# PARTIAL ORDERING OF WEAK MUTUALLY UNBIASED BASES IN FINITE QUANTUM SYSTEMS

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# Abstract


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Partial ordering of weak mutually unbiased bases.

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There has being an enormous work on finite quantum systems with variables in $\mathcal{Z}_d$, especially on mutually unbiased bases. The reason for this is due to its wide areas of application. We focus on partial ordering of weak mutually unbiased bases. In it, we studied a partial ordered relation which exists between a subsystem $\Lambda(q)$ and a larger system $\Lambda(d)$ and also, between a subgeometry $\mathcal{G}_q$ and larger geometry $\mathcal{G}_d$. Furthermore, we show an isomorphism between:

(i) the set $\{\mathbf{G}_d\}$ of subgeometries of a finite geometry $\mathcal{G}_d$ and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

(ii) the set $\{\mathbf{h}_d\}$ of subspaces of a finite Hilbert space $\mathcal{H}_d$ and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

(iii) the set $\{\Upsilon(d)\}$ of subsystems of a finite quantum system $\Lambda(d)$ and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

We conclude this work by showing a duality between lines in finite geometry $\mathcal{G}_d$ and weak mutually unbiased bases in finite dimensional Hilbert space $\mathcal{H}_d$.

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# Declaration

A portion of the work presented in this thesis has been published in the article: **S.O.Oladejo, C.Lei, and A.Vourdas** "Partial Ordering of Weak Mutually Unbiased Bases", J. Phys A.: Math. Theor.,47:485204 (2014)

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# List of notations

| Notation | Definition |
|---|---|
| $mod\ d$ | modulo $d$. |
| $q\|d$ | $q$ divides $d$. |
| $\psi(d)$ | Dedekind psi function, psi ($d$). |
| $\varphi(d)$ | Euler phi function, phi ($d$). |
| $\sigma_n(d)$ | Divisor function to power $n$ of $d$. |
| $a \prec b$ | $a$ is partially ordered in $b$. |
| $a \cong b$ | $a$ is isomorphic to $b$. |
| $\mathcal{Z}_d$ | integer modulo $d$. |
| $\{\mathcal{D}(d)\}$ | set of divisors of $d$. |
| $A \subseteq B$ | A is a subset of B. |
| $A \subset B$ | A is a proper subset of B. |
| $a \in \mathcal{Z}_d$ | $a$ is an element in $\mathcal{Z}_d$. |
| $\mathcal{Z}_d \ni a$ | $\mathcal{Z}_d$ contains element $a$. |
| $d = \prod_{\mathbf{j=1}}^{\mathbf{k}} \mathbf{p_j}$ | $d$ is a product of prime, $\mathbf{p_1} \times ... \times \mathbf{p_k}$. |
| $\mathbf{1}$ | Identity operator |
| $\mathcal{H}_\infty$ | Infinite dimensional Hilbert space |
| $\mathcal{H}_d$ | $d-$ dimensional Hilbert space |

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# Chapter 1

# Introduction

## 1.1 Background knowledge

Over the years, a lot of work has been done on finite quantum systems $\Lambda(d)$ with variables in $\mathcal{Z}_d$ [1-6] and mutually unbiased bases $[7-22]$. It is known that for $d$ a prime, there exists $d+1$ mutually unbiased bases. However, if $d$ is a non-prime number, the number of mutually unbiased bases is still an unanswered question up till today. A weaker concept called weak mutually unbiased bases was formulated by $[23, 24]$. It is noted here that the difference between weak mutually unbiased bases and mutually unbiased bases is that in mutually unbiased bases,

$$|\langle \mathcal{X}_{\Theta_\mathbf{i}}; n | \mathcal{X}_{\Theta_\mathbf{j}}; m\rangle| = \frac{1}{\sqrt{d}} \text{ for } d \text{ a prime } m, n \in \mathcal{Z}_d \text{ and } \Theta_\mathbf{i} \neq \Theta_\mathbf{j}. \tag{1.1}$$

whereas, in weak mutually unbiased bases,

$$|\langle \mathcal{X}_{\Theta_{\mathbf{i}}}; n|\mathcal{X}_{\Theta_{\mathbf{j}}}; m\rangle| = \frac{1}{\sqrt{q}};\ \text{or } 0,\ \text{for } q|d\ m, n \in \mathcal{Z}_d \text{ and } \Theta_{\mathbf{i}} \neq \Theta_{\mathbf{j}}. \tag{1.2}$$

In our work, we consider the dimension $d$ to be the products,

$$d = \prod_{\mathbf{j=1}}^{\mathbf{k}} \mathbf{p_j} \tag{1.3}$$

where $\mathbf{p_j}$ are prime numbers. We use Chinese remainder theorem to establish a bijection between $\mathcal{Z}_d$ and $\prod_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{Z}_{\mathbf{p_j}}$. This concept was used in [25, 26] to express a quantum state in $\Lambda(d)$ in terms of states in $\Lambda(\mathbf{p_1}), ..., \Lambda(\mathbf{p_k})$.The weak mutually unbiased bases are obtained by taking the tensor products of mutually unbiased bases in prime factor Hilbert spaces $\mathcal{H}_{\mathbf{p_j}}$. For $\mathbf{p_j}$ a prime number, there exists $\psi(\mathbf{p_j})$ mutually unbiased bases in $\Lambda(\mathbf{p_j})$ and for this reason there exists $\psi(d) = \prod_{\mathbf{j=1}}^{\mathbf{k}} \psi(\mathbf{p_j})$ weak mutually unbiased bases in $\Lambda(d)$. An essential characteristic of weak mutually unbiased bases is that they are connected to the features of lines in phase space $\mathcal{Z}_d \times \mathcal{Z}_d$ of $d-$dimensional finite quantum system $\Lambda(d)$. Most of the research carried out in finite geometry in the past focused on near-linear geometry $[27-29]$. This is built on the well known axiom that "two lines have at most one point in common". However in this work, we focus on non-near- linear finite geometry. We define our geometry $\mathcal{G}_d$ as;

$$\mathcal{G}_d = \mathcal{Z}_d \times \mathcal{Z}_d \tag{1.4}$$

where $\mathcal{Z}_d$ is ring of integer modulo $d$.

Two lines can have $q$ points in common (where $q|d$) and hence, the above axiom is violated. Lines with $q$ points are called sublines. In [24], it was stated that there exists a duality between lines in $\mathcal{G}_d$ and weak mutually unbiased bases in a finite dimensional Hilbert space $\mathcal{H}_d$. More works on finite geometries for finite dimensional quantum systems are presented in $[30-33]$.

For $q|d, \mathcal{Z}_q$ is a subgroup of $\mathcal{Z}_d$. Furthermore, the quantum states of $\Lambda(q)$ could be embedded in $\Lambda(d)$. This means that $\Lambda(q)$ is a subsystem of $\Lambda(d)$.

We show that, there exists a partial order between a subsystem and a larger quantum system and between a subgeometry and a larger geometry. We establish an isomorphism between them and also, demonstrate in detail the link between finite quantum systems and the geometries of their phase space. This is show-cased in our work by showing a duality between lines in phase space $\mathcal{G}_d$ and weak mutually unbiased bases in Hilbert space $\mathcal{H}_d$.

This work is divided into six chapters as follows: chapter two is introduction, it focuses on quantum systems with infinite dimensional Hilbert space. In chapter three, we discuss quantum systems with finite dimensional Hilbert space. The concept of partial ordering of weak mutually unbiased bases is discussed in chapter four. In chapter five, we demonstrate the concept of partial ordering of weak mutually unbiased bases using examples. Finally in chapter six, we present the conclusion of our work.

# Chapter 2

# Quantum systems with infinite Hilbert space $\mathcal{H}_\infty$

## 2.1 Introduction

Quantum mechanics uses Hilbert space $\mathcal{H}$ with extra conditions that:

(i) only vectors of norm 1 correspond to physical state.

(ii) vectors differ only by a phase value (which is a complex number of modulus 1) correspond to the same physical state.

## 2.2 Fourier transform

Let $g(t)$ be a complex function with respect to time $t$ over the interval $-\infty < t < \infty$, the Fourier transform $G(f)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}dt. \tag{2.1}$$

An important concept is working towards retrieving the function $G(f)$ from $g(t)$. Its inverse, $g(t)$ is defined as

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{2\pi i f t} df. \tag{2.2}$$

This transform is very essential in many areas of applied physics and mathematics.

