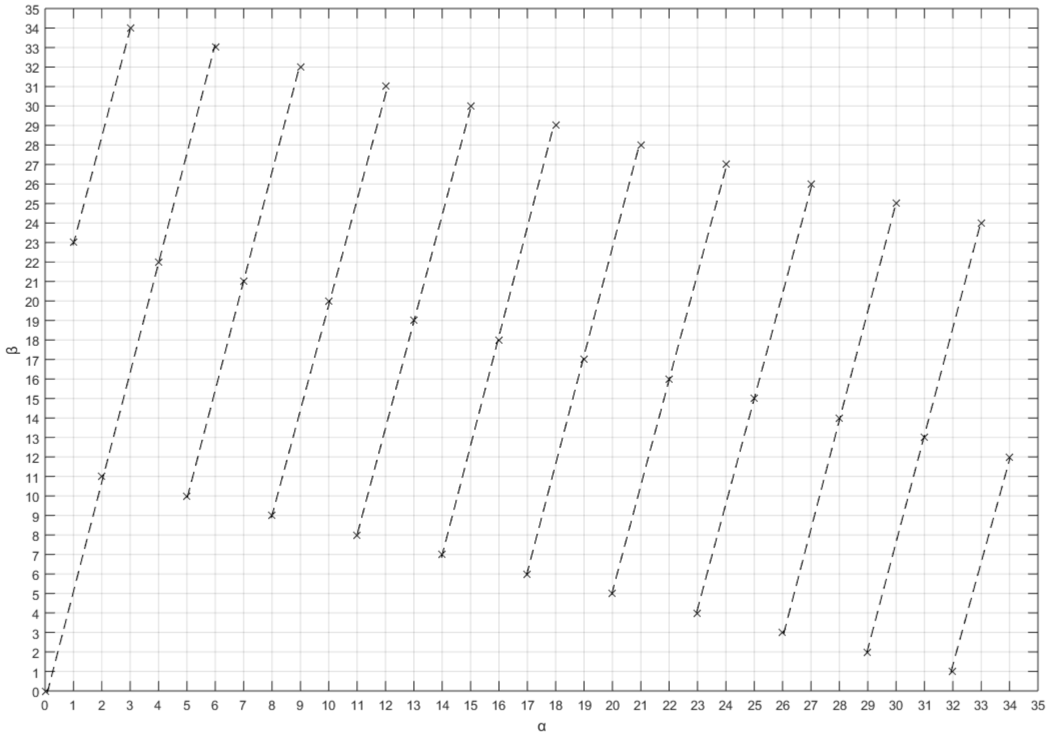

Fig. 4.1 The maximal line $\mathbf{M}(2,11)$ in the geometry $\Pi(35)$. * shows each of the 35 points. Since $\mathbb{Z}(35) \times \mathbb{Z}(35)$ is torodial, the figure shows a single line from the origin (other lines are continuation of the line from the origin).

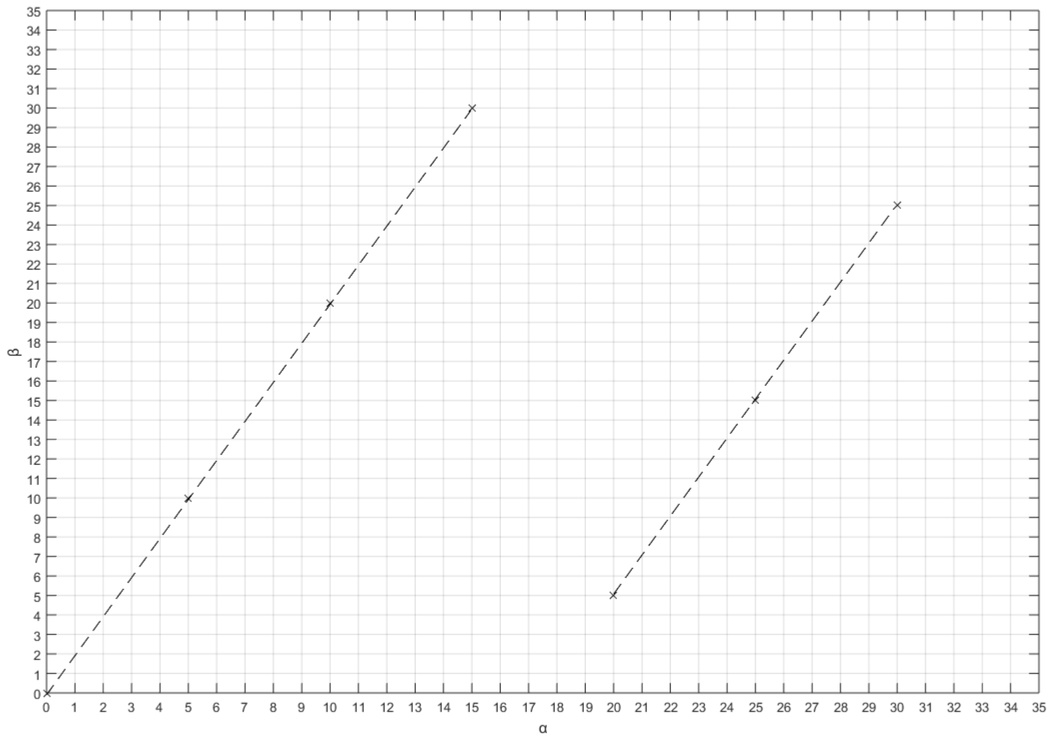

Fig. 4.2 The subline $\mathbf{M}(10, 20)$ in the geometry $\Pi(35)$. 'x' shows each of the '7' points. Since $\mathbb{Z}(35) \times \mathbb{Z}(35)$ is torodial , the figure shows a single line from the origin (other lines are continuation of the line from the origin).

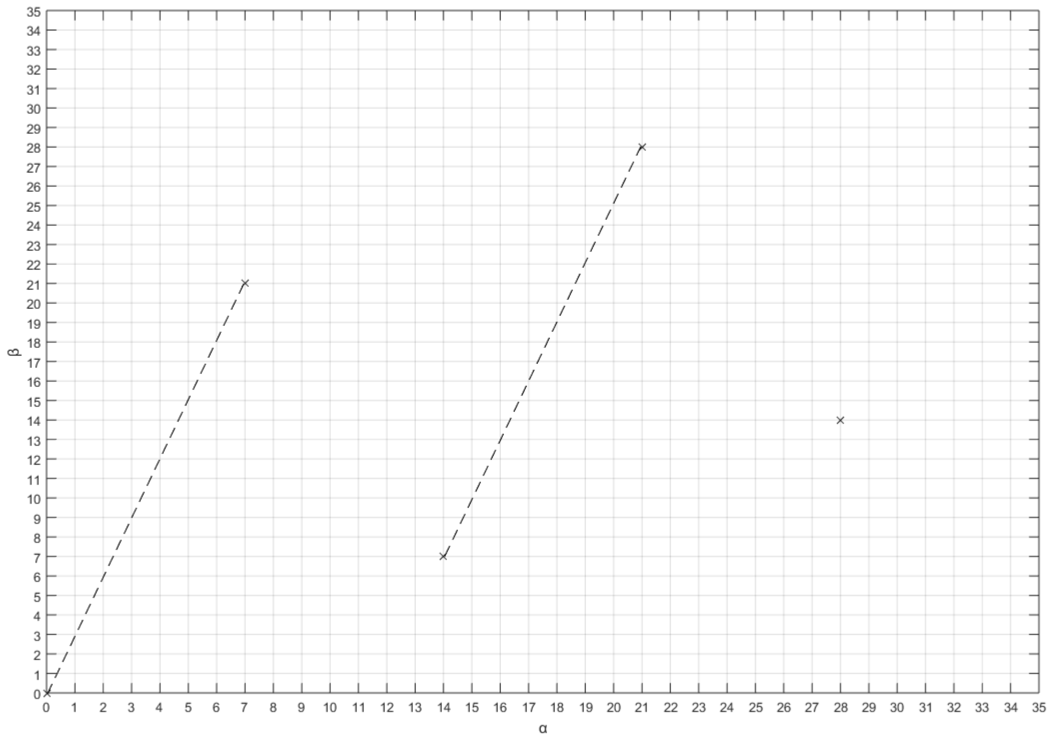

Fig. 4.3 The subline $\mathbf{M}(14, 7)$ in the geometry $\Pi(35)$. ‘x’ shows each of the ‘7’ points. Since $\mathbb{Z}(35) \times \mathbb{Z}(35)$ is torodial , the figure shows a single line from the origin (other lines are continuation of the line from the origin).

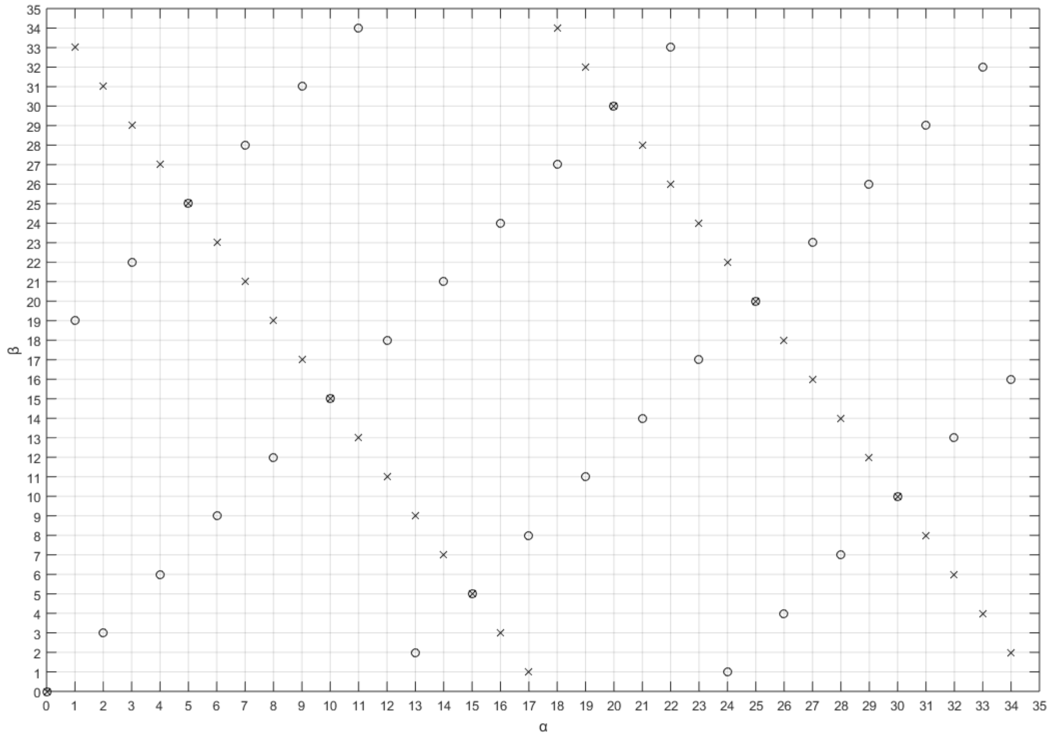

Fig. 4.4 The line $\mathbf{M}(1, 33)$ (crosses) and the line $\mathbf{M}(1, 19)$ (circles) in the geometry $\Pi(35)$.
The two lines have in common seven points $(0, 0)$, $(5, 25)$, $(10, 15)$, $(15, 5)$, $(20, 30)$, $(25, 30)$, $(30, 10)$.

Chapter 5

Weak Mutually Unbiased Bases

In this chapter, we introduce a weaker concept of the mutually unbiased bases in the non-prime dimension d, called the weak mutually unbiased bases (WMUBs). For a non-prime d we use the factorization of quantum systems discussed in chapter 3, where by a quantum system $H(d)$ factorizes into $H(d_1)$ and $H(d_2)$ i.e $H(d) = H(d_1) \otimes H(d_2)$. Weak mutually unbiased bases are expressed as tensor products of mutually unbiased bases in $H(d_n)$. [57, 58] has a detailed formalism for weak mutually unbiased bases.