## 2.3 Parity and displacement operator

Mathematics related to coherent states is presented through the use of unitary operator. This is called displacement operator.
It is defined as;

$$D(\mathbf{z}) \equiv \exp(\mathbf{z}\hat{b}^\dagger - \mathbf{z}^*\hat{b}), \tag{2.3}$$

where $\mathbf{z} = \frac{\mathbb{X}+i\mathbb{P}}{\sqrt{2}}$ is a complex number, $\hat{b} = \frac{\hat{X}+i\hat{P}}{\sqrt{2}}$, and $\hat{b}^\dagger = \frac{\hat{X}-i\hat{P}}{\sqrt{2}}$ are ladder operators. Eq.(2.3) is expressed as

$$\begin{aligned} D(\mathbb{X},\mathbb{P}) &\equiv \exp\left\{\frac{(\mathbb{X}+i\mathbb{P})(\hat{X}-i\hat{P})}{2} - \frac{(\mathbb{X}-i\mathbb{P})(\hat{X}+i\hat{P})}{2}\right\} \\ &= \exp\{i\mathbb{P}\hat{X} - i\mathbb{X}\hat{P}\} \end{aligned} \tag{2.4}$$

$\hat{X}$ and $\hat{P}$ represent position and momentum operators respectively, $\exp\{i\mathbb{P}\hat{X}\}$ and $\exp\{-i\mathbb{X}\hat{P}\}$ denote unitary translation operator in position space and momentum space respectively, and these operators do not commute.

$D(\mathbf{z})$ form a Heisenberg-Weyl group with the composition law

$$D(\mathbf{z}_1)D(\mathbf{z}_2) = e^{i\mathbf{z}_1\mathbf{z}_2^*}D(\mathbf{z}_1 + \mathbf{z}_2). \tag{2.5}$$

The displaced parity operator is defined as

$$\mathbf{P}(\mathbb{X}, \mathbb{P}) = D(2\mathbb{X}, 2\mathbb{P})\mathbf{P}(0, 0). \tag{2.6}$$

where

$$\mathbf{P}(0, 0) = \int_{-\infty}^{+\infty} d\mathbb{X}|\mathbb{X}\rangle\langle-\mathbb{X}|. \tag{2.7}$$

This operator carries out transformation around the point $(0, 0)$.

## 2.4 Coherent states

This is the eigenstate of the annihilation operator $\hat{b}$ such that,

$$\hat{b}|\mathbf{z}\rangle = \mathbf{z}|\mathbf{z}\rangle \tag{2.8}$$

It is the unitary transformation of vacuum state $|0\rangle$ which can be defined as a superposition

$$|\mathbf{z}\rangle = \sum_{n=0}^{\infty} c_n|n\rangle; \tag{2.9}$$

where $c_n = \frac{\mathbf{z}^n}{\sqrt{n!}} e^{-\frac{1}{2}|\mathbf{z}|^2}$.

The canonical coherent state is expressed as

$$|\mathbf{z}\rangle = e^{\mathbf{z}\hat{b}^\dagger - \mathbf{z}^*\hat{b}}|0\rangle \tag{2.10}$$

Using Baker-Campbell Hausdorff formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-1/2[\hat{A},\hat{B}]} \tag{2.11}$$

we obtain;

$$e^{\frac{-|\mathbf{z}|^2}{2}} e^{\mathbf{z}\hat{b}^\dagger} e^{-\mathbf{z}^*\hat{b}}|0\rangle = e^{\frac{-|\mathbf{z}|^2}{2}} \sum_{n=0}^{\infty} \frac{(\mathbf{z}\hat{b}^\dagger)^n}{n!}|0\rangle = e^{\frac{-|\mathbf{z}|^2}{2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbf{z}^n |n\rangle. \tag{2.12}$$

## 2.5 Wigner and Weyl functions in infinite systems

The Wigner function is a quasi-probability distribution which can be obtained by taking the trace of the parity operator, $\mathbf{P}(\mathbb{X}, \mathbb{P})$. We express the Wigner function $\mathcal{W}(\mathbb{X}, \mathbb{P})$ as

$$\mathcal{W}(\mathbb{X}, \mathbb{P}) = Tr[\hat{\rho}\mathbf{P}(\mathbb{X}, \mathbb{P})] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\Phi \left\langle \mathbb{X} + \frac{\Phi}{2} \middle| \hat{\rho} \middle| \mathbb{X} - \frac{\Phi}{2} \right\rangle e^{-i\mathbb{P}\Phi} \tag{2.13}$$

where $\mathbf{P}(\mathbb{X}, \mathbb{P})$ is defined in eq.(2.6) and $\hat{\rho}$ represents a density matrix. Specifically, suppose the density matrix denotes a pure state that is;

$$\hat{\rho} = |\xi\rangle\langle\xi| \tag{2.14}$$

then

$$\mathcal{W}(\mathbb{X},\mathbb{P}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\Phi \xi^*\left(\mathbb{X}+\frac{\Phi}{2}\right)\xi\left(\mathbb{X}-\frac{\Phi}{2}\right)e^{-i\mathbb{P}\Phi} \tag{2.15}$$

If the Wigner function is normalized, it looks the same way as the classical probability distribution

$$\int_{-\infty}^{+\infty} d\mathbb{X}d\mathbb{P}\mathcal{W}(\mathbb{X},\mathbb{P}) = \int_{-\infty}^{+\infty} d\mathbb{X}d\Phi\xi^*\left(\mathbb{X}+\frac{\Phi}{2}\right)\xi\left(\mathbb{X}-\frac{\Phi}{2}\right)\delta(\Phi) = Tr[\hat{\rho}] = 1 \tag{2.16}$$

The Weyl function is related to the displacement operator and it is defined with respect to the phase space $\mathbb{X},\mathbb{P}$ as,

$$\widetilde{\mathcal{W}}(\mathbb{X},\mathbb{P}) = Tr[\hat{\rho}D(\mathbb{X},\mathbb{P})] = \int_{-\infty}^{+\infty} d\Phi\left\langle \Phi-\frac{\mathbb{X}}{2}\middle|\hat{\rho}\middle|\Phi+\frac{\mathbb{X}}{2}\right\rangle e^{i\mathbb{P}\Phi} \tag{2.17}$$

It is a representation of a quantum operator and density matrix in its phase space.

## 2.6 Conclusion

Quantum systems in infinitely dimensional Hilbert space $\mathcal{H}_\infty$ was focused on in this chapter. It is a space of infinite dimension which is fundamental to a quantum description of nearly all physical systems. It has an infinite number of bases such that if we represent a set of bases states by $\{|n\rangle | n = 0, \pm 1, ...\}$ where $n$ denotes the position of $n^{th}$ atom, then any state of the system can be expressed as

$$|\phi\rangle = \sum_{n=-\infty}^{+\infty} c_n |n\rangle. \tag{2.18}$$

In an infinite dimensional Hilbert space, a particle at position $\mathbb{X}$ will be in basis state $|\mathbb{X}\rangle$ where $-\infty < \mathbb{X} < \infty$. Hence, there will be a continuous range of such basis states and as a result the completeness relation is expressed in integral form as;

$$|\phi\rangle = \int_{-\infty}^{+\infty} |\mathbb{X}\rangle\langle\mathbb{X}|\phi\rangle d\mathbb{X}. \tag{2.19}$$

# Chapter 3

# Quantum systems with finite Hilbert space $\mathcal{H}_d$

## 3.1 Introduction

In this chapter, we focus our writing on $d$ dimensional Hilbert space in a finite quantum system, we represent it as $\mathcal{H}_d$. Here, we define an orthonormal basis of position state, this is denoted by $|\mathbb{X}_d; m\rangle$ and $m \in \mathcal{Z}_d$. It is apparent that the state $|\mathbb{X}_d; m\rangle$ satisfies the relation

$$\langle \mathbb{X}_d; m|\mathbb{X}_d; n\rangle = \delta(m, n), \quad \sum_{m=0}^{d-1} |\mathbb{X}_d; m\rangle\langle \mathbb{X}_d; m| = \mathbf{1}_d \tag{3.1}$$

where $\delta(m, n)$ represents the Kronecker delta, which satisfies the condition

$$\delta(m, n) = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{3.2}$$

### 3.1.1 Fourier transform

We define a finite Fourier transform as

$$\mathcal{F}_d = d^{-\frac{1}{2}} \sum_{m,n}^{d-1} \omega(mn)|\mathbb{X}_d; m\rangle\langle\mathbb{X}_d; n|, \quad \omega(m) \equiv \omega^m = \exp\left[i\frac{2\pi m}{d}\right]. \tag{3.3}$$

This transform is an essential tool in the study of representation theory of group and also, it is relevant to topics in representation of vector as linear combination of orthonormal basis.

In a complex plane, Fourier transform is associated to a group known as $d^{th}$ root of unity,$\{e^{2\pi ij/d} : 0 \leq j \leq d-1\}$ with each root of unity $e^{2\pi ij/d}$ associated with rotation $\mathcal{Z} \to e^{2\pi ij/d}\mathcal{Z}$ on the complex plane.

We confirm that

$$\mathcal{F}_d\mathcal{F}_d^\dagger = \mathcal{F}_d^\dagger\mathcal{F}_d = \mathbf{1}_d \text{ and } \mathcal{F}_d^4 = \mathbf{1}_d \tag{3.4}$$

### 3.1.2 Position and momentum states in finite quantum systems

In quantum mechanics, two mathematical objects play essential role. They are the wave function and operator.