We present an explicit construction for a complete set weak mutually bases for non-prime d using symplectic transformations. We end by establishing a duality between lines in finite geometry $\Pi(d)$ and weak mutually unbiased bases in $H(d)$

5.1 Definition

A set of k orthonormal bases $|\mathbb{X}_n; b\rangle in\ H(d)$, where $b \in \mathbb{Z}(d)$ and $n = 0, \ldots, k-1$. Such bases are called weak unbiased bases if for any pair of them the overlap

$$|\langle \mathbb{X}_n; b | \mathbb{X}_m; a \rangle| = 1/\sqrt{p} \tag{5.1}$$

where $1/p$ is 0 or p is a divisor of d:

5.2 Construction of Weak Mutually Unbiased Bases

In this section we give a detailed construction of weak mutually unbiased bases in $H(d)$ where $d = d_1 \times d_2$ and d_1, d_2 are primes. We proceed to introduce our notation used in later sections. We make use of the symplectic transformations introduced earlier, acting on position states . In general we combine mutually unbiased bases in $H(d_1)$ with mutually unbiased bases in $H(d_2)$ to derive the set of weak mutually unbiased bases.

For mutually unbiased bases $|\mathbb{X}_1(\vartheta_1);\overline{a}_1\rangle \in H(d_1)$ and $|\mathbb{X}_2(\vartheta_2);\overline{a}_2\rangle \in H(d_1)$ we write the corresponding weak mutually unbiased bases $|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$ as

$$|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X}_1(\vartheta_1);\overline{a}_1\rangle \otimes |\mathbb{X}_2(\vartheta_2);\overline{a}_2\rangle \tag{5.2}$$

The each mutually unbiased bases $|\mathbb{X}_n(\vartheta_n);\overline{a}_n\rangle$ is given as

$$|\mathbb{X}_n(\vartheta_n);\overline{a}_n\rangle = S(0,-1|1,\vartheta_i)\,|\mathbb{X}_n;\overline{a}_n\rangle\,; \qquad \overline{a}_n \in \mathbb{Z}(d_n). \tag{5.3}$$

In the special case $\vartheta_1 = \vartheta_2 = -1$ we get

$$|\mathfrak{X}(-1,-1);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X}_1(-1);\overline{a}_1\rangle \otimes |\mathbb{X}_2(-1);\overline{a}_2\rangle = |\mathbb{X}_1;\overline{a}_1\rangle \otimes |\mathbb{X}_2;\overline{a}_2\rangle \tag{5.4}$$

In the special case $\vartheta_1 = \vartheta_2 = 0$, which is the momentum bases we get

$$|\mathfrak{X}(0,0);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X}_1(0);\overline{a}_1\rangle \otimes |\mathbb{X}_2(0);\overline{a}_2\rangle = |\mathbb{P}_1;a_1\rangle \otimes |\mathbb{P}_2;a_2\rangle \tag{5.5}$$

We also note the $\vartheta_i = -1$, in which case $|\mathbb{X}_n(-1);\overline{a}_n\rangle$ is just the position state $|\mathbb{X}_n;\overline{a}_n\rangle$.

Therefore $\vartheta_n = -1, ..., d_n - 1$.

The overlap of two vectors in two different bases, is 0 or $1/p$ where p is a divisor of d:

$$|\langle \mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2 \, | \mathfrak{X}(\vartheta_1', \vartheta_2'); \overline{b}_1, \overline{b}_2 \rangle \, |^2 = \frac{1}{p} \text{ or } 0; \quad p|d. \tag{5.6}$$

In place of the requirement which defines mutually unbiased bases, which is that the square of the absolute value of the overlap is $1/d$ in mutually unbiased bases we have the weaker requirement that it is $1/p$ or 0. And that is why we call them weak mutually unbiased bases.

There are $\psi(d) = (d_1 + 1)(d_2 + 1)$ weak mutually unbiased bases.

To express the weak mutually unbiased bases in terms of the symplectic transformation acting on them we can relabel the $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ as follows:

•

$$\begin{aligned}
&|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle = S(0, -1|1, \vartheta_1) \, |\mathbb{X}; \overline{a}_1\rangle \otimes S(0, -1|1, \vartheta_2) \, |\mathbb{X}; \overline{a}_2\rangle \\
&= S(0, -\rho^{-1}|\rho, \vartheta) \, |\mathbb{X}; a\rangle = S(0, -1|1, \rho^{-1}\vartheta) S(\rho^{-1}, 0|0, \rho) \, |\mathbb{X}; a\rangle \\
&= S(0, -1|1, \rho^{-1}\vartheta) \, |\mathbb{X}; a\rho^{-1}\rangle \equiv |\mathbb{X}(1, \rho^{-1}\vartheta); a\rho^{-1}\rangle \\
&\vartheta = \vartheta_1 s_1 + \vartheta_2 s_2; \quad \rho = d_1 + d_2; \quad \vartheta_i = 0, ..., d_n - 1
\end{aligned} \tag{5.7}$$

Here we have used Eq.(3.68), and a is related to $\overline{a}_1, \overline{a}_2$ through Eq.(3.58).

•

$$|\mathfrak{X}(-1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X};\overline{a}_1\rangle \otimes S(0,-1|1,\vartheta_2)\,|\mathbb{X};\overline{a}_2\rangle$$
$$= |\mathbb{X};\overline{a}_1\rangle \otimes S(0,-1|1,\vartheta_2)\,|\mathbb{X};\overline{a}_2\rangle = S(\eta,\zeta|\rho,\vartheta)\,|\mathbb{X};a\rangle = |\mathbb{X}(d_1,s_1+\vartheta_2 s_2);a\rangle$$
$$\eta = s_1;\quad \zeta = -s_2 d_1^{-1};\quad \rho = d_1;\quad \vartheta = s_1+\vartheta_2 s_2 \tag{5.8}$$

Here we used Eq.(3.69).

In a similar way using Eq.(3.70) we get

$$|\mathfrak{X}(\vartheta_1,-1);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X}(d_2,s_2+\vartheta_1 s_1);a\rangle \tag{5.9}$$

• and by definition,

$$|\mathfrak{X}(-1,-1);\overline{a}_1,\overline{a}_2\rangle = |\mathbb{X};\overline{a}_1\rangle \otimes |\mathbb{X};\overline{a}_2\rangle = |\mathbb{X}(0,1);a\rangle \tag{5.10}$$

There are $\psi(d) = (d_1+1)(d_2+1)$ weak mutually unbiased bases which can be broken down as follows; - $d_1 d_2$ states in Eq.(5.7) (which have already been introduced in Eq.(3.81))
- d_1 states in Eq.(5.8)
- d_2 states in Eqs(5.9)
- and one state in Eq.(5.10) .

We have expressed the WMUBs in two different notations.
In the first notation we have four different cases which shows the different symplectic

transformations acting on the position states:

$$\begin{aligned}
&|\mathbb{X}(1, \rho^{-1}\vartheta); m\rho^{-1}\rangle = S(0, -1|1, \vartheta)\,|\mathbb{X}; a\rho^{-1}\rangle \\
&|\mathbb{X}(d_1, s_1 + \vartheta_2 s_2); a\rangle = S(s_1, -s_2 d_1^{-1}|d_1, s_1 + \vartheta_2 s_2)\,|\mathbb{X}; a\rangle \\
&|\mathbb{X}(d_2, s_2 + \vartheta_1 s_1); a\rangle = S(s_2, -s_1 d_2^{-1}|d_2, s_2 + \vartheta_1 s_1)\,|\mathbb{X}; a\rangle \\
&|\mathbb{X}(0, 1); a\rangle = |\mathbb{X}; \overline{a}_1\rangle \otimes |\mathbb{X}; \overline{a}_2\rangle
\end{aligned} \tag{5.11}$$

In the second notation we expressed ϑ in its factorized form $(\vartheta_1, \vartheta_2)$. Therefore in the first notation

$$\mathbf{W}(\rho, \vartheta) = \{|\mathbb{X}(\rho, \vartheta); a\rangle\}, \tag{5.12}$$

where ρ takes the values $1, d_1, d_2, 0$ and in the second notation $\mathfrak{W}(\vartheta_1, \vartheta_2)$ is the basis

$$\mathfrak{W}(\vartheta_1, \vartheta_2) = \{|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle\}; \qquad \vartheta_n = -1, ..., d_n - 1. \tag{5.13}$$

The overlap of Eq.(5.6) for vectors in two bases $\mathfrak{W}(\vartheta_1, \vartheta_2)$ and $\mathfrak{W}(\vartheta_1', \vartheta_2')$ takes one of the two values $\frac{p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')}{d}$ or 0. We express this as

$$(\mathfrak{W}(\vartheta_1, \vartheta_2), \mathfrak{W}(\vartheta_1', \vartheta_2')) = \frac{p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')}{d} \text{ or } 0 \tag{5.14}$$

where the different possible values of $p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')$ is given in Eq(4.20).

This is consistent with the definition of weak mutually unbiased bases given at the beginning of this chapter.

5.3 The duality between weak mutually unbiased bases in $H(d)$ and lines in $\Pi(d)$

In this section we discuss the concept of duality that exists in $H(d)$ and lines in $\Pi(d)$. We have previously discussed the properties of lines and in particular maximal lines in $\Pi(d)$. We have also seen explicit construction using symplectic transformation. As discussed in the earlier section, we can construct weak mutually unbiased bases in $H(d)$ using similar symplectic transformation on position states.
Weak mutually unbiased bases are expressed as product of mutually unbiased bases in each prime dimensional Hilbert space $H(d_n)$. Using the same concept of factorization, we express lines in $\Pi(d)$ as factorized prime factor lines in $\Pi(d_n)$. Overall we have established the following correspondence

1. The lines $\mathbf{M}(1, \rho^{-1}\vartheta)$, $\mathbf{M}(p_1, s_1 + s_2\vartheta_2)$, $\mathbf{M}(p_2, s_2 + s_2\vartheta_1)$ of Eqs(4.17, 4.18, 4.19) corresponds to basis $|\mathbb{X}(1, \rho^{-1}\vartheta); a\rho^{-1}\rangle$, $|\mathbb{X}(p_1, s_1 + s_2\vartheta_2); a\rangle$, $|\mathbb{X}(p_2, s_2 + s_2\vartheta_1); a\rangle$ in Eq(5.11) respectively. This is due to the fact that the symplectic transformations used for the lines are the same as the symplectic transformation for the basis.

2. There are $\psi(d) = (d_2 + 1)(d_1 + 1)$ maximal lines through the origin in $\Pi(d)$. This corresponds to the set of $\psi(d)$ weak mutually unbiased bases in $H(d)$.

From above we can conclude that there exists a bijective map (duality) between the lines in $\Pi(d)$ and the weak mutually unbiased bases in $H(d)$ as follows:

$$\mathfrak{W}(\vartheta_1, \vartheta_2) \leftrightarrow \mathfrak{M}(\vartheta_1, \vartheta_2). \tag{5.15}$$

Example We an equivalent of example 4.4.1 in terms of bases is the $\mathfrak{W}(1, 4)$ and

$\mathfrak{W}(3,4)$. In this case, we have $\vartheta_2 = \vartheta_2'$, therefore $p(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') = d_2$

$$(\mathfrak{W}(1,4), \mathfrak{W}(3,4)) = \frac{7}{35} \text{ or } 0. \tag{5.16}$$

Table A shows explicitly this duality for the case $d = 35$.

In later chapters, we show as our novel result, that there is another bijective map between these two sets and the set of zeros, in an analytic representation approach to weak mutually unbiased bases.

5.4 Summary

In this chapter the weak mutually unbiased bases in H(d) was studied. We gave a detailed construction of weak mutually unbiased bases for non-prime d dimension. We established the correspondence which explains the duality between maximal lines through the origin and weak mutually unbiased bases.

Chapter 6

Analytic representation of weak mutually unbiased bases

In this chapter we present an analytic representation of weak mutually unbiased bases using Theta function. Theta functions are Gaussian functions wrapped on a circle. Symplectic transformations on Gaussian functions in a real line, give Gaussian functions. We present an analogous for Theta function. This helps to prove that the zeros of the analytic representation for the state $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ are on a straight line.

6.1 Analytic Representations of Finite Quantum Systems

Let $|f\rangle$ be an arbitrary state

$$\begin{aligned} |f\rangle &= \sum_a f_a \, |X;a\rangle \;= \sum_a \widetilde{f}_a \, |P;a\rangle \, ; \qquad \sum_a |g_a|^2 = 1 \\ \widetilde{g}_a &= d^{-1/2} \sum_b \omega(-ab) g_b \end{aligned} \tag{6.1}$$

We use the notation (star indicates complex conjugation)

$$|f^*\rangle = \sum_a f_a^* \, |X;a\rangle\,; \qquad \langle f| = \sum_a f_a^* \, \langle X;a|\,; \qquad \langle f^*| = \sum_a f_a \, \langle X;a| \tag{6.2}$$

The analytic representation of state $|f\rangle$ is defined as,

$$F(z) = \pi^{-1/4} \sum_{a=0}^{d-1} f_a^* \, \Theta_3 \left[\frac{\pi a}{d} - z\frac{\pi}{d}; \frac{i}{d} \right] \tag{6.3}$$

where Θ_3 is Theta function [45]

6.1.1 Theta Function

Theta functions are function defined for any two complex variables u and τ as

$$\Theta_3(u,\tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi\tau n^2 + i2nu) \tag{6.4}$$

Properties of Theta function

- $$\Theta_3[u;\tau] = \Theta_3[u + \pi m + \pi n\tau; \tau] \exp(2inu + i\pi\tau n^2) \tag{6.5}$$

- $$[\Theta_3[u;\tau]]^* = \Theta_3[u;\tau] \tag{6.6}$$

- They are quasi-periodic with the following periodicity conditions are

$$\begin{aligned} \Theta_3(u+\pi,\tau) &= \Theta_3(u,\tau+2) = \Theta_3(u,\tau) \\ \Theta_3(u+\tau\pi,\tau) &= \Theta_3(u,\tau)\exp[-i(\pi\tau+2u)] \end{aligned} \tag{6.7}$$

- The zeros of Theta functions are given by

$$\zeta_{SR} = (2S-1)\frac{\pi}{2} + (2R-1)\frac{i\pi}{2d}. \tag{6.8}$$

- An important property which is useful in our work is given below as

$$\Theta_3(u,\tau) = (-i\tau)^{-1/2} \exp\left[\frac{u^2}{i\pi\tau}\right] \Theta_3\left(\frac{u}{\tau}, \frac{-1}{\tau}\right), \tag{6.9}$$

The function $F(z)$ satisfies the following periodicity conditions

$$\begin{aligned} F(z+d) &= F(z) \\ F(z+id) &= F(z)\exp\left(-\pi d - 2i\pi z\right). \end{aligned} \tag{6.10}$$

We define $F(z)$ on a square area Γ on the complex plane

$$\Gamma_{SR} = [Sd, (S+1)d) \times [Rd, (R+1)d] \tag{6.11}$$

where (S, R) are integers labelling the cell.

The analytic representation of the scalar product of any two states is given by

$$\langle f_2 | f_1^* \rangle = \frac{\sqrt{2\pi}}{d^{5/2}} \int_{\mathfrak{S}} dz_R dz_I \exp\left(\frac{-2\pi}{d} z_I^2\right) F_1(z) F_2(z^*) = \sum_{a\in\mathbb{Z}(d)} g_{2a}^* g_{1a}^* \tag{6.12}$$

We have the orthogonality relation expressed as

$$\frac{\sqrt{2}}{d^{5/2}} \int_S exp\left(\frac{-2\pi}{d} z_I^2\right) d\mu(z) \Theta_3\left[\frac{\pi b}{d} - z\frac{\pi}{d}; \frac{i}{d}\right] \Theta_3\left[\frac{\pi a}{d} - z^* \frac{i}{d}; \frac{\pi}{d}\right] = \delta(a,b) \tag{6.13}$$

where $\delta(a,b)$ is Kronecker's delta, is proved as follows

Proof. Using the definition of Theta function, we write Eq.(6.13)

$$\begin{aligned}
&= \frac{\sqrt{2}}{d^{5/2}} \int_0^d dz_R dz_I \exp\left(\frac{-2\pi}{d} z_I^2\right) \\
&\times \sum_{k,l=-\infty}^{\infty} \exp\left[\frac{-\pi k^2}{d} + 2ik\frac{\pi b}{d} - 2ik(z_R + iz_I)\frac{\pi}{d}\right] \\
&\times \exp\left[\frac{-\pi l^2}{d} + 2il\frac{\pi a}{d} - 2ik(z_R - iz_I)\frac{\pi}{d}\right] \qquad (6.14)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{d^{5/2}} \int_0^d dz_R dz_I \exp\left(\frac{-2\pi}{d} z_I^2\right) \sum_{k,l=-\infty}^{\infty} exp\left[\frac{2i\pi}{d}(ka+lb)\right] \\
&\times \exp\left[\frac{-\pi k^2}{d} - \frac{\pi l^2}{d} - 2ik\frac{\pi}{d} z_R - 2il\frac{\pi}{d} z_R\right] \\
&\times \exp\left[2\frac{\pi}{d} k z_I - 2\frac{\pi}{d} l z_I\right] \qquad (6.15)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{d^{5/2}} \sum_{k,l=-\infty}^{\infty} \exp\left[\frac{2i\pi}{d}(ka+lb)\right] \exp\left[\frac{-\pi k^2}{d} - \frac{\pi l^2}{d}\right] \\
&\times \int_0^d dz_I \exp\left[-\frac{2\pi}{d} z_I^2 + 2\frac{\pi}{d} k z_I - 2\frac{\pi}{d} l z_I\right] \\
&\times \int_0^d dz_R \exp\left[-i\frac{2\pi}{d}(k+l) z_R\right] \qquad (6.16)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{d^{5/2}} \sum_{k,l=-\infty}^{\infty} exp\left[\frac{2i\pi}{d}(ka+lb)\right] exp\left[\frac{-\pi}{d}(l^2+k^2)\right] \\
&\times \int_0^d dz_I \exp\left[-\frac{2\pi}{d}(z_I^2 - kz_I + lz_I)\right] \\
&\times \int_0^d dz_R \exp\left[-i\frac{2\pi}{d}(k+l) z_R\right] \qquad (6.17)
\end{aligned}$$

Since

$$\int_0^d dz_R\ exp\left[-i\frac{2\pi}{d}(k+l)z_R\right] = d\delta(k,-l) \tag{6.18}$$

Inserting Eq.(6.18) into Eq.(6.17), we have

$$\begin{aligned}
&= \frac{\sqrt{2}}{d^{3/2}}\sum_{k,l=-\infty}^{\infty}\exp\left[\frac{2i\pi}{d}(ka+lb)\right]\exp\left[\frac{-\pi}{d}(l^2+k^2)\right] \\
&\times\int_0^d dz_I\ exp\left[-\frac{2\pi}{d}(z_I^2-kz_I+lz_I)\right]\delta(k,-l) \\
&= \frac{\sqrt{2}}{d^{3/2}}\sum_{k=-\infty}^{\infty}\exp\left[\frac{2i\pi}{d}k(b-a)\right] \\
&\times\int_0^d dz_I\exp\left[-\frac{2\pi}{d}(z_I-k)^2\right]
\end{aligned} \tag{6.19}$$

Let $k = k_0 + Nd$ for $0 \le k_0 \le d-1$

$$= \frac{\sqrt{2}}{d^{3/2}}\sum_{k_0=0}^{d_1} exp\left[\frac{2i\pi}{d}k_0(b-a)\right]\sum_{N=-\infty}^{\infty}\int_0^d exp\left[\frac{-2\pi}{d}(z_I-(k_0+Nd))^2\right] \tag{6.20}$$

which gives

$$\begin{aligned}
&= d^{-1}\sum_{k_0=0}^{d_1} exp\left[\frac{2i\pi}{d}k_0(b-a)\right] \\
&= \delta(b,a)
\end{aligned} \tag{6.21}$$

In Eq(6.21) above, we have used

$$\begin{aligned}
\sum_{N=-\infty}^{\infty}\int_0^d \exp\left[\frac{-2\pi}{d}(z_I-(k_0+Nd))^2\right] &= \int_{-\infty}^{\infty} exp\left[\frac{-2\pi}{d}(z_I-k_0)^2\right] \\
&= \sqrt{\frac{d}{2}}
\end{aligned} \tag{6.22}$$

□

Using this orthogonality relation we can derive the coefficients f_a in terms of $F(z)$

$$f_a = \frac{\sqrt{2}}{d^{5/2}} \int_S exp\left(\frac{-2\pi}{d} z_I^2\right) d\mu(z) \Theta_3 \left[\frac{\pi a}{d} - z\frac{\pi}{d}; \frac{i}{d}\right] F(z*) \tag{6.23}$$

The position eigenstates $|\mathbb{X}; a\rangle$ is represented analytically by

$$|\mathbb{X}; a\rangle \to \pi^{-1/4} \Theta_3 \left[\frac{\pi a}{d} - z\frac{\pi}{d}; \frac{i}{d}\right] \tag{6.24}$$

We derive the corresponding representation for the momentum eigenstates as follows

Proof. Taking the Fourier of Eq(6.24), we have

$$\begin{aligned}
|\mathbb{P}; m\rangle \quad \to \quad & \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \omega(am) \Theta_3 \left[\frac{\pi a}{d} - z\frac{\pi}{d}; \frac{i}{d}\right] \qquad (6.25) \\
&= \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \omega(am) \exp\left[\frac{-\pi n^2}{d} + \frac{2in\pi a}{d} - \frac{2inz\pi}{d}\right] \\
&= \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \omega(am + an) \exp\left[\frac{-\pi n^2}{d} - \frac{2inz\pi}{d}\right] \\
&= \sqrt{d} \sum_{n=-m+dN}^{d-1} \exp\left[\frac{-\pi n^2}{d} - \frac{2inz\pi}{d}\right] \\
&= \sqrt{d} \sum_{n=-m+dN}^{d-1} \exp\left[\frac{-\pi(-m + dN)^2}{d} - \frac{2i(-m + dN)z\pi}{d}\right] \\
&= \sqrt{d} \sum_{N}^{d-1} \exp\left[\frac{-\pi m^2}{d} + \frac{2imz\pi}{d}\right] \Theta\left[-i\pi m - z\pi\right]
\end{aligned}$$

□

We make use of Eq(6.9) in Eq(6.25) to define the momentum correspondingly as

$$|\mathbb{P}; a\rangle \leftarrow \pi^{-1/4} \exp\left(\frac{-1}{2} z^2\right) \Theta_3 \left[\frac{\pi a}{d} - z\frac{\pi}{d}; \frac{i}{d}\right] \tag{6.26}$$

6.2 Analytic representation of the weak mutually unbiased bases

The proposition below is our main result for this section, which gives the analytic representation of weak mutually unbiased bases. We start by presenting a Lemma which will be needed in the proposition.

Lemma 6.2.1.

$$\prod_n \omega[\sigma(\overline{a}_n, \overline{b}_n, \vartheta_n)] = \omega[\rho^{-1}\sigma(a,b,\vartheta)]; \quad \rho^{-1} = d_2^{-1}s_1 + d_1^{-1}s_2 \ (\mathrm{mod}\ d). \tag{6.27}$$

where $\sigma(a,k,n) = -ak + 2^{-1}\vartheta j^2$ *(see Eq.(3.77)).*

Proof. We use Eqs.(3.60) to prove that

$$ba\rho^{-1} = \overline{a}_1\overline{b}_1 d_2 + \overline{a}_2\overline{b}_2 d_1; \quad \vartheta_1(\overline{b}_1)^2 d_2 + \vartheta_2(\overline{b}_2)^2 d_1 = \rho^{-1}\vartheta b^2. \tag{6.28}$$

From these relations follows Eq.(6.27). □

Proposition 6.2.2. *The analytic representation of the state* $|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$ *where* $\vartheta_n = -1,...,d_n - 1$ *and* $\overline{a}_n \in \mathbb{Z}(d_n)$*, is given by:*

1. *For* $\vartheta_n = 0,...,d_n - 1$ *and* $\overline{a}_n \in \mathbb{Z}(d_n)$*, the analytic representation of the state* $|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$ *is given by*

$$F(z) = \pi^{-1/4}\exp\left(-\frac{\pi}{d}z^2\right)\Theta_3(u;\tau)$$

where

$$\tau = \frac{i - \vartheta\rho^{-1}(d+1)}{d}; \qquad u = -\pi\rho^{-1}\left(\frac{\overline{a}_1}{d_1} + \frac{\overline{a}_2}{d_2}\right) + i\frac{\pi z}{d_1 d_2} \tag{6.29}$$

ρ^{-1} is a constant and $\vartheta = \vartheta_1 s_1 + \vartheta_2 s_2$ according to Eq(3.58). s_n is a constant given in Eqs.(3.68),(3.59).