### 3.1.3 Wave function

This is a probability amplitude which describes the state of a quantum system and its behaviour. It is a mathematical function of position and time which forms an abstract vector space.

It can be expressed in terms of a set of other functions. For example $\Phi$ may be expressed as a linear combination of other functions

$$|\Phi\rangle = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + ... + a_n|n\rangle \tag{3.5}$$

where $|\mathrm{i}\rangle$ are called the basis functions and $a_\mathbf{i}$ are coefficients (numbers). Often, there is a limited set of basis functions needed to describe any particular wave function, such a set is referred to as a complete basis set. Usually, the members of these sets are orthogonal and can be chosen to be normalised, that is,

$$\langle \mathrm{i}|\mathrm{j}\rangle = 0, \quad \text{and } \langle \mathrm{i}|\mathrm{i}\rangle = 1 \tag{3.6}$$

### 3.1.4 Position and momentum operator

The position and momentum operator is represented as $x_d$ and $p_d$ where,

$$x_d = \sum_{m=0}^{d-1} m|\mathbb{X}_d; m\rangle\langle\mathbb{X}_d; m|, \tag{3.7}$$

and

$$p_d = \sum_{m=0}^{d-1} m|\mathbb{P}_d; m\rangle\langle\mathbb{P}_d; m|. \tag{3.8}$$

Given a position state $|\mathbb{X}_d; m\rangle$, the momentum state is obtained by acting the Fourier transform $\mathcal{F}_d$ on the position states $|\mathbb{X}_d; m\rangle$.

$$|\mathbb{P}_d; m\rangle = \mathcal{F}_d|\mathbb{X}_d; m\rangle = d^{-\frac{1}{2}} \sum_{n=0}^{d-1} \omega(mn)|\mathbb{X}_d; n\rangle \tag{3.9}$$

In this work, $|\mathbb{P}_d; m\rangle$ represents momentum state.

Hence, it is deduced that the momentum and position wave functions are Fourier transform pairs. It is obvious from the above that a state $|\mathcal{S}_d\rangle$ in Hilbert space $\mathcal{H}_d$ can be expressed as

$$|\mathcal{S}_d\rangle = \sum_{n=0}^{d-1} \mu_n|\mathbb{X}_d; n\rangle = \sum_{m=0}^{d-1} \lambda_m|\mathbb{P}_d; m\rangle; \quad \mu_n = d^{-\frac{1}{2}} \sum_{m=0}^{d-1} \lambda_m \omega(mn). \tag{3.10}$$

The functions $\{\mu_n\}$ and $\{\lambda_m\}$ stand for wave functions for the state $|\mathcal{S}_d\rangle$ in position and momentum representations respectively.

From eqs.(3.3) and (3.7) we confirm that;

$$\mathcal{F}_d x_d \mathcal{F}_d^\dagger = p_d \tag{3.11}$$

Generally, Fourier transform is connected closely to uncertainty principle. It states that, two distribution of a function of a variable and its Fourier transform cannot both be measured exactly at the same time. Suppose we represent the position and momentum of a Harmonic oscillator as $\hat{X}_d$ and $\hat{P}_d$ respectively. $\hat{X}_d$ and $\hat{P}_d$ do not commute and they satisfy the relation

$$[\hat{X}_d, \hat{P}_d] = i\mathbf{1}_d \tag{3.12}$$

In a finite quantum system, both position and momentum take values in integer modulo $d$. Hence, the position-momentum phase space is a torroidal lattice $\mathcal{Z}_d \times \mathcal{Z}_d$.

### 3.1.5 Parity and displacement operator

The phase space displacement operator is defined as

$$Z_d = \exp\left[i\frac{2\pi}{d}x_d\right], \quad X_d = \exp\left[-i\frac{2\pi}{d}p_d\right] \tag{3.13}$$

(where $x_d$ and $p_d$ represent unitary operators defined earlier in eqs.(3.7) and (3.8)). They perform displacement along $\mathbb{P}$ and $\mathbb{X}$ axes respectively in the phase space. We confirm that for $\beta, \alpha \in \mathcal{Z}_d$;

$$Z_d^{\beta}|\mathbb{P}_d; m\rangle = |\mathbb{P}_d; m+\beta\rangle; \quad Z_d^{\beta}|\mathbb{X}_d; m\rangle = \omega^{m\beta}|\mathbb{X}_d; m\rangle, \tag{3.14}$$

and

$$X_d^{\alpha}|\mathbb{P}_d; m\rangle = \omega^{-m\alpha}|\mathbb{P}_d; m\rangle, \quad X_d^{\alpha}|\mathbb{X}_d; m\rangle = |\mathbb{X}_d; m+\alpha\rangle \tag{3.15}$$

Acting Fourier operator on eq.(3.13), we confirm the following relations,

$$\mathcal{F}_d X_d \mathcal{F}_d^{\dagger} = Z_d, \quad \mathcal{F}_d Z_d \mathcal{F}_d^{\dagger} = X_d^{-1} \tag{3.16}$$

The displacement operators satisfy the conditions,

$$X_d^d = Z_d^d = \mathbf{1}_d \text{ and } X_d^{\alpha} Z_d^{\beta} = Z_d^{\beta} X_d^{\alpha} \omega^{-\beta\alpha} \tag{3.17}$$

The general displacement operators are defined as

$$D(\beta,\alpha) = Z_d^{\beta} X_d^{\alpha} \omega^{-2^{-1}\beta\alpha} \tag{3.18}$$

($2^{-1}$ exists only in $\mathcal{Z}_d$ with odd $d$).

They are unitary operators associated with Heisenberg-Weyl group in finite quantum systems. From eq.(3.18),

$$D(\beta_1,\alpha_1)D(\beta_2,\alpha_2) = Z_d^{\beta_1} X_d^{\alpha_1} \omega^{-2^{-1}\beta_1\alpha_1} Z_d^{\beta_2} X_d^{\alpha_2} \omega^{-2^{-1}\beta_2\alpha_2}. \tag{3.19}$$

From eq.(3.19) above we have

$$D(\beta_1,\alpha_1)D(\beta_2,\alpha_2) = D(\beta_1+\beta_2)D(\alpha_1+\alpha_2)\omega^{-2^{-1}(\beta_1\alpha_2-\beta_2\alpha_1)} \tag{3.20}$$

Also, we confirm from eqs.(3.14) and (3.15) that

$$D(\beta,\alpha)|\mathbb{X}_d;m\rangle = \omega^{(2^{-1}\beta\alpha+\beta m)}|\mathbb{X}_d;m+\alpha\rangle \tag{3.21}$$

$$D(\beta,\alpha)|\mathbb{P}_d;m\rangle = \omega^{(-2^{-1}\beta\alpha-\alpha m)}|\mathbb{P}_d;m+\beta\rangle \tag{3.22}$$

Parity operator around the origin, $\mathbf{P}_d(0,0)$ is defined as

$$\mathbf{P}_d(0,0) = \mathcal{F}_d^2, \quad [\mathbf{P}_d(0,0)]^2 = \mathbf{1}_d \tag{3.23}$$

It has $1,-1$ as its eigenvalues. Acting the operator $\mathbf{P}_d(0,0)$ on position and momentum states $|\mathbb{X}_d; m\rangle$ and $|\mathbb{P}_d; m\rangle$ gives

$$\begin{aligned}
&\mathbf{P}_d(0,0)|\mathbb{X}_d; m\rangle = |\mathbb{X}_d; -m\rangle, \quad \mathbf{P}_d(0,0)|\mathbb{P}_d; m\rangle = |\mathbb{P}_d; -m\rangle. \\
\text{Also,}\quad &\mathbf{P}_d(0,0)x_d[\mathbf{P}_d(0,0)]^\dagger = -x_d, \quad \mathbf{P}_d(0,0)p_d[\mathbf{P}_d(0,0)]^\dagger = -p_d \\
\text{and}\quad &\mathbf{P}_d(0,0)Z_d[\mathbf{P}_d(0,0)]^\dagger = Z_d^\dagger, \quad \mathbf{P}_d(0,0)X_d[\mathbf{P}_d(0,0)]^\dagger = X_d^\dagger \qquad (3.24)
\end{aligned}$$

The parity operator about a point $(\beta, \alpha)$ is called a displaced parity operator. It is defined as;

$$\begin{aligned}
\mathbf{P}_d(\beta,\alpha) &= D(\beta,\alpha)\mathbf{P}_d(0,0)[D(\beta,\alpha)]^\dagger = D(2\beta,2\alpha)\mathbf{P}_d(0,0) \\
&= \mathbf{P}_d(0,0)[D(2\beta,2\alpha)]^\dagger \qquad (3.25)
\end{aligned}$$

and

$$[\mathbf{P}_d(\beta,\alpha)]^2 = \mathbf{1}_d \qquad (3.26)$$

## 3.2 Symplectic transformation in finite quantum systems

In this section, we discuss the concept of symplectic transformation in the phase space $\mathcal{Z}_d \times \mathcal{Z}_d$ of a finite quantum system. Here, we define the unitary transformations;

$$X'_d = \mathbf{S}_d X_d \mathbf{S}_d^\dagger = X_d^\xi Z_d^\eta \omega^{2^{-1}\xi\eta} = D(\eta, \xi),$$
$$Z'_d = \mathbf{S}_d Z_d \mathbf{S}_d^\dagger = X_d^\zeta Z_d^\tau \omega^{2^{-1}\zeta\tau} = D(\tau, \zeta),$$
$$\xi\tau - \eta\zeta = 1(mod(d)) \tag{3.27}$$

where $\xi, \tau, \eta, \zeta$ are integers in $\mathcal{Z}_d$. It is evident that this transformation preserve eq.(3.17). Hence $X'_d$ and $Z'_d$ can also be used as displacement operator. As an analogy, eq.(3.27) has three independent variables $\xi, \eta, \zeta$ while the fourth variable $\tau$ is a constraint which can be linked to the existence of inverse of the elements in $\mathcal{Z}_d$.