2. *For $\vartheta_1 = -1$ and $\vartheta_2 = 0, ..., d_2 - 1$, the corresponding analytic representation of $|\mathfrak{X}(-1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ is given by*

$$F(z) = \pi^{-1/4} \exp\left(\frac{-\pi d_2 w^2}{d_1}\right) \Theta_3(u; \tau)$$
$$\text{where} \tag{6.30}$$
$$\tau = \frac{-\vartheta_2(d_2+1) + id_1}{d_2}; \quad u = -\frac{\pi \overline{a}_2}{d_2} + i\pi w; \quad w = \frac{z}{d_2} - \overline{a}_1.$$

We have an equivalent representation for $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$, that is the case $\vartheta_1 = 0, ..., d_1 - 1$ and $\vartheta_2 = -1$, which is

$$F(z) = \pi^{-1/4} \exp\left(\frac{-\pi d_1 w^2}{d_2}\right) \Theta_3(u; \tau)$$
$$\text{where} \tag{6.31}$$
$$\tau = \frac{-\vartheta_1(d_1+1) + id_2}{d_1}; \quad u = -\frac{\pi \overline{a}_1}{d_1} + i\pi w; \quad w = \frac{z}{d_1} - \overline{a}_2.$$

(3) in the case $\vartheta_1 = \vartheta_2 = -1$

$$|\mathfrak{X}(-1,-1); a\rangle = |\mathbb{X}; a\rangle \quad \rightarrow \quad F(z) = \pi^{-1/4} \Theta_3\left[\frac{\pi a}{d} - z\left(\frac{\pi}{d}\right); \frac{i}{d}\right] \tag{6.32}$$

Proof.

(1) Using Eq.(3.77) with $\rho = 1$ we get

$$|\mathbb{X}(\vartheta_n); \overline{a}_1\rangle = \frac{1}{\sqrt{d_n}} \sum_{b_n=0}^{d_n-1} \omega[\sigma(\overline{a}_n, \overline{b}_n, \vartheta_n)] |\mathbb{X}; \overline{b}_n\rangle \tag{6.33}$$

Therefore

$$
\begin{aligned}
|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle &= \sum_k |\mathbb{X};k\rangle\langle\mathbb{X};k|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle \\
&= \sum_k |\mathbb{X};k\rangle[\langle\mathbb{X}_1;\overline{k}_1|\otimes\langle\mathbb{X}_2;\overline{k}_2|]|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle \\
&= \sum_k |\mathbb{X};k\rangle\left[\langle\mathbb{X}_1;\overline{k}_1|\mathbb{X}_1(\vartheta_1);\overline{a}_1\rangle\right]\left[\langle\mathbb{X}_2;\overline{k}_2|\mathbb{X}_2(\vartheta_2);\overline{a}_2\rangle\right] \\
&= \sum_k \frac{1}{\sqrt{d}}\prod_n \omega[\sigma(\overline{a}_n,\overline{k}_n,\vartheta_n)]|\mathbb{X};k\rangle. \qquad (6.34)
\end{aligned}
$$

We then use Eq.(6.27) and we get

$$
|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle = \sum_k \frac{1}{\sqrt{d}}\omega[\rho^{-1}\sigma(a,k,\vartheta)]|\mathbb{X};j\rangle. \qquad (6.35)
$$

Eq.(6.3) gives us the analytic representation of the state $|\mathbb{X};j\rangle$ and together with lemma 3.8.1 , we represent $|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$ with the sum

$$
\frac{\pi^{-1/4}}{\sqrt{d}}\sum_k \omega[-\rho^{-1}\sigma(a,k,\vartheta)]\Theta_3\left[\frac{\pi j}{d} - z\left(\frac{\pi}{d}\right);\frac{i}{d}\right] \qquad (6.36)
$$

Next we need to prove that this sum of Theta functions is equal to the single Theta function shown on the right hand side of Eq.(6.29).

We use the property of Theta functions in Eq.(6.9) and Eq.(6.4) and we have

$$
\begin{aligned}
\Theta_3\left[\frac{\pi k}{d} - z\left(\frac{\pi}{d}\right);\frac{i}{d}\right] &= \sqrt{d}\exp\left[\frac{-\pi k^2}{d} + 2k\left(\frac{\pi}{d}\right)z - \left(\frac{\pi}{d}z^2\right)\right] \\
&\times \quad \Theta_3\left[-i\pi k + i\pi z; id\right] \\
&= \sqrt{d}\exp\left[\frac{-\pi k^2}{d} + 2k\left(\frac{\pi}{d}\right)z - \left(\frac{\pi}{d}z^2\right)\right] \\
&\times \quad \sum_{n=-\infty}^{\infty}\exp\left[-\pi d n^2 + 2n\pi j - 2n\pi z\right]. \qquad (6.37)
\end{aligned}
$$

We replace 2^{-1} with $\frac{d+1}{2}$ in $\sigma(a,k,\vartheta)$ since d is odd. Therefore the sum

$$\frac{\pi^{-1/4}}{\sqrt{d}}\sum_k \omega[-\rho^{-1}\sigma(a,k,\vartheta)]\Theta_3\left[\frac{\pi j}{d} - z\left(\frac{\pi}{d}\right); \frac{i}{d}\right]$$

becomes

$$\begin{aligned}
&= \pi^{-1/4}\exp\left[\frac{-\pi}{d}z^2\right]\sum_{n=-\infty}^{\infty}\sum_{k=0}^{d-1}\exp\left[\frac{-\pi}{d}(-k+nd)^2\right] \\
&\exp\left[-\frac{i\pi\vartheta\rho^{-1}(d+1)}{d}(-k+nd)^2\right]\exp\left[-\frac{2i\pi m\rho^{-1}}{d}(-k+nd)\right] \\
&\times \quad \exp\left[-2(-k+nd)\left(\frac{\pi}{d}\right)z\right]
\end{aligned} \tag{6.38}$$

We now change variable into $N = nd - k$ which is possible since n takes all integer values in $\mathbb{Z}(d)$, the variable N takes all integer values .

Therefore the above sum becomes

$$\pi^{-1/4}\exp\left[\frac{-\pi}{d}z^2\right]\sum_{N=-\infty}^{\infty}\exp\left[\frac{-\pi}{d}N^2 - \frac{i\pi\vartheta\rho^{-1}(d+1)}{d}N^2 - \frac{2i\pi m\rho^{-1}N}{d} - 2N\left(\frac{\pi}{d}\right)z\right]. \tag{6.39}$$

This is the result in Eq.(6.29).

(2) $|\mathfrak{X}(-1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$ can also be expressed as the sum

$$\sum_b \delta(\overline{b}_1,\overline{a}_1)\omega(\sigma(\overline{a}_2,\overline{b}_2,\vartheta_2))|\mathbb{X};b\rangle. \tag{6.40}$$

where $b = \overline{b}_1 d_2 + \overline{b}_2 d_1$. Summation over k is equivalent to summation over both $\overline{b}_1, \overline{b}_2$.

Using the Eq.(6.3) which gives the analytic representation of the state $|\mathbb{X};b\rangle$, the sum in Eq(6.40) becomes

$$\frac{\pi^{-1/4}}{\sqrt{d}}\sum_{\overline{b}_1}\sum_{\overline{b}_2}\delta(\overline{b}_1,\overline{a}_1)\omega(-\sigma(\overline{a}_2,\overline{b}_2,\vartheta_2))\Theta_3\left[\frac{\pi b}{d}-z\left(\frac{\pi}{d}\right);\frac{i}{d}\right]$$
$$=\frac{\pi^{-1/4}}{\sqrt{d}}\sum_{\overline{b}_2}\omega(-\sigma(\overline{a}_2,\overline{b}_2,\vartheta_2))\Theta_3\left[\frac{\pi(\overline{a}_1d_2+\overline{k}_2d_1)}{d}-z\left(\frac{\pi}{d}\right);\frac{i}{d}\right]$$
$$=\frac{\pi^{-1/4}}{\sqrt{d}}\sum_{\overline{b}_2}\omega(-\sigma(\overline{a}_2,\overline{b}_2,\vartheta_2))\sqrt{d}\exp\left[\frac{-\pi(\overline{a}_1d_2+\overline{b}_2d_1)^2}{d}+2(\overline{a}_1d_2+\overline{b}_2d_1)\left(\frac{\pi}{d}\right)z-\left(\frac{\pi}{d}z^2\right)\right]$$
$$\times\Theta_3\left[-i\pi(\overline{a}_1d_2+\overline{b}_2d_1)+i\pi z;id\right]$$
$$=\pi^{-1/4}\exp\left[\frac{-\pi(\overline{a}_1d_2-z)^2}{d_1d_2}\right]\sum_{n=-\infty}^{\infty}\sum_{\overline{b}_2}\exp\left[-\frac{i\pi\vartheta_2(d_2+1)}{d_2}(nd_2-\overline{b}_2)^2-\frac{2i\pi\overline{a}_2}{d_2}(nd_2-\overline{b}_2)\right]$$
$$\times\exp\left[-\frac{\pi d_1}{d_2}(nd_2-\overline{b}_2)^2+2\pi\left(\overline{a}_1-\frac{z}{d_2}\right)(nd_2-\overline{b}_2)\right] \tag{6.41}$$

With a change of variable $nd_2-\overline{b}_2$ into N, and we have the Theta function in Eq.(6.30).

(3) The proof of Eq.(6.32) follows from the definition of the analytic representation given in Eq.(6.3).

□

Remark τ in Eq.(6.29) is in terms of $\vartheta\rho^{-1}$ which is an integer modulo d. Consequently, τ is defined up to an integer multiple of $d+1$. Since $d+1$ is an even integer, the Θ_3 does not change (Eq.(6.4)).

6.3 Zeros of the analytic representation of the WMUBs

In this section consider the states in WMUB $|\mathfrak{X}(\vartheta_1,\vartheta_2);\overline{a}_1,\overline{a}_2\rangle$, and using proposition 6.2.2 which is their analytic representation. We show that the zeros of their analytic representation are on a straight line.

Proposition 6.3.1. *The d zeros of the analytic representation of the vector $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ where $\vartheta_n = -1, ..., d_n - 1$ and $\overline{a}_n \in \mathbb{Z}(d_n)$, are on a straight line and they are given by:*

(1)

$$\begin{aligned}
&\varphi(\vartheta_1, \vartheta_2; \overline{a}_1, \overline{a}_2; R) = \kappa - i\lambda\kappa + \mu \\
&\textit{For } \vartheta_n = 0, ..., d_n - 1, \quad \textit{for } n = 1, 2, \\
&\kappa = \frac{2R-1}{2}; \quad \lambda = -\rho^{-1}\vartheta d - \rho^{-1}\vartheta; \quad \mu = \frac{-id2S + id - i\rho^{-1}2a}{2} \\
&\textit{where} \\
R = K+1, ..., K+d; &\quad a = \overline{a}_1 d_2 + \overline{a}_2 d_1; \quad \vartheta = \vartheta_1 s_1 + \vartheta_2 s_2
\end{aligned} \tag{6.42}$$

where ρ^{-1}, s_n are constants given in Eqs.(??), (3.59). To ensure we have the zeros in the cell Γ, we make a convenient choice for K and S .

(2)

$$\begin{aligned}
&\varphi(-1, \vartheta_2; \overline{a}_1, \overline{a}_2; R) = \kappa d_1 - i\lambda'\kappa + \mu' \\
&\textit{For } \vartheta_1 = -1 \quad \textit{and } \vartheta_2 = 0, \ldots, d_2 - 1 \\
&\kappa = \frac{2R-1}{2}; \quad \lambda = -\vartheta_2 - \vartheta_2 d_2; \quad \mu' = \overline{a}_1 d_2 - i\overline{a}_2 - id_2\left(\frac{2S-1}{2}\right) \\
&\textit{where} \\
R = K_1+1, ..., K_1+d_2; &\quad a = \overline{a}_1 d_2 + \overline{a}_2 d_1; \quad S = K_2+1, ..., K_2+d_1
\end{aligned} \tag{6.43}$$

An equivalent result holds for the case $\vartheta_2 = -1$ *and* $\vartheta_1 = 0, ..., d_1 - 1$ *which is*

$$\begin{aligned}
&\varphi(\vartheta_1, -1; \overline{a}_1, \overline{a}_2; R) = \kappa d_1 - i\lambda'\kappa + \mu' \\
&\textit{For } \vartheta_1 = 0, \ldots, d_2 - 1 \quad \textit{and } \vartheta_2 = -1 \\
&\kappa = \frac{2R-1}{2}; \quad \lambda = -\vartheta_1 - \vartheta_2 d_2; \quad \mu' = \overline{a}_2 d_1 - i\overline{a}_1 - id_1\left(\frac{2S-1}{2}\right) \\
&\textit{where} \\
&R = K_1 + 1, ..., K_1 + d_2; \qquad a = \overline{a}_2 d_1 + \overline{a}_1 d_2; \qquad S = K_2 + 1, ..., K_2 + d_1
\end{aligned} \tag{6.44}$$

To ensure we have the zeros in the cell Γ*, we make a convenient choice for* K_1 *and* K_2*.*

(3)

$$\begin{aligned}
&\varphi(-1, -1; \overline{a}_1, \overline{a}_2; N) = -i\kappa + \mu'' \\
&\textit{For } \vartheta_1 = \vartheta_2 = -1 \\
&\mu'' = \frac{2a - S2d + d + i2d}{2}; \quad \kappa = \frac{2R-1}{2} \\
&\textit{where} \\
&R = K + 1, ..., K + d \qquad a = \overline{a}_1 d_2 + \overline{a}_2 d_1
\end{aligned} \tag{6.45}$$

Appropriate choices of the 'winding integers' K, S*, locate the zeros in the desirable cell.*

Proof. (1) We use the single Theta function representation of $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ given in Eq.(6.29). Therefore, according to Eq(eq1) the zeros when $\vartheta = \vartheta_1 s_1 + \vartheta_2 s_2 =$

$0, ..., d-1$ are:

$$-\frac{\pi\rho^{-1}a}{d} + i\varphi\left(\frac{\pi}{d}\right) = (2S-1)\frac{\pi}{2} + (2R-1)\frac{\pi\tau}{2}$$

using the definition of τ *and* we solve for φ to give;

$$\varphi = \frac{2R-1}{2} + \frac{(2R-1)(-\vartheta\rho^{-1}(d+1))}{2} - \frac{-i(2S-1)d}{2} - i\rho^{-1}a \qquad (6.46)$$

We get the result of Eqs.(6.42).

where R, S are integers, and τ is given in Eq.(6.29).

(2) We use the single Theta function representation of $|\mathfrak{X}(-1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ in In Eq.(6.30).

Therefore, according to Eq(eq1) the zeros in the case $\vartheta_2 = 0, ..., d_2 - 1$ are:

$$-\frac{\pi\overline{a}_2}{d_2} + i\varphi\left(\frac{\pi}{d_2}\right) - i\pi\overline{a}_1 = (2S-1)\frac{\pi}{2} + (2R-1)\frac{\pi\tau}{2}$$

using the definition of τ *and* we solve for φ to give;

$$\varphi = \frac{2R-1}{2}d_1 + \frac{(2R-1)(i\vartheta_2(d_2+1))}{2} - \frac{-i(2S-1)d_2}{2} - i\overline{a}_2 + \overline{a}_1 d_2 \qquad (6.47)$$

and we get the result of Eqs.(6.43).

R, S are integers, and τ is given in Eq.(6.30).

For similar result for the case $\vartheta_1 = 0, ..., d_2 - 1$ and $\vartheta_2 = -1$, we use the single Theta function representation of $|\mathfrak{X}(\vartheta_1, -1); \overline{a}_1, \overline{a}_2\rangle$ in In Eq.(6.31). Therefore, according to Eq(6.49) the zeros are:

$$-\frac{\pi\overline{a}_1}{d_1} + i\varphi\left(\frac{\pi}{d_1}\right) - i\pi\overline{a}_2 = (2S-1)\frac{\pi}{2} + (2R-1)\frac{\pi\tau}{2}$$

using the definition of τ *and* we solve for φ to give;

$$\varphi = \frac{2R-1}{2}d_2 + \frac{(2R-1)(i\vartheta_1(d_1+1))}{2} - \frac{-i(2S-1)d_1}{2} - i\overline{a}_1 + \overline{a}_2 d_2 \qquad (6.48)$$

and we get the result of Eqs.(6.44).

R, S are integers, and τ is given in Eq.(6.31).

(3) in the case $\vartheta_1 = \vartheta_2 = -1$, the zeros of the Theta function in Eq.(6.32) give

$$\frac{\pi a}{d} - \zeta\left(\frac{\pi}{d}\right) = (2M-1)\frac{\pi}{2} + (2N-1)\frac{i\pi}{2d} \tag{6.49}$$

and from this follows Eq.(6.45).

□

In the previous chapter we have described the various vectors using two different notations. The first notation reflects the Symplectic transformation and the second notation uses the factorization of ϑ into ϑ_1 and ϑ_2. We use this same method to express the zeros corresponding to the various vectors.

Using the first notation for the four different cases, we have

$$\begin{aligned}
&\varphi'(1, \rho^{-1}\vartheta; a\rho^{-1}) \text{ for } \vartheta = \vartheta_1 s_1 + \vartheta_2 s_2;\\
&\varphi'(d_1, s_1 + \vartheta_2 s_2; a); \text{ for } \vartheta_1 = -1 \ \vartheta_2 = 0, ..., d_2 - 1\\
&\varphi'(d_2, s_2 + \vartheta_1 s_1; a) \text{ for } \vartheta_1 = 0, ..., d_1 - 1 \ \vartheta_2 = -1\\
&\varphi'(0, 1; a) \text{ for } \vartheta_1 = \vartheta_2 = -1
\end{aligned} \tag{6.50}$$

where ; $a = \overline{a}_1 d_2 + \overline{a}_2 d_1$; $\quad \vartheta_n = 0, ..., d_n - 1$

The following notation corresponds to the second notation which reflects the factorization of ϑ.

$$
\begin{aligned}
&\varphi(\vartheta_1, \vartheta_2; \overline{a}_1, \overline{a}_2) \text{ for } \vartheta = \vartheta_1 s_1 + \vartheta_2 s_2 \\
&\varphi(-1, \vartheta_2; \overline{a}_1, \overline{a}_2) \text{ for } \vartheta_1 = -1 \ \vartheta_2 = 0, ..., d_2 - 1 \\
&\varphi(\vartheta_1, -1; \overline{a}_1, \overline{a}_2) \text{ for } \vartheta_1 = 0, ..., d_1 - 1 \ \vartheta_2 = -1 \\
&\varphi(-1, -1; \overline{a}_1, \overline{a}_2) \text{ for } \vartheta_1 = \vartheta_2 = -1
\end{aligned} \tag{6.51}
$$

This is identical to the notation given in Eqs(5.7),(5.8),(5.9),(5.10) for the various vectors.

We describe the 'line' of the d zeros corresponding to $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ as follows;

$$
\mathfrak{K}(\vartheta_1, \vartheta_2; \overline{a}_1, \overline{a}_2) = \{\varphi(\vartheta_1, \vartheta_2; \overline{a}_1, \overline{a}_2; R);\ R = 1, ..., d\}; \quad \vartheta_n \in \mathbb{Z}(d_n) \tag{6.52}
$$

Using the first notation which reflects the symplectic transformation, the lines of the d zeros are written as

$$
\begin{aligned}
&\mathfrak{K}'(1, \rho^{-1}\vartheta; a) = \{\varphi'(1, \rho^{-1}\vartheta; a; R);\ R = 1, ..., d\} \\
&\mathfrak{K}'(d_1, s_1 + \vartheta_2 s_2; a) = \{\varphi'(d_1, s_1 + \vartheta_2 s_2; a; R);\ R = 1, ..., d\} \\
&\mathfrak{K}'(d_2, s_2 + \vartheta_1 s_1; a) = \{\varphi'(d_2, s_2 + \vartheta_1 s_1; a; R);\ R = 1, ..., d\} \\
&\mathfrak{K}'(0, 1; a) = \{\varphi'(0, 1; a; R);\ R = 1, ..., d\}
\end{aligned} \tag{6.53}
$$

6.4 The d^2 zeros in the cell Γ, of all d vectors in the basis $\mathfrak{W}(\vartheta_1, \vartheta_2)$

We have considered the d vectors in the weak mutually unbiased bases using the analytic representation in the previous section. We now give an precise expression which defines the d^2 zeros of the vectors. This reflects that the choices of ϑ_1, ϑ_2 does not change the set of the d^2 zeros and therefore could be represented as a lattice of zeros $\chi(d)$

Proposition 6.4.1. *In the basis $\mathfrak{W}(\vartheta_1, \vartheta_2)$, all the d^2 zeros in the cell Γ is given as;*

$$\gamma(m,n) = mr + in) + \frac{1}{2}(1+i); \quad m,n = 0, ..., d-1. \tag{6.54}$$

and they do not depend on $(\vartheta_1, \vartheta_2)$. We denote as $\chi(d)$ the lattice of these zeros.

Proof. For the four different cases in Proposition 6.3.1, we consider what happens in the real and imaginary axis :

(1) In the case $\vartheta_n = 0, ..., d_n - 1$ for $i = 1, 2$, we use the straight line of zeros given in Eq. (6.42). N takes all values $1, ..., d$ in the real axis. As N changes, $i\rho^{-1}a$ generates all the required values $1, ..., d$ in the imaginary axis.

This is possible due to the fact that ρ^{-1} is invertible, then as a takes all values in $\mathbb{Z}(d)$, then $\rho^{-1}a$ also takes all values in $\mathbb{Z}(d)$.

(2) In the case $\vartheta_1 = -1$ and $\vartheta_2 = 0, ..., d_2 - 1$, we use the straight line of zeros given Eq.(6.43). All the values $1, ..., d$ in the real axis are generated by the term $Nd_1 + \overline{a}_1 d_2$. This is possible due to the fact that Nd_1 gives the integer multiples of d_1 and $\overline{a}_1 d_2$ gives the 'in between' values. We note that d_2 is an invertible element within $\mathbb{Z}(d_1)$.

All required values $1, ..., d$ in the imaginary axis corresponding to $Rd_1 + \overline{a}_1 d_2$, is generated by the term $i(d_2 S + \overline{a}_2)$.

This is possible due to the fact that Sd_2 gives the integer multiples of d_2 and $\overline{a}_2$ gives the 'in between' values.

We give an equivalent explanation for the case $\vartheta_2 = -1$ and $\vartheta_1 = 0, ..., d_1 - 1$.

(3) In the case $\vartheta_1 = \vartheta_2 = -1$ we use the straight line of zeros given in Eq.(6.45). The a takes all values $1, ..., d$ in the real axis.

For each a, the R gives all required values $1, ..., d$ in the imaginary axis.

We have been able to justify that the above proof do not depend on the value of $(\vartheta_1, \vartheta_2)$. $\square$

6.5 Summary

In this chapter, we studied the analytic representation of finite quantum systems. Important relations such as the scalar product, orthogonality were defined. A basic definition of Theta function was given and we gave some its important properties. A detailed analytic representation of weak mutually unbiased bases are given with the proofs. The proof shows that a sum of Theta function can be represented with a single Theta function. For each analytic representation we consider the four different cases based on the factorization of ϑ into ϑ_1 and ϑ_2. For each cases of ϑ_1 and ϑ_2 we gave an analytic representation.