In our work, we study the symplectic transformation at an applied level and we show that it forms a group. This is expressed by showing first that if we couple two of the transformations above, it gives another transformation of the same type. Thus;

$$\mathbf{S}_d(\xi_1, \eta_1, \zeta_1)\mathbf{S}_d(\xi_2, \eta_2, \zeta_2) = \mathbf{S}_d(p, r, s)$$
$$p = \xi_1\xi_2 + \eta_1\zeta_2$$
$$r = \xi_1\eta_2 + \eta_1\xi_2^{-1}(1 + \eta_2\zeta_2)$$
$$s = \xi_2\zeta_1 + \zeta_2\xi_1^{-1}(1 + \eta_1\zeta_1). \tag{3.28}$$

Suppose we define a unitary operator $\mathbf{S}_d(\xi, \eta|\zeta, \tau)$ as

$$\begin{aligned} \mathbf{S}_d(\xi, \eta|\zeta, \tau) &= \mathbf{S}_d(\xi_1, \eta_1|\zeta_1, \tau_1)\mathbf{S}_d(\xi_2, \eta_2|\zeta_2, \tau_2) \\ &= \mathbf{S}_d(\xi_2, \eta_2|\zeta_2, \tau_2)\mathbf{S}_d(\xi_1, \eta_1|\zeta_1, \tau_1) \end{aligned} \tag{3.29}$$

Analytically, symplectic operator $\mathbf{S}_d(\xi, \eta, \zeta)$ can be defined from eq.(3.28) as

$$\begin{pmatrix} 1 & 0 \\ \zeta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \zeta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix} = \begin{pmatrix} \zeta_3(1+\zeta_1\zeta_2) & \zeta_3\zeta_2 \\ \zeta_1\zeta_3^{-1} & \zeta_3^{-1} \end{pmatrix} \tag{3.30}$$

where

$$\zeta_1 = \xi\zeta(1+\eta\zeta)^{-1}, \zeta_2 = \xi^{-1}\eta(1+\eta\zeta), \zeta_3 = \xi(1+\eta\zeta)^{-1} \tag{3.31}$$

If we substitute eq.(3.31) into eq.(3.30) that is $\zeta_1, \zeta_2, \zeta_3$ into the matrix, we get

$$\begin{pmatrix} \xi & \eta \\ \zeta & \tau \end{pmatrix} \tag{3.32}$$

In the case of non-prime $d$, eq.(3.31) is still valid when $\xi \in \mathcal{Z}_d^*$ and $(1+\eta\zeta) \in \mathcal{Z}_d$.

In general, the operator $\mathbf{S}_d(\xi, \eta|\zeta, \tau)$ is expressed as;

$$\mathbf{S}_d(\xi, \eta|\zeta, \tau) D(\beta, \alpha) [\mathbf{S}_d(\xi, \eta|\zeta, \tau)]^\dagger = D(\beta\tau + \alpha\eta, \beta\zeta + \alpha\xi) \tag{3.33}$$

The prove of eq.(3.33) is established as follows; from eq.(3.18)

$$\mathbf{S}_d D(\beta, \alpha) \mathbf{S}_d^\dagger = \mathbf{S}_d Z_d^\beta X_d^\alpha \omega^{-2^{-1}\beta\alpha} \mathbf{S}_d^\dagger = \mathbf{S}_d Z_d^\beta \mathbf{S}_d^\dagger \mathbf{S}_d X_d^\alpha \mathbf{S}_d^\dagger \tag{3.34}$$

From eq.(3.27) we have

$$\begin{aligned} &\text{(i)} \quad \mathbf{S}_d Z_d^{\beta} \mathbf{S}_d^{\dagger} = X_d^{\beta\zeta} Z_d^{\beta\tau} \omega^{2^{-1}\beta^2\tau\zeta} = D(\beta\tau, \beta\zeta) \\ &\text{(ii)} \quad \mathbf{S}_d X_d^{\alpha} \mathbf{S}_d^{\dagger} = X_d^{\alpha\xi} Z_d^{\alpha\eta} \omega^{2^{-1}\xi\alpha^2\eta} = D(\alpha\eta, \alpha\xi) \end{aligned} \tag{3.35}$$

If we substitute eq.(3.35) into eq.(3.34) we get

$$X_d^{\beta\zeta+\alpha\xi} Z_d^{\beta\tau+\alpha\eta} \omega^{2^{-1}\beta^2\tau\zeta} \omega^{2^{-1}\alpha^2\xi\eta} = D(\beta\tau + \alpha\eta, \beta\zeta + \alpha\xi) \tag{3.36}$$

where $\omega^{2^{-1}\beta^2\tau\zeta}$ and $\omega^{2^{-1}\alpha^2\xi\eta}$ represent phase factors.

The symplectic operator $\mathbf{S}_d(\xi, \eta, \zeta)$ changes a basis $|\mathbb{X}_d; m\rangle$ and $|\mathbb{P}_d; m\rangle$ into a new one

$$|\mathbb{X}'_d; m\rangle = \mathbf{S}_d(\xi, \eta, \zeta)|\mathbb{X}_d; m\rangle, \quad |\mathbb{P}'_d; m\rangle = \mathbf{S}_d(\xi, \eta, \zeta)|\mathbb{P}_d; m\rangle. \tag{3.37}$$

Also, it transforms the displacement operators $X_d$ and $Z_d$ into $X'_d$ and $Z'_d$ respectively as defined in eq.(3.27). In general, the displacement operators $D(\beta, \alpha)$ are transformed into

$$D'(\beta, \alpha) \equiv \mathbf{S}_d D(\beta, \alpha) \mathbf{S}_d^{\dagger} = D(\beta\tau + \alpha\eta, \beta\zeta + \alpha\xi). \tag{3.38}$$

$D'(\beta, \alpha)$ perform displacement operators by $(\beta, \alpha)$ in the $\mathbb{P}' - \mathbb{X}'$ bases, this is the same as performing a displacement by $(\beta\tau + \alpha\eta, \beta\zeta + \alpha\xi)$ in the $\mathbb{P} - \mathbb{X}$ bases.

### 3.2.1 Marginal properties of displacement operator

Due to lack of inverse of integer 2 in quantum systems with even dimension, we consider here systems with odd $d$. Its displacement operator satisfies the following relations;

$$d^{-1}\sum_{\alpha=0}^{d-1} D(\beta,\alpha) = |\mathbb{P}_d; 2^{-1}\beta\rangle\langle\mathbb{P}_d; -2^{-1}\beta|$$
$$d^{-1}\sum_{\beta=0}^{d-1} D(\beta,\alpha) = |\mathbb{X}_d; 2^{-1}\alpha\rangle\langle\mathbb{X}_d; -2^{-1}\alpha| \tag{3.39}$$

and in addition,

$$d^{-1}\sum_{\beta,\alpha=0}^{d-1} D(\beta,\alpha) = \mathbf{P}_d(0,0) \tag{3.40}$$

### 3.2.2 Marginal properties of displaced parity operator

If we act with the parity $\mathbf{P}_d(0,0)$ on the right hand side of eq.(3.39) and considering eq.(3.25), we confirm that;

$$d^{-1}\sum_{\alpha=0}^{d-1} \mathbf{P}_d(\beta,\alpha) = |\mathbb{P}_d; \beta\rangle\langle\mathbb{P}_d; \beta|$$
$$d^{-1}\sum_{\beta=0}^{d-1} \mathbf{P}_d(\beta,\alpha) = |\mathbb{X}_d; \alpha\rangle\langle\mathbb{X}_d; \alpha| \tag{3.41}$$

and in addition,

$$d^{-1}\sum_{\beta,\alpha=0}^{d-1} \mathbf{P}_d(\beta,\alpha) = \mathbf{1}_d \tag{3.42}$$

in systems with odd $d$, $Tr\mathbf{P}_d(\beta,\alpha) = 1$.

## 3.3 Wigner and Weyl functions

In fundamental problems of quantum mechanics, Wigner and Weyl functions play an essential role [40]. Also, in applied problems of quantum optics, equations in quantum mechanics can be expressed in terms of the Wigner (or Weyl) function. This is due to the fact that all the information in the density matrix is also contained in the corresponding Wigner function.The Wigner function is closely related to displaced parity operator while the Weyl function is related to the displacement operator.

### 3.3.1 Definition

Suppose we represent an arbitrary operator as $\Phi$. We define the Wigner function $\mathcal{W}(\Phi;\beta,\alpha)$ as

$$\mathcal{W}(\Phi;\beta,\alpha) = Tr[\Phi\mathbf{P}_d(\beta,\alpha)] \tag{3.43}$$

If $\Phi$ is an arbitrary Hermitian operator, the Wigner function is real, however for a non-Hermitian operator, the Wigner function is complex.The function is the Fourier transform of the matrix elements of the operator $\Phi$;

$$\begin{aligned}\mathcal{W}(\Phi;\beta,\alpha) &= \omega^{2\beta\alpha}\sum_{l=0}^{d-1}\omega^{-2\beta l}\Phi_{\mathbb{X}}(l,2\alpha-l)\\ &= \omega^{-2\beta\alpha}\sum_{l=0}^{d-1}\omega^{2\alpha l}\Phi_{\mathbb{P}}(l,2\beta-l)\end{aligned} \tag{3.44}$$

where $\Phi_{\mathbb{X}}(a,l) \equiv \langle \mathbb{X}; a|\Phi|\mathbb{X}; l\rangle$

### 3.3.2 Weyl function

The Weyl function corresponding to the operator $\Phi$ is expressed in terms of displacement operator. It is defined as the Fourier transform of the operator $\Phi$ and satisfies the relation

$$\widetilde{\mathcal{W}}(\Phi;\beta,\alpha) \equiv Tr[\Phi D(\beta,\alpha)] \tag{3.45}$$

$$\begin{aligned}\widetilde{\mathcal{W}}(\Phi;\beta,\alpha) &= \omega^{2^{-1}\beta\alpha}\sum_{l=0}^{d-1}\omega^{\beta l}\Phi_{\mathbb{X}}(l,\alpha+l)\\ &= \omega^{-2^{-1}\beta\alpha}\sum_{l=0}^{d-1}\omega^{-\alpha l}\Phi_{\mathbb{P}}(l,\beta+l)\end{aligned} \tag{3.46}$$

where $\Phi_{\mathbb{X}}(a,l)$ and $\Phi_{\mathbb{P}}(a,l)$ are defined earlier in eq.(3.44).

$$\widetilde{\mathcal{W}}(\Phi;\beta,\alpha) = \omega^{2^{-1}\beta\alpha}\sum_{l=0}^{d-1}\omega^{\beta l}\langle \mathbb{X}_d; l|\mathbb{X}_d; 0\rangle\langle \mathbb{X}_d; 0|\mathbb{X}_d; \alpha+l\rangle \tag{3.47}$$

## 3.4 Factorization of big system in terms of its components

Factorizing very large integers could be very tedious in terms of computational costs. The fast Fourier transform [58] addressed this set back by factorizing large dimensional Hilbert space in terms of component spaces. In this work, we use the fast Fourier transform scheme of Good [34] where by a large integer $d$ is expressed as,

$$d = \mathbf{p_1} \times ... \times \mathbf{p_k} \tag{3.48}$$

where $\mathbf{p_1}, ..., \mathbf{p_k}$ are relatively prime with respect to each other.

This method is built on Chinese remainders theorem and was used by [1] to factorize a large quantum system as products of many small quantum systems. In this work, we use the same concept in chapter four to factorize lines in finite geometry $\mathcal{G}_d$ in terms of prime factor lines and large quantum system in terms of $\mathbf{k}$ subsystems. This was carried out by defining two bijections between $\mathcal{Z}_d$ and $\mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}}$. That is;

$$\mathcal{Z}_d \longleftrightarrow \mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}} \tag{3.49}$$

The first map was used for position while the second stands for the momentum state. The derivation of the two bijections are established as follows; suppose an arbitrary integer $d$ can be factored as;

$$d = \mathbf{r_j p_j}, \text{and} \;\; \mathbf{t_j r_j} = 1(mod\ \mathbf{p_j}) \tag{3.50}$$

where $\mathbf{t_j}$ is the inverse of $\mathbf{r_j}$ in $\mathcal{Z}_{\mathbf{p_j}}$, in addition, we define

$$\mathbf{s_j} = \mathbf{t_j r_j}, \;\text{where}\; \mathbf{s_j} = \mathbf{t_j r_j} \in \mathcal{Z}_d \tag{3.51}$$

and it is an integer multiple of $\mathbf{p_j}$ plus 1.

The first $1-1$ map between $\mathcal{Z}_d$ and $\mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}}$ is

$$\beta \longleftrightarrow (\beta_{\mathbf{1}}, ..., \beta_{\mathbf{k}}),\ \beta_{\mathbf{j}} = \beta(mod\ \mathbf{p_j}),\ \beta = \sum_{\mathbf{j=1}}^{\mathbf{k}} \beta_{\mathbf{j}}\mathbf{s_j} \tag{3.52}$$

The second $1-1$ map between $\mathcal{Z}_d$ and $\mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}}$ is

$$\beta \longleftrightarrow (\bar{\beta_{\mathbf{1}}}, ..., \bar{\beta_{\mathbf{k}}}),\ \bar{\beta_{\mathbf{j}}} = \beta\mathbf{t_j}(mod\ \mathbf{p_j}),\ \beta = \sum_{\mathbf{j=1}}^{\mathbf{k}} \bar{\beta_{\mathbf{j}}}\mathbf{r_j}(mod(d)). \tag{3.53}$$

Suppose we define

$$\omega(\mathbf{r_j s_j}) = \omega_{\mathbf{j}} \equiv \exp\left(i\frac{2\pi}{\mathbf{p_j}}\right),\ \text{where for } \mathbf{j} \neq \text{i} \rightarrow \omega(\mathbf{r_j s_i}) = 1. \tag{3.54}$$

In relation to the above equation, we confirm that

$$\omega(\beta\alpha) = \prod_{\mathbf{j}=1}^{\mathbf{k}} \omega(\beta_{\mathbf{j}}\bar{\alpha_{\mathbf{j}}}) \tag{3.55}$$

As an illustration, suppose $d = 10, \mathbf{p_1} = 2$, and $\mathbf{p_2} = 5$, then $\mathbf{r_1} = 5$, $\mathbf{r_2} = 2, \mathbf{t_1} = 1, \mathbf{t_2} = 3, \mathbf{s_1} = 5$, and $\mathbf{s_2} = 6$.

Hence,

$$\beta = 5\beta_{\mathbf{1}} + 6\beta_{\mathbf{2}} = 5\bar{\beta}_{\mathbf{1}} + 2\bar{\beta}_{\mathbf{2}}. \tag{3.56}$$

For example $\beta = 7$ in $\mathcal{Z}_{10}$ corresponds to $\beta_{\mathbf{1}} = 1, \beta_{\mathbf{2}} = 2$ according to mapping of eq.(3.52) and $(\bar{\beta}_{\mathbf{1}} = 1, \bar{\beta}_{\mathbf{2}} = 1)$ according to eq.(3.53).

Also if we substitute $\beta = 7$ and $\alpha = 7$ in $\mathcal{Z}_{10}$ into eq.(3.55)

It is confirmed accordingly. That is,

$$\begin{aligned} \exp\left[\frac{2\pi i \times 7 \times 7}{10}\right] &= \exp\left[\frac{2\pi i \times 1 \times 1}{2}\right] \exp\left[\frac{2\pi i \times 2 \times 1}{5}\right] \\ \exp\left[\frac{98\pi i}{10}\right] &= \exp\left[\frac{8\pi i}{10}\right] \end{aligned} \tag{3.57}$$

Also, we consider a quantum system with a $d-$ dimensional Hilbert space $\mathcal{H}_d$. We factorize $d$ the same way as in eq.(3.48) above (that is, $d = \mathbf{p_1} \times ... \times \mathbf{p_k}$) where $\mathbf{p_1}, ..., \mathbf{p_k}$ are relatively prime with respect to each other.