We considered the states in the WMUBs $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$ and we proved that the zeros of their analytic representation are on a straight line. We gave an explicit expression which describes this line. We used the two different notations (one which reflects the Symplectic transformation and the other which reflect the factorized ϑ) given in the previous chapter for the zeros corresponding to the vector $|\mathfrak{X}(\vartheta_1, \vartheta_2); \overline{a}_1, \overline{a}_2\rangle$. A line of the d zeros corresponding to the WMUB was described as $\mathfrak{K}(\vartheta_1, \vartheta_2; \overline{a}_1, \overline{a}_2)$ using the set of the d zeros. Finally the d^2 zeros in the cell Γ, of al the d vectors in the basis $\mathfrak{W}(\vartheta_1, \vartheta_2)$ was considered. We gave an explicit expression for the d^2 zeros and the results show that they do not depend on ϑ_1, ϑ_2. Hence, we conclude that they fom a lattice $\chi(d)$.

Chapter 7

The triality between lines in finite geometries, WMUBs, and the zeros of their analytic representation

Earlier in chapter five, we have established the duality between WMUBs in $H(d)$ and lines in $\Pi(d)$. In this chapter we introduce the set of the parallel lines of zeros in the cell Γ of the d vectors in a weak mutually unbiased basis. We then characterize these sets by the slope of the lines it contains. This allows us to establish the triality between these three different structures.

Definition

We define the set of the d parallel lines of zeros in Γ, of the d vectors in a weak mutually unbiased basis as

$$\mathfrak{N}(\vartheta_1, \vartheta_2) = \{\mathfrak{K}(a; \vartheta_1, \vartheta_2) | a \in \mathbb{Z}(d)\}; \quad \vartheta_n \in \mathbb{Z}(d_n) \tag{7.1}$$

We use the first notation which corresponds to Eq(6.53) to express this as

$$
\begin{aligned}
&\mathbf{N}(1, \rho^{-1}\vartheta) = \{\mathfrak{K}'(1, \rho^{-1}\vartheta; a) | a \in \mathbb{Z}(d)\} \\
&\mathbf{N}(d_1, s_1 + \vartheta_2 s_2) = \{\mathfrak{K}'(d_1, s_1 + \vartheta_2 s_2; a) | a \in \mathbb{Z}(d)\} \\
&\mathbf{N}(d_2, s_2 + \vartheta_1 s_1) = \{\mathfrak{K}'(d_2, s_2 + \vartheta_1 s_1; a) | a \in \mathbb{Z}(d)\} \\
&\mathbf{N}(0, 1) = \{\mathfrak{K}'(0, 1; a) | a \in \mathbb{Z}(d)\}. \qquad (7.2)
\end{aligned}
$$

Each of these sets is defined by the slope of the lines it contains.

In the theorem below, we use the slopes of these lines. We also define slopes of a line $L(\alpha, \beta)$ in $\Pi(d)$ as $\frac{\alpha}{\beta}$. Two lines $\mathbf{M}(\alpha, \beta)$ and $\mathbf{M}(\alpha', \beta')$ have the same slope if

$$\alpha\beta' - \alpha'\beta = 0 \ (\text{mod } d). \qquad (7.3)$$

Theorem 7.0.1.

(1) There exists a triality between three rather different structures which are

(i) the weak mutually unbiased bases in $H(d)$,

(ii) the non-near linear finite geometry $\Pi(d)$ associated with the phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$,

(iii) the lattice $\chi(d)$ in the cell Γ, which we also regard as a non-near linear finite geometry $\mathbb{Z}(d)$. This is summarized as :

$$\mathfrak{W}(\vartheta_1, \vartheta_2) \leftrightarrow \mathfrak{M}(\vartheta_1, \vartheta_2) \leftrightarrow \mathfrak{N}(\vartheta_1, \vartheta_2) \qquad (7.4)$$

(2) Within this triality we see the following correspondence between (i) the overlap $(\mathfrak{W}(\vartheta_1, \vartheta_2), \mathfrak{W}(\vartheta_1', \vartheta_2'))$ between vectors in the WMUBs (ii) common points between two lines $\mathfrak{M}(\vartheta_1, \vartheta_2)$ and $\mathfrak{M}(\vartheta_1', \vartheta_2')$ and (iii) common points between lines of zeros $\mathfrak{K}(a; \vartheta_1, \vartheta_2)$ in $\mathfrak{N}(\vartheta_1, \vartheta_2)$ and $\mathfrak{K}(a; \vartheta_1', \vartheta_2')$ in $\mathfrak{N}(\vartheta_1', \vartheta_2')$.

All these give the same value $p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')$, where $p(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_2')$ has been given in Eq.(4.20).

Proof.

(1) The bijective map between basis $\mathfrak{W}(\vartheta_1, \vartheta_2)$ and lines $\mathfrak{M}(\vartheta_1, \vartheta_2)$ was explained earlier in (Eq.(5.15)). All we need to do now is to extend this to a triality with lines of zeros such that there is a bijective map between $\mathfrak{M}(\vartheta_1, \vartheta_2)$ and $\mathfrak{N}(\vartheta_1, \vartheta_2)$.

We make use of the slopes which describes the two lines (i.e. lines in the geometry $\Pi(d)$ and lines of the set of zeros in $\chi(d)$). We consider the following four cases:

1. In the case $\vartheta_n = 0, ..., d_n - 1$,
   the slope of $\mathfrak{N}(\vartheta_1, \vartheta_2)$ in $\chi(d)$ is $\rho^{-1}\vartheta(d+1)$ as given in Eq.(6.42).
   For the line $\mathfrak{M}(\vartheta_1, \vartheta_2) = \mathbf{M}(1, \rho^{-1}\vartheta)$ in $\Pi(d)$, Eq.(4.13) gives the expression for the line with the slope equal to $\rho^{-1}\vartheta$.
   Since all varaibles are modulo d;

$$\rho^{-1}\vartheta(d+1) = \rho^{-1}\vartheta \tag{7.5}$$

   This proves that the slopes are equal.

2. In the case $\vartheta_1 = -1$ and $\vartheta_2 = 0, ..., d_2 - 1$, the slope of $\mathfrak{N}(-1, \vartheta_2)$ in $\chi(d)$ is given in Eq.(6.43) as $\frac{\vartheta_2(1+d_2)}{d_1}$.
   Equivalently we have the slope of the line $\mathfrak{M}(-1, \vartheta_2) = \mathbf{M}(d_1, s_1 + s_2\vartheta_2)$ in $\Pi(d)$ given as $\frac{s_1+s_2\vartheta_2}{d_1}$ in Eq.(4.14) . These two slopes are equal according to Eq.(7.3).

   Consequently for the case $\vartheta_2 = -1$ and $\vartheta_1 = 0, ..., d_1 - 1$, the slope of $\mathfrak{N}(-1, \vartheta_2)$ in $\chi(d)$ is given in Eq.(6.44) as $\frac{\vartheta_2(1+d_2)}{d_1}$.
   Equivalently we have the slope of the line $\mathfrak{M}(\vartheta_1, -1) = \mathbf{M}(d_2, s_2 + s_1\vartheta_1)$ in

$\Pi(d)$ given as $\frac{s_1+s_2\vartheta_2}{d_1}$ in Eq.(4.15) .

These two slopes are equal according to Eq.(7.3).

3. In the case $\vartheta_1 = \vartheta_2 = -1$,

   From Eq.(6.45) we can infer that the $\mathfrak{N}(-1,-1)$ in $\chi(d)$ is perpendicular and also the line $\mathfrak{L}(-1,-1) = \mathbf{M}(0,1)$ in in $\Pi(d)$ is represents the original position vector which is also perpendicular.

(2) In proposition 4.4.2 we have proved that two lines $\mathfrak{M}(\vartheta_1,\vartheta_2)$ and $\mathfrak{L}(\vartheta_1',\vartheta_2')$ have in common $p(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$ points.

The analogous for basis which is $(\mathfrak{W}(\vartheta_1,\vartheta_2),\mathfrak{W}(\vartheta_1',\vartheta_2')) = p(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')/d$ was also shown in Eq.(5.14). Now we prove an equivalent result for the lines of zeros.

We use lines $\mathfrak{W}(\vartheta_1,\vartheta_2)$ and $\mathfrak{W}(\vartheta_1',\vartheta_2')$ and assume that they have p points in common (where $p|d$).

We show that the lines $\mathfrak{K}(a;\vartheta_1,\vartheta_2)$ and $\mathfrak{K}(a;\vartheta_1',\vartheta_2')$ also have p points in common, i.e., that $\varphi(a,\vartheta_1,\vartheta_2,R) = \varphi(a,\vartheta_1',\vartheta_2',R')$ for p pairs (R,R'). We now give a detailed proof only for the case that all $\vartheta_n,\vartheta_n' = 0,...,d-1$. The proof in the other cases is similar.

In this case, using Eq.(4.13) we conclude that there exist p pairs (κ,κ') such that

$$(\kappa,\kappa\rho^{-1}\vartheta) = (\kappa',\kappa'\rho^{-1}\vartheta');\ \text{where } \vartheta = \vartheta_1 s_1 + \vartheta_2 s_2. \tag{7.6}$$

From this we infer that $\kappa = \kappa' \pmod d$ and $\kappa\rho^{-1}\vartheta = \kappa'\rho^{-1}\vartheta' \pmod d$.

Using Eq.(6.42) we show that $\kappa = \kappa' \pmod d$ and $\kappa\rho^{-1}\vartheta = \kappa'\rho^{-1}\vartheta' \pmod d$ implies that

$$\varphi(a,\vartheta_1,\vartheta_2,R) = \varphi(a,\vartheta_1',\vartheta_2',R'). \tag{7.7}$$

for each of the p pairs (κ, κ').

Therefore lines $\mathfrak{K}(a; \vartheta_1, \vartheta_2)$ and $\mathfrak{K}(a; \vartheta'_1, \vartheta'_2)$ also have p points in common.

□

For the case $d = 35$, this triality is shown explicitly in Table A

7.0.1 Example

We consider an example of two sets of lines of zeros in $\chi(25)$, which is analogous to example given for basis and lines (in this case $d_1 = 5$ and $d_2 = 7$). They are the $\mathfrak{N}(1,4)$ and $\mathfrak{N}(3,4)$. We take the line of zeros $\mathfrak{K}(10;1,4)$ from the set $\mathfrak{N}(1,4)$, and the line of zeros $\mathfrak{K}(10;3,4)$ from the set $\mathfrak{N}(3,4)$ (i.e., we take as an example, $a = 10$). The lines $\mathfrak{K}(10;1,4)$ and $\mathfrak{K}(10;3,4)$ which have in common the $d_2 = 7$ zeros:

$$\begin{aligned}
R = 3 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 2.5 + i17.5 \\
R = 3 + d_1 = 8 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 7.5 + i7.5 \\
R = 3 + 2d_1 = 13 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 12.5 + i32.5 \\
R = 3 + 3d_1 = 18 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 17.5 + i22.5 \\
R = 3 + 4d_1 = 23 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 22.5 + i12.5 \\
R = 3 + 5d_1 = 28 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 27.5 + i2.5 \\
R = 3 + 6d_1 = 33 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 32.5 + i27.5
\end{aligned} \tag{7.8}$$

If we regard the $2.5+i17.5$ as 'origin', these three points have coordinates $(0,0)$, $(5,25)$, $(10,15)$, $(15,5)$, $(20,30)$, $(25,30)$ and $(30,10)$ which are exactly the same as in the example in chapter 4. Figs.4.4, 7.1 show this equivalence.

We also consider the case $a = 17$.