At this point, we define an isomorphism between $\mathcal{H}_d$ and product of Hilbert space $\mathcal{H}_{\mathbf{p_1}} \otimes ... \otimes \mathcal{H}_{\mathbf{p_k}}$

We establish the mapping between the position bases and momentum bases in $\mathcal{H}_d$ and $\mathcal{H}_{\mathbf{p_1}} \otimes ... \otimes \mathcal{H}_{\mathbf{p_k}}$ as

$$\begin{aligned} |\mathbb{X}_d; \beta\rangle &\longleftrightarrow |\mathbb{X}_{\mathbf{p_1}}; \bar{\beta_1}\rangle \otimes ... \otimes |\mathbb{X}_{\mathbf{p_k}}; \bar{\beta_k}\rangle \\ |\mathbb{P}_d; \beta\rangle &\longleftrightarrow |\mathbb{P}_{\mathbf{p_1}}; \beta_1\rangle \otimes ... \otimes |\mathbb{P}_{\mathbf{p_k}}; \beta_k\rangle \end{aligned} \tag{3.58}$$

Example

$$\begin{aligned} |\mathbb{X}_{10}; 7\rangle &\longleftrightarrow |\mathbb{X}_2; 1\rangle \otimes |\mathbb{X}_5; 1\rangle \\ |\mathbb{P}_{10}; 7\rangle &\longleftrightarrow |\mathbb{P}_2; 1\rangle \otimes |\mathbb{P}_5; 2\rangle \end{aligned} \tag{3.59}$$

The phase space toroidal lattice in $\mathcal{Z}_d \times \mathcal{Z}_d$ can be factored as multi-dimensional torroidal lattice phase space

$$(\mathcal{Z}_{\mathbf{p_1}} \times \mathcal{Z}_{\mathbf{p_1}}) \times ... \times (\mathcal{Z}_{\mathbf{p_k}} \times \mathcal{Z}_{\mathbf{p_k}}). \tag{3.60}$$

Also the displacement operator $D(\beta, \alpha)$ is expressed as;

$$D(\beta, \alpha) = \prod_{\mathbf{j}=1}^{\mathbf{k}} D_{\mathbf{j}}(\beta_{\mathbf{j}}, \bar{\alpha}_{\mathbf{j}}) \tag{3.61}$$

where $\beta, \alpha, \beta_{\mathbf{j}}$ and $\bar{\alpha}_{\mathbf{j}}$ are related to eqs.(3.52) and (3.53) respectively.

## 3.5 Conclusion

In this chapter, we focus our discussion on quantum systems with finite dimensional Hilbert space $\mathcal{H}_d$ with variables in $\mathcal{Z}_d$. Its phase space $\mathcal{Z}_d \times \mathcal{Z}_d$ forms a torroidal lattice.

The displacement operator and parity operator along with their properties were mentioned. They are related to Weyl and Wigner function respectively. Furthermore, symplectic transformation in the phase space was discussed with its properties. Also, we reviewed factorization of a big system as products of many small quantum systems. This concept was used in our work to factorize large dimensional Hilbert space in finite quantum systems as products of many small dimension Hilbert spaces $\mathcal{H}_{\mathbf{p_j}}$.

# Chapter 4

# Partial ordering of weak mutually unbiased bases

## 4.1 Introduction

In this chapter, we discuss the concept of partial ordered relation that exists between a finite geometry and the set of its subgeometries. By convention, an Euclidean geometry has an infinite number of points, lines, and planes as well as an appreciable collection of growing theorems. A small sized geometry that has a finite number of elements is called a finite geometry.

## 4.2 Partial ordered relation

Suppose we define $A$ as a non- empty set, a relation $R$ on $A$ expressed as $R \subseteq A \times A$ is called a partial ordered relation if:

(i) $\forall\, a \in A; a \leq a$ ( reflexivity)

(ii) $\forall\ a, b \in A$; if $a \leq b$ and $b \leq a \Rightarrow a = b$ (antisymmetric)

(iii) $\forall\ a, b, c \in A$; if $a \leq b$ and $b \leq c\ \Rightarrow a \leq c$ (transitivity)

### 4.2.1 Definition

A finite geometry $\mathcal{G}_d$ is defined as a geometry which consists of a collection of finite set of points and lines. Mathematically, it is expressed as $\mathcal{G}_d = (\mathbb{P}(d), \mathbf{L}_d)$. It is the combination of the set of points, $\mathbb{P}(d)$ and lines, $\mathbf{L}_d$ where;

$$\mathbb{P}(d) = \{(\mu, \nu) | \mu, \nu \in \mathcal{Z}_d\} \tag{4.1}$$

A line $L(x, y)$ through the origin is defined as

$$L(x, y) = \{(\alpha x, \alpha y) | \alpha \in \mathcal{Z}_d\}; x, y \in \mathcal{Z}_d. \tag{4.2}$$

## 4.3 The non-near-linear geometry $\mathcal{G}_d$ and its subgeometries $\mathcal{G}_q$

Suppose we define two lines $L_1(x, y)$ and $L_2(u, v)$ in $\mathcal{G}_d$ (where $u, v, x, y \in \mathcal{Z}_d$). If every pair of lines, $L_1(x, y)$ and $L_2(u, v)$ have at most one point in common then the geometry $\mathcal{G}_d$ is called a near-linear geometry $[27-29]$. For $d$ a prime, $\mathcal{Z}_d$ is a field of integers modulo $d$. In this case, there is no other divisors of $d$ apart from the trivial ones which is the integer, 1 and $d$ itself. Hence, there are no other subgeometries apart from $\mathcal{G}_1$ and $\mathcal{G}_d$.

For $d$ a non-prime, $\mathcal{Z}_d$ is a ring of integers modulo $d$. In this case, there are non-trivial divisors $\mathcal{D}(d)$ and for $q|d$, $L_1(x,y), L_2(u,v) \in \mathcal{G}_d$ have $q$ points in common. This type of geometry $\mathcal{G}_d$ is called a non-near-linear geometry. This is the focus of our work.

In a finite geometry, each geometrical element represents a finite set. In a $d-$ dimensional non-near-linear geometry, there exists $\sigma_0(d)$ subgeometries. $\mathcal{G}_d$ denotes a finite geometry of dimension $d$, it is defined as;

$$\mathcal{G}_d = \mathcal{Z}_d \times \mathcal{Z}_d \tag{4.3}$$

where $\mathcal{Z}_d$ is a ring of integer modulo $d$.

A line $L(x,y)$ through the origin is defined in eq.(4.2) above. In this work, $d$ is expressed as product of prime numbers to power one as defined in eq.(3.48). Mathematically, $\mathcal{Z}_d$ is a cyclic module. The notation $\mathbf{L}_d$ represents the set of all lines through the origin in $\mathcal{G}_d$.

We define $\mathcal{Z}_d^*$ as a reduced system residue modulo $d$. It is a set which consists of the invertible elements in the ring of integer modulo $d$. Its cardinality is $\varphi(d)$ where

$$\varphi(d) = d\prod_{\mathbf{j=1}}^{\mathbf{k}}\left(1 - \frac{1}{\mathbf{p_j}}\right). \tag{4.4}$$

We represent the greatest common divisor of two elements $x$ and $y$ in $\mathcal{Z}_d$ as $\mathcal{GCD}(x,y)$. In $\mathcal{G}_d$, lines with $d$ points are called maximal lines. The total number of maximal lines in a finite geometry $\mathcal{G}_d$ is $\psi(d)$ [23], where

$$\psi(d) = d\prod_{\mathbf{j=1}}^{\mathbf{k}}\left(1+\frac{1}{\mathbf{p_j}}\right). \tag{4.5}$$

We define the set $\{\mathcal{D}(d)\}$ of divisors of $d$. It is a partially ordered set with divisibility as partial order.

Let

$$\mathbb{Z}_d = \{\mathcal{Z}_q | q \in \mathcal{D}(d)\} \tag{4.6}$$

be a set which contains all subgroups of $\mathcal{Z}_d$. It is a partially ordered set with divisibility as partial order. The divisibility $\mathcal{D}(d)$ and $\mathbb{Z}_d$ are isomorphic to each other.

If $q|d$, the elements of $\mathcal{Z}_q$ are enclosed in $\mathcal{Z}_d$ as follows;

$$\mathcal{Z}_q \ni m \to \frac{dm}{q} \in \mathcal{Z}_d. \tag{4.7}$$

Also, $\mathcal{Z}_q \subset \mathcal{Z}_d$, this implies $\mathcal{Z}_q \prec \mathcal{Z}_d$. Likewise $\mathcal{G}_q$ is a subgeometry of $\mathcal{G}_d$ (that is $\mathcal{G}_q \prec \mathcal{G}_d$).

Below, we compile the previous results of [23, 24] in the following propositions:

(i) In geometry $\mathcal{G}_d$, lines with $d$ points are called maximal lines and there are $\psi(d)$ of such lines.

Example: a finite geometry $\mathcal{G}_6$ has 12 maximal lines.

(ii) If

$$b \in \mathcal{Z}_d^* \text{ then } L(bx, by) = L(x, y). \tag{4.8}$$

Example: $L(1,2)$ in $\mathcal{G}_{42}$ is the same as $L(25,8)$ in $\mathcal{G}_{42}$ thus

$L(1,2) = \{(0,0)(1,2)(2,4)(3,6)(4,8)(5,10)(6,12)(7,14)(8,16)(9,18)$
$(10,20)(11,22)(12,24)(13,26)(14,28)(15,30)(16,32)(17,34)(18,36)$
$(19,38)(20,40)(21,0)(22,2)(23,4)(24,6)(25,8)(26,10)(27,12)(28,14)$
$(29,16)(30,18)(31,20)(32,22)(33,24)(34,26)(35,28)(36,30)(37,32)$
$(38,34)(39,36)(40,38)(41,40)\}$.

$L(25,8) = \{(0,0)(25,8)(8,16)(33,24)(16,32)(41,40)(24,6)(7,14)(32,22)$
$(15,30)(40,38)(23,4)(6,12)(31,20)(14,28)(39,36)(22,2)(5,10)(30,18)$
$(13,26)(38,34)(21,0)(4,8)(29,16)(12,24)(37,32)(20,40)(3,6)(28,14)$
$(11,22)(36,30)(19,38)(2,4)(27,12)(10,20)(35,28)(18,36)(1,2)$
$(26,10)(9,18)(34,26)(17,34)\}$.

If

$$b \in \mathcal{Z}_d - \mathcal{Z}_d^* \text{ then } L(bx,by) \prec L(x,y) \tag{4.9}$$

Example: $L(14,28) = \{(0,0)(14,28)(28,14)\}$.

There exists a 1-1 map between the set of points of lines $L(14,28)$ and $L(1,2)$ in $\mathcal{G}_{42}$. This is expressed as $L(14,28) \prec L(1,2)$.

Hence, if $\mathcal{GCD}(x,y) \in \mathcal{Z}_d^*$ then $L(x,y)$ is a maximal line in $\mathcal{G}_d$, however if $\mathcal{GCD}(x,y) \in \mathcal{Z}_d - \mathcal{Z}_d^*$ then $L(x,y)$ is a subline in $\mathcal{G}_d$.

(iii) Suppose we define a line in finite geometry $\mathcal{G}_d$ as

$$L(x,y) = L(\alpha x, \alpha y). \tag{4.10}$$

$L(x, y)$ is the same as

$$L(\mathbf{w}x, \mathbf{w}y) = \{(\alpha\mathbf{w}x, \alpha\mathbf{w}y)|\alpha = 0, 1..., d-1\} \text{ in } \mathcal{G}_{\mathbf{w}d} \tag{4.11}$$

at the same time the line $L(\mathbf{w}x, \mathbf{w}y)$ is a subline of

$$L(x, y) = \{(\alpha' x, \alpha' y)|\alpha' = 0, 1..., \mathbf{w}d - 1\}. \tag{4.12}$$

in a big-size geometry $\mathcal{G}_d$.

Example: suppose $d = 7, \mathbf{w} = 6$, $L(1, 4)$ in $\mathcal{G}_7$ is

$$L(1, 4) = \{(0, 0)(1, 4)(2, 1)(3, 5)(4, 2)(5, 6)(6, 3)\} \tag{4.13}$$

this is the same as $L(6, 24)$ in $\mathcal{G}_{42}$ that is,

$$L(6, 24) = \{(0, 0)(6, 24)(12, 6)(18, 30)(24, 12)(30, 36)(36, 18)\} \tag{4.14}$$

$L(6, 24)$ is a subline of the maximal line $L(1, 4)$ in $\mathcal{G}_{42}$.

Where $L(1, 4)$ in $\mathcal{G}_{42}$ is defined as

$L(1, 4) = \{(0, 0)(1, 4)(2, 8)(3, 12)(4, 16)(5, 20)(6, 24)(7, 28)(8, 32)(9, 36)$
$(10, 40)(11, 2)(12, 6)(13, 10)(14, 14)(15, 18)(16, 22)(17, 26)(18, 30)(19, 34)$
$(20, 38)(21, 0)(22, 4)(23, 8)(24, 12)(25, 16)(26, 20)(27, 24)(28, 28)(29, 32)$
$(30, 36)(31, 40)(32, 2)(33, 6)(34, 10)(35, 14)(36, 18)(37, 22)$
$(38, 26)(39, 30)(40, 34)(41, 38)\}$.

(iv) If two maximal lines have $q$ points in common where $q|d$, the $q$ points produce a subline $L(z, w)$ where $z, w \in \frac{d}{q}\mathcal{Z}_q$, and

$$\frac{d}{q}\mathcal{Z}_q = \left\{0, \frac{d}{q}, .., \frac{d(q-1)}{q}\right\}. \tag{4.15}$$

If we consider the subgeometry $\mathcal{G}_q$, the subline $L(z, w)$ in $\mathcal{G}_d$ is a maximal line in $\mathcal{G}_q$. There exists $\psi(q)$ maximal lines in subgeometry $\mathcal{G}_q$ of finite geometry $\mathcal{G}_d$.

Example: In $\mathcal{G}_{42}$, the intersection of maximal lines $L(1, 2)$ and $L(1, 4)$ gives a subline $L(21, 0)$ with two points, $(0, 0)$ and $(21, 0)$ that is,

$$L(1, 2) \cap L(1, 4) = L(21, 0) = \{(0, 0)(21, 0)\}. \tag{4.16}$$

and $L(21, 0) \cong L(1, 0)$ in $\mathcal{G}_2$.

Hence from propositions (i)-(iv) above, we deduce that, if $L(x, y)$ is a maximal line in $\mathcal{G}_d$, then $L(x, y) \cong L(\mathbf{w}x, \mathbf{w}y)$ in $\mathcal{G}_{\mathbf{w}d}$. The maximal line in $\mathcal{G}_{\mathbf{w}d}$ which contains $L(\mathbf{w}x, \mathbf{w}y)$ is $L(a, b)$ with

$$a = \frac{x}{\mathcal{GCD}(x, y)}, \text{and} \quad b = \frac{y}{\mathcal{GCD}(x, y)}. \tag{4.17}$$

Example: suppose $d = 7, \mathbf{w} = 6, a = 1, b = 4, L(1, 4)$ in $\mathcal{G}_7 \cong L(6, 24) \prec L(1, 4)$ in $\mathcal{G}_{42}$.

More examples are shown in chapter five.

## 4.4 Factorization of lines in terms of prime factor lines

In this section, we discuss the factorization of lines in terms of their prime factors. This concept was built on Chinese remainders theorem as discussed in chapter three. It was used by [34] in fast Fourier transform to factorize integer $d$ as products of its factors $\mathbf{p_1} \times ... \times \mathbf{p_k}$ where $\mathbf{p_1}, ..., \mathbf{p_k}$ are pairwise coprime. We used this concept to factorize lines $L(a, b)$ in $\mathcal{Z}_d \times \mathcal{Z}_d$ as products of lines in $(\mathcal{Z}_{\mathbf{p_1}} \times \mathcal{Z}_{\mathbf{p_1}}) \times ... \times (\mathcal{Z}_{\mathbf{p_k}} \times \mathcal{Z}_{\mathbf{p_k}})$ where $\mathbf{p}$ is a prime number, in our work, we label it as prime factor lines. Using the methods discussed in eqs.(3.50 − 3.53) There exists a bijection map between $(\mathcal{Z}_d \times \mathcal{Z}_d)$ and $[(\mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}})] \times [(\mathcal{Z}_{\mathbf{p_1}} \times ... \times \mathcal{Z}_{\mathbf{p_k}})]$.
That is;

$$(a, b) \longleftrightarrow (a_{\mathbf{1}}, ...a_{\mathbf{k}}, \bar{b}_{\mathbf{1}}, ..., \bar{b}_{\mathbf{k}}). \tag{4.18}$$

In our work, we use eq.(3.52) to represent $a$, it is linked to positions and eq.(3.53) to represent $b$, it is linked to momenta. Hence a maximal line $L(a, b)$ in $\mathcal{G}_d$ can be expressed as

$$L(a, b) = L(a_1, \bar{b}_1) \times ... \times L(a_{\mathbf{k}}, \bar{b}_{\mathbf{k}}); \quad L(a_{\mathbf{j}}, \bar{b}_{\mathbf{j}}) \in \mathcal{Z}_{\mathbf{p_j}} \times \mathcal{Z}_{\mathbf{p_j}} \tag{4.19}$$

We label the lines $L(a_{\mathbf{j}}, \bar{b}_{\mathbf{j}})$ as prime factor lines in our work.
Example: the maximal line $L(1, 2) \in \mathcal{Z}_{42} \times \mathcal{Z}_{42}$ is expressed as products of prime factor lines $L(1, 0) \times L(1, 1) \times L(1, 5) \in ((\mathcal{Z}_2 \times \mathcal{Z}_2) \times (\mathcal{Z}_3 \times \mathcal{Z}_3) \times (\mathcal{Z}_7 \times \mathcal{Z}_7))$

using eqs.(3.52) and (3.53) as follows,

we factorise the ordinate 1 in line $L(1,2)$ as,

$$1 \longleftrightarrow (1,1,1) \tag{4.20}$$

and likewise, the ordinate 2 in $L(1,2)$ is factorized as,

$$2 \longleftrightarrow (0,1,5) \tag{4.21}$$

Therefore;

$$L(1,2) = L(1,0) \times L(1,1) \times L(1,5) \tag{4.22}$$

In this work, we call eq.(4.19) factorization of line in terms of prime factor lines. It is related to expression of integers as products of their primes. The geometry $\mathcal{G}_d$ is related to the set $\{\mathcal{D}(d)\}$ of divisors of $d$, the subline $L(z,w)$ in $\frac{d}{q}\mathcal{Z}_q \times \frac{d}{q}\mathcal{Z}_q$ (where $z, w \in \frac{d}{q}\mathcal{Z}_q$) is related to common divisor between two or more integers and $L(0,0)$ corresponds to the only single line in finite geometry $\mathcal{G}_1$ which contains only one point $(0,0)$.