The lines $\mathfrak{K}(17;1,4)$ and $\mathfrak{K}(17;3,4)$have in common the $d_2 = 7$ zeros:

$$\begin{aligned}
R = 3 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 2.5 + i31.5\\
R = 3 + d_1 = 8 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 7.5 + i21.5\\
R = 3 + 2d_1 = 13 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 12.5 + i11.5\\
R = 3 + 3d_1 = 18 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 17.5 + i1.5\\
R = 3 + 4d_1 = 23 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 22.5 + i26.5\\
R = 3 + 5d_1 = 28 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 27.5 + i16.5\\
R = 3 + 6d_1 = 33 \;&\rightarrow\; \varphi(10;1,4,R) = \varphi(10;3,4;R) = 32.5 + i6.5
\end{aligned} \tag{7.9}$$

Again we regard the $2.5 + i31.5$ as 'origin', and these seven points have coordinates $(0,0)$, $(5,25)$, $(10,15)$, $(15,5)$, $(20,30)$, $(25,30)$ and $(30,10)$ as above and as in the example chapter 4.

It is seen that for any a, the lines $\mathfrak{K}(a;\vartheta_1,\vartheta_2)$ in $\mathfrak{N}(\vartheta_1,\vartheta_2)$ and $\mathfrak{K}(a;\vartheta_1',\vartheta_2')$ in $\mathfrak{N}(\vartheta_1',\vartheta_2')$ have p points in common (where $p = 1, d_1, d_2$).

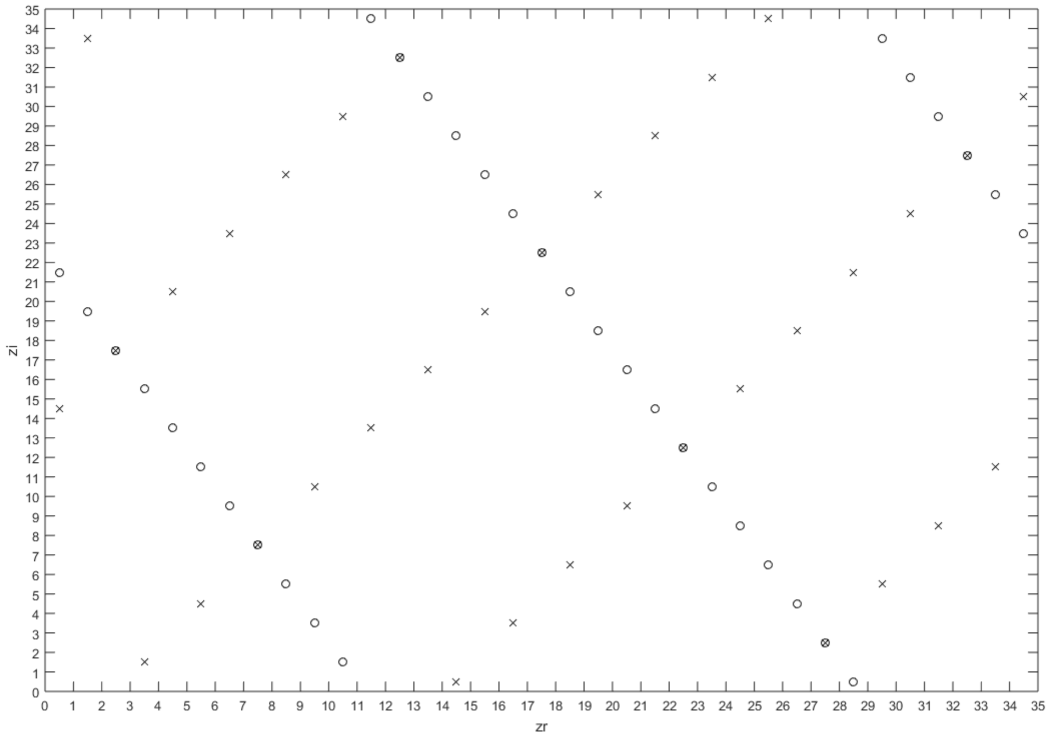

Fig. 7.1 The lines of zeros of $\mathfrak{K}(10;1,4)$ (crosses) and line of zeros $\mathfrak{K}(10;3,4)$ (circles) in the complex plane. The two lines have in common the zeros $(2.5+i17.5),(7.5+i7.5),(12.5+i32.5),(17.5+i22.5),(22.5+i12.5),(27.5+i2.5),(32.5+i27.5)$. zi and zr represents the imaginary and real axes respectively.

The precise correspondence of the various quantities involved in this triality, is summarized below.

- We have $\psi(d)$ WMUB $\mathfrak{W}(\vartheta_1,\vartheta_2)$, $\psi(d)$ maximal lines through the origin $\mathfrak{M}(\vartheta_1,\vartheta_2)$ and $\psi(d)$ sets $\mathfrak{N}(\vartheta_1,\vartheta_2)$ of parallel lines of zeros.
- We have d orthogonal vectors in each WMUB $\mathfrak{W}(\vartheta_1,\vartheta_2)$, d points in each $\mathfrak{M}(\vartheta_1,\vartheta_2)$ and d parallel lines of zeros in each set $\mathfrak{N}(\vartheta_1,\vartheta_2)$ with each line containing d zeros.
- We have overlap of two basis $(\mathfrak{W}(\vartheta_1,\vartheta_2),\mathfrak{W}(\vartheta_1',\vartheta_2')) = \frac{p(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')}{d}$, two lines

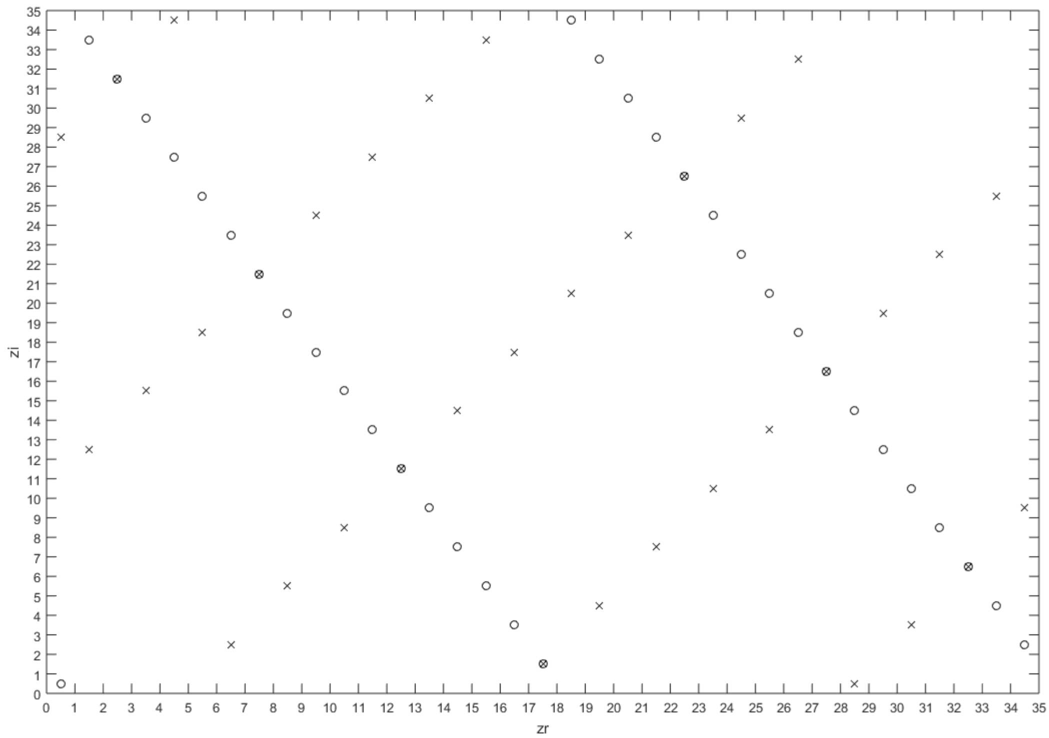

Fig. 7.2 The lines of zeros of $\mathfrak{K}(17;1,4)$ (crosses) and line of zeros $\mathfrak{K}(17;3,4)$ (circles) in the complex plane. The two lines have in common the zeros $(2.5+i31.5), (7.5+i21.5), (12.5+i11.5), (17.5+i1.5), (22.5+i26.5), (27.5+i16.5), (32.5+i6.5)$. zi and zr represents the imaginary and real axes respectively.

$\mathfrak{M}(\vartheta_1,\vartheta_2)$ and $\mathfrak{M}(\vartheta_1',\vartheta_2')$ have in common $p(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$ points and for any a, the lines $\mathfrak{K}(a;\vartheta_1,\vartheta_2)$ in $\mathfrak{N}(\vartheta_1,\vartheta_2)$ and $\mathfrak{K}(a;\vartheta_1',\vartheta_2')$ in $\mathfrak{N}(\vartheta_1',\vartheta_2')$ have $p(\vartheta_1,\vartheta_2|\vartheta_1',\vartheta_2')$ points in common.

7.1 Summary

In this chapter we have introduced the set of the parallel lines of zeros $\mathfrak{N}(\vartheta_1,\vartheta_2)$ in Γ. These sets are then characterized by the slopes of the lines contained in it. We have expressed these sets in the two different notations, which are analogous to the notions for lines and WMUBs. We defined two lines as equal if their slopes are equal. A triality between the weak mutually unbiased bases in $H(d)$, the non-near linear finite geometry $\Pi(d)$ associated with the phase space $\mathbb{Z}(d)\times\mathbb{Z}(d)$ and the lattice $\chi(d)$ in the cell Γ was established and proved. This also confirmed that this is a a non-near linear finite geometry $\mathbb{Z}(d)$.

Within this triality, we show that there is a correspondence in overlap between two basis in the WMUB, common points between two lines in non-near linear finite geometry $\Pi(d)$ and common lines of zeros between two different sets of lines of zeros in the lattice $\chi(d)$.

In the example we gave in this chapter, we considered two sets of lines of zeros in $\chi(d)$ and we show their common zeros. These zeros are related to the 'R' which defines them as given in Eq(6.42, 6.43, 6.44, 6.45). This example is equivalent to the example given for maximal lines. In fact by choosing one of the zeros as origin, we see that these zeros are in fact same as coordinates which indicates the common points for the lines. These comparison can be seen in Figs.4.4, 7.1.

Chapter 8

Conclusion and future work

8.1 Conclusion

We have considered quantum systems with variables in $\mathbb{Z}(d)$ and three different structures. Precisely we have used $d = d_1 \times d_2$ throughout this work.

First is $\mathbb{Z}(d) \times \mathbb{Z}(d)$ which is a non-near-linear geometry. Straight lines in it have more than one point in common. There is the existence of 'sublines' with p points where p is a divisor of d.

Second is the concept of weak mutually unbiased bases, which it fits naturally to the concept of rings. The construction of weak mutually unbiased bases is based on tensor product of the mutually unbiased bases. In the case that d is prime, $\mathbb{Z}(d)$ is a field, and it is known that there are $d + 1$ such bases.

We establish a duality between these two structures as follows

1. The lines $\mathbf{M}(1, \rho^{-1}\vartheta)$, $\mathbf{M}(d_1, s_1 + s_2\vartheta_2)$, $\mathbf{M}(d_2, s_2 + s_2\vartheta_1)$ of Eq(4.17, 4.18, 4.19) corresponds to basis $|\mathbb{X}(1, \rho^{-1}\vartheta); a\rho^{-1}\rangle$, $|\mathbb{X}(d_1, s_1 + s_2\vartheta_2); a\rangle$, $|\mathbb{X}(d_2, s_2 + s_2\vartheta_1); a\rangle$ in Eq(5.11) respectively. This due to the fact that the symplectic transformations used for the lines are same as the symplectic transformation for the basis.

2. There are $\psi(d) = (d_2 + 1)(d_1 + 1)$ maximal lines through the origin in $\Pi(d)$. This corresponds to set of $\psi(d)$ weak mutually unbiased bases in $H(d)$.

We then consider a very different problem of the analytic representation of the weak mutually unbiased bases based on Theta functions, and its zeros. This extended the concept of duality to a triality, with the involvement of the zeros of analytic functions that represent the quantum states.

We have then shown that there is a triality between this analytic representation and its zeros, the finite geometry in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space, and the weak mutually unbiased bases.

The appearance of lines of zeros of WMUBs in as a third component in this triality is absolutely surprising, and it reaffirms the important role of analytic functions in the description of quantum system.

Existing work in the general area of mutually unbiased bases is based on discrete mathematics. The present work links them with the theory of analytic functions and provides an alternate approach to the problem of mutually unbiased bases generally.

8.2 Future work

Due to the importance of mutually unbiased bases in ongoing research in quantum information, we can extend this work to find the correspondence to mutually orthogonal Latin squares especially in higher dimensions. We can use the concept of weak mutually unbiased bases to construct a 'weaker' structure for Latin squares which accommodates higher non-prime dimensions.

Our work with Theta functions can be extended the work to find properties of 'factorized' Theta functions which will reflect on their zeros.

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Appendix A

For non prime d correspondence between lines in the $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space, WMUB in the Hilbert space $H(d)$ and zeros in the lattice $\chi(d)$

$\mathcal{G}(21)$	$H(21)$	$\mathfrak{Z}(21)$
$\mathbf{W}(0,1) = \mathfrak{W}(-1,-1)$	$\mathbf{M}(0,1) = \mathfrak{M}(-1,-1)$	$\mathbf{N}(0,1) = \mathfrak{N}(-1,-1)$
$\mathbf{W}(1,0) = \mathfrak{W}(0,0)$	$\mathbf{M}(1,0) = \mathfrak{M}(0,0)$	$\mathbf{N}(1,0) = \mathfrak{N}(0,0)$
$\mathbf{W}(1,1) = \mathfrak{W}(2,5)$	$\mathbf{M}(1,1) = \mathfrak{M}(2,5)$	$\mathbf{N}(1,1) = \mathfrak{N}(2,5)$
$\mathbf{W}(1,2) = \mathfrak{W}(4,3)$	$\mathbf{M}(1,2) = \mathfrak{M}(4,3)$	$\mathbf{N}(1,2) = \mathfrak{N}(4,3)$
$\mathbf{W}(1,3) = \mathfrak{W}(1,1)$	$\mathbf{M}(1,3) = \mathfrak{M}(1,1)$	$\mathbf{N}(1,3) = \mathfrak{N}(1,1)$
$\mathbf{W}(1,4) = \mathfrak{W}(3,6)$	$\mathbf{M}(1,4) = \mathfrak{M}(3,6)$	$\mathbf{N}(1,4) = \mathfrak{N}(3,6)$
$\mathbf{W}(1,5) = \mathfrak{W}(0,4)$	$\mathbf{M}(1,5) = \mathfrak{M}(0,4)$	$\mathbf{N}(1,5) = \mathfrak{N}(0,4)$
$\mathbf{W}(1,6) = \mathfrak{W}(2,2)$	$\mathbf{M}(1,6) = \mathfrak{M}(2,2)$	$\mathbf{N}(1,6) = \mathfrak{N}(2,2)$
$\mathbf{W}(1,7) = \mathfrak{W}(4,0)$	$\mathbf{M}(1,7) = \mathfrak{M}(4,0)$	$\mathbf{N}(1,7) = \mathfrak{N}(4,0)$
$\mathbf{W}(1,8) = \mathfrak{W}(1,5)$	$\mathbf{M}(1,8) = \mathfrak{M}(1,5)$	$\mathbf{N}(1,8) = \mathfrak{N}(1,5)$
$\mathbf{W}(1,9) = \mathfrak{W}(3,3)$	$\mathbf{M}(1,9) = \mathfrak{M}(3,3)$	$\mathbf{N}(1,9) = \mathfrak{N}(3,3)$
$\mathbf{W}(1,10) = \mathfrak{W}(0,1)$	$\mathbf{M}(1,10) = \mathfrak{M}(0,1)$	$\mathbf{N}(1,10) = \mathfrak{N}(0,1)$
$\mathbf{W}(1,11) = \mathfrak{W}(2,6)$	$\mathbf{M}(1,11) = \mathfrak{M}(2,6)$	$\mathbf{N}(1,11) = \mathfrak{N}(2,6)$
$\mathbf{W}(1,12) = \mathfrak{W}(4,4)$	$\mathbf{M}(1,12) = \mathfrak{M}(4,4)$	$\mathbf{N}(1,12) = \mathfrak{N}(4,4)$

$\mathcal{G}(21)$	$H(21)$	$\mathfrak{Z}(21)$
$\mathbf{W}(1,13) = \mathfrak{W}(1,2)$	$\mathbf{M}(1,13) = \mathfrak{M}(1,2)$	$\mathbf{N}(1,13) = \mathfrak{N}(1,2)$
$\mathbf{W}(1,14) = \mathfrak{W}(3,0)$	$\mathbf{M}(1,14) = \mathfrak{M}(3,0)$	$\mathbf{N}(1,14) = \mathfrak{N}(3,0)$
$\mathbf{W}(1,15) = \mathfrak{W}(0,5)$	$\mathbf{M}(1,15) = \mathfrak{M}(0,5)$	$\mathbf{N}(1,15) = \mathfrak{N}(0,5)$
$\mathbf{W}(1,16) = \mathfrak{W}(2,3)$	$\mathbf{M}(1,16) = \mathfrak{M}(2,3)$	$\mathbf{N}(1,16) = \mathfrak{N}(2,3)$
$\mathbf{W}(1,17) = \mathfrak{W}(4,1)$	$\mathbf{M}(1,17) = \mathfrak{M}(4,1)$	$\mathbf{N}(1,17) = \mathfrak{N}(4,1)$
$\mathbf{W}(1,18) = \mathfrak{W}(1,6)$	$\mathbf{M}(1,18) = \mathfrak{M}(1,6)$	$\mathbf{N}(1,18) = \mathfrak{N}(1,6)$
$\mathbf{W}(1,19) = \mathfrak{W}(3,4)$	$\mathbf{M}(1,19) = \mathfrak{M}(3,4)$	$\mathbf{N}(1,19) = \mathfrak{N}(3,4)$
$\mathbf{W}(1,20) = \mathfrak{W}(0,2)$	$\mathbf{M}(1,20) = \mathfrak{M}(0,2)$	$\mathbf{N}(1,20) = \mathfrak{N}(0,2)$
$\mathbf{W}(1,21) = \mathfrak{W}(2,0)$	$\mathbf{M}(1,21) = \mathfrak{M}(2,0)$	$\mathbf{N}(1,21) = \mathfrak{N}(2,0)$
$\mathbf{W}(1,22) = \mathfrak{W}(4,5)$	$\mathbf{M}(1,22) = \mathfrak{M}(4,5)$	$\mathbf{N}(1,22) = \mathfrak{N}(4,5)$
$\mathbf{W}(1,23) = \mathfrak{W}(1,3)$	$\mathbf{M}(1,23) = \mathfrak{M}(1,3)$	$\mathbf{N}(1,23) = \mathfrak{N}(1,3)$
$\mathbf{W}(1,24) = \mathfrak{W}(3,1)$	$\mathbf{M}(1,24) = \mathfrak{M}(3,1)$	$\mathbf{N}(1,24) = \mathfrak{N}(3,1)$
$\mathbf{W}(1,25) = \mathfrak{W}(0,6)$	$\mathbf{M}(1,25) = \mathfrak{M}(0,6)$	$\mathbf{N}(1,25) = \mathfrak{N}(0,6)$
$\mathbf{W}(1,26) = \mathfrak{W}(2,4)$	$\mathbf{M}(1,26) = \mathfrak{M}(2,4)$	$\mathbf{N}(1,26) = \mathfrak{N}(2,4)$
$\mathbf{W}(1,27) = \mathfrak{W}(4,2)$	$\mathbf{M}(1,27) = \mathfrak{M}(4,2)$	$\mathbf{N}(1,27) = \mathfrak{N}(4,2)$
$\mathbf{W}(1,28) = \mathfrak{W}(1,0)$	$\mathbf{M}(1,28) = \mathfrak{M}(1,0)$	$\mathbf{N}(1,28) = \mathfrak{N}(1,0)$
$\mathbf{W}(1,29) = \mathfrak{W}(3,5)$	$\mathbf{M}(1,29) = \mathfrak{M}(3,5)$	$\mathbf{N}(1,29) = \mathfrak{N}(3,5)$
$\mathbf{W}(1,30) = \mathfrak{W}(0,3)$	$\mathbf{M}(1,30) = \mathfrak{M}(0,3)$	$\mathbf{N}(1,30) = \mathfrak{N}(0,3)$
$\mathbf{W}(1,31) = \mathfrak{W}(2,1)$	$\mathbf{M}(1,31) = \mathfrak{M}(2,1)$	$\mathbf{N}(1,31) = \mathfrak{N}(2,1)$
$\mathbf{W}(1,32) = \mathfrak{W}(4,6)$	$\mathbf{M}(1,32) = \mathfrak{M}(4,6)$	$\mathbf{N}(1,32) = \mathfrak{N}(4,6)$
$\mathbf{W}(1,33) = \mathfrak{W}(1,4)$	$\mathbf{M}(1,33) = \mathfrak{M}(1,4)$	$\mathbf{N}(1,33) = \mathfrak{N}(1,4)$
$\mathbf{W}(1,34) = \mathfrak{W}(3,2)$	$\mathbf{M}(1,34) = \mathfrak{M}(3,2)$	$\mathbf{N}(1,34) = \mathfrak{N}(3,2)$
$\mathbf{W}(5,21) = \mathfrak{W}(-1,0)$	$\mathbf{M}(5,21) = \mathfrak{M}(-1,0)$	$\mathbf{N}(5,21) = \mathfrak{N}(-1,0)$

$\mathcal{G}(21)$	$H(21)$	$\mathfrak{Z}(21)$
$\mathbf{W}(5,1) = \mathfrak{W}(-1,1)$	$\mathbf{M}(5,1) = \mathfrak{M}(-1,1)$	$\mathbf{N}(5,1) = \mathfrak{N}(-1,1)$
$\mathbf{W}(5,16) = \mathfrak{W}(-1,2)$	$\mathbf{M}(5,16) = \mathfrak{M}(-1,2)$	$\mathbf{N}(5,16) = \mathfrak{N}(-1,2)$
$\mathbf{W}(5,31) = \mathfrak{W}(-1,3)$	$\mathbf{M}(5,31) = \mathfrak{M}(-1,3)$	$\mathbf{N}(5,31) = \mathfrak{N}(-1,3)$
$\mathbf{W}(5,11) = \mathfrak{W}(-1,4)$	$\mathbf{M}(5,11) = \mathfrak{M}(-1,4)$	$\mathbf{N}(5,11) = \mathfrak{N}(-1,4)$
$\mathbf{W}(5,26) = \mathfrak{W}(-1,5)$	$\mathbf{M}(5,26) = \mathfrak{M}(-1,5)$	$\mathbf{N}(5,26) = \mathfrak{N}(-1,5)$
$\mathbf{W}(5,6) = \mathfrak{W}(-1,6)$	$\mathbf{M}(5,6) = \mathfrak{M}(-1,6)$	$\mathbf{N}(5,6) = \mathfrak{N}(-1,6)$
$\mathbf{W}(7,15) = \mathfrak{W}(0,-1)$	$\mathbf{M}(7,15) = \mathfrak{M}(0,-1)$	$\mathbf{N}(7,15) = \mathfrak{N}(0,-1)$
$\mathbf{W}(7,1) = \mathfrak{W}(1,-1)$	$\mathbf{M}(7,1) = \mathfrak{M}(1,-1)$	$\mathbf{N}(7,1) = \mathfrak{N}(1,-1)$
$\mathbf{W}(7,22) = \mathfrak{W}(2,-1)$	$\mathbf{M}(7,22) = \mathfrak{M}(2,-1)$	$\mathbf{N}(7,22) = \mathfrak{N}(2,-1)$
$\mathbf{W}(7,5) = \mathfrak{W}(3,-1)$	$\mathbf{M}(7,5) = \mathfrak{M}(3,-1)$	$\mathbf{N}(7,5) = \mathfrak{N}(3,-1)$
$\mathbf{W}(7,29) = \mathfrak{W}(4,-1)$	$\mathbf{M}(7,29) = \mathfrak{M}(4,-1)$	$\mathbf{N}(7,29) = \mathfrak{N}(4,-1)$