If a subline is common to many lines, it is similar to common divisor of many integers.

The set of all lines denoted by $\mathbf{L}_d$ through the origin in $\mathcal{G}_d$ is a partially ordered set with subline as partial order. Its cardinality is,

$$|\mathbf{L}_d| = \sum_{\mathbf{w} \in \mathcal{D}(d)} \psi(\mathbf{w}) \tag{4.23}$$

## 4.5 Symplectic $Sp(2, \mathcal{Z}_d)$ group on $\mathcal{G}_d$

Suppose we define the matrices

$$\mathcal{M}(z, w|y, x) \equiv \begin{pmatrix} z & w \\ y & x \end{pmatrix} \tag{4.24}$$

where $\det(\mathcal{M}) = zx - wy = 1(mod\ (d)); z, w, y, x \in \mathcal{Z}_d$.

$\mathcal{M}$ form a group called symplectic $Sp(2, \mathcal{Z}_d)$ group.

If we act $\mathcal{M}(z, w|y, x)$ on all points of line $L(a, b)$ in $\mathcal{Z}_d \times \mathcal{Z}_d$ where $d$ is prime, this produces all the points of the line $L(za + wb, ya + xb)$. We write it as $\mathcal{M}(z, w|y, x)L(a, b)$.

Example: suppose $d = 3$, if we substitute $z = 2, w = 1, y = 1, x = 1$ into $\mathcal{M}$ then act it on $L(a, b)$ where $a = 0$ and $b \in \mathcal{Z}_3$, this yields all the points in line $L(1, 1)$

Thus;

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a + b \end{pmatrix}$$

where

$$L(1, 1) \text{ in } \mathcal{G}_3 = \{(0, 0)(1, 1)(2, 2)\}. \tag{4.25}$$

In our work, we define $\mathcal{M}(z, w|y, x) \equiv \mathcal{M}(0, 1| - 1, \Theta), \Theta = -1, ..., d - 1$ such that for $d$ a prime, acting $\mathcal{M}(0, 1| - 1, \Theta)$ on the line $L(0, 1)$, we obtain all the lines (maximal lines) through the origin. Here, we label the lines as

$$\Theta = -1 \rightarrow \Gamma(-1) = L(0, 1). \tag{4.26}$$

$$\Theta = 0, ..., d-1 \rightarrow \Gamma(\Theta) = \mathcal{M}(0,1|-1,\Theta)L(0,1) = L(1,\Theta). \tag{4.27}$$

In this work, we take the rule that if $\Theta = -1$, then $\mathcal{M}(0,1|-1,\Theta)$ is substituted by $\mathbf{1}$.

### 4.5.1 Partial ordering of the finite geometry

In this subsection, we illustrate how smaller geometries are enclosed in larger geometry via partial ordering. Here, we use the symplectic matrix defined in eq. (4.24) to factorize lines in finite geometry $\mathcal{G}_d$ in terms of prime factor lines using eqs.(3.52) and (3.53). Thus, $Sp(2, \mathcal{Z}_d)$ is factorized as $Sp(2, \mathcal{Z}_{\mathbf{p_1}}) \times ... \times Sp(2, \mathcal{Z}_{\mathbf{p_k}})$,
where

$$\mathcal{M}(z, w|y, x) = \bigotimes_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{M}(z_{\mathbf{j}}, w_{\mathbf{j}}\mathbf{r_j}|\bar{y}_{\mathbf{j}}, x_{\mathbf{j}}). \tag{4.28}$$

Note that; $z_{\mathbf{j}}, w_{\mathbf{j}}, x_{\mathbf{j}}$ are linked to $z, w, x$ which is related to $\beta_{\mathbf{j}}$ in eq.(3.52) and $\bar{y}_{\mathbf{j}}$ is linked to $y$ which is related to $\bar{\beta}_{\mathbf{j}}$ in eq.(3.53).
Example; $d = 42 \equiv 2 \times 3 \times 7$; suppose $z = 17, w = 12, y = 7, x = 5$
then $\mathcal{M}(17, 12|7, 5)$ is factorized in terms of eq.(4.28) using eqs.(3.52) and (3.53) as;

$$\mathcal{M}(17, 12|7, 5) = \mathcal{M}(1, 0|1, 1) \bigotimes \mathcal{M}(2, 0|2, 2) \bigotimes \mathcal{M}(3, 2|0, 5) \tag{4.29}$$

As a result, we define

$$\mathcal{M}(0,1|-1,\Theta) = \bigotimes_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{M}(0,\mathbf{r_j}|-\mathbf{t_j},\Theta_\mathbf{j}) \tag{4.30}$$

where $\Theta_\mathbf{j} = \Theta(mod\ \mathbf{p_j})$.

Example: suppose $d = 42, \mathbf{p_1} = 2, \mathbf{p_2} = 3, \mathbf{p_3} = 7, \mathbf{t_1} = 1, \mathbf{t_2} = 2, \mathbf{t}_3 = 6, \mathbf{r_1} = 21, \mathbf{r_2} = 14$, and $\mathbf{r_3} = 6$, then

$$\mathcal{M}(0,1|-1,\Theta) = \bigotimes_{\mathbf{j=1}}^{3} \mathcal{M}(0,\mathbf{r_j}|-\mathbf{t_j},\Theta_\mathbf{j}) \tag{4.31}$$

and $\mathcal{M}(0,1|-1,\Theta) = \mathcal{M}(0,21|-1,\Theta_1) \bigotimes \mathcal{M}(0,14|-2,\Theta_2) \bigotimes \mathcal{M}(0,6|-6,\Theta_3)$.

The set $\{\mathbf{G}_d\}$ of subgeometries of $\mathcal{G}_d$ with the partial order subgeometry is isomorphic to partially ordered set $\{\mathcal{D}(d)\}$.

Here, we define the eq.(4.19) in terms of eqs.(3.52) and (3.53) thereafter, we define the new labelling method which we derived in this section. This is important due to the fact that it relates to the notation in the weak mutually unbiased bases which we discussed later in this chapter. It demonstrates the concept of isomorphism between partial ordered relation that exists in finite geometries and partial ordered relation in finite quantum systems.

In the following lemma, we express eq.(4.19) in terms of eqs. (4.26) and (4.27) using the new labelling method.

**Proposition 1.** *Suppose we consider $\mathcal{G}_d$, its maximal lines are expressed thus:*

*(i)* $\forall\ a_{\mathbf{j}} \neq 0$; *then*

$$L(a,b) = L(1, a_1^{-1}\bar{b}_1) \times ... \times L(1, a_{\mathbf{k}}^{-1}\bar{b}_{\mathbf{k}}) \tag{4.32}$$

$$= \Gamma(\Theta_1) \times ... \times \Gamma(\Theta_{\mathbf{k}}) = \Gamma(\Theta_1, ..., \Theta_{\mathbf{k}}). \tag{4.33}$$

$$\Theta_{\mathbf{j}} = a_{\mathbf{j}}^{-1}\bar{b}_{\mathbf{j}}, \quad a_{\mathbf{j}}, \bar{b}_{\mathbf{j}} \in \mathcal{Z}_{p_{\mathbf{j}}}, \mathbf{j} = 1, .., \mathbf{k} \tag{4.34}$$

*(ii) Suppose* $a_{\mathbf{j}} = 0$ , $L(0, \bar{b}_{\mathbf{j}}) = L(0,1) = \Gamma(-1)$ *and hence* $\Theta_{\mathbf{j}} = -1$ *in eq.*(4.34).

*Proof.* Suppose $a_{\mathbf{j}} \neq 0$, the line $L(a_{\mathbf{j}}, \bar{b}_{\mathbf{j}})$ is expressed as $L(1, a_{\mathbf{j}}^{-1}\bar{b}_{\mathbf{j}})$. That is $a$ belongs to the field of integer $\mathcal{Z}_d$ since every element in $\mathcal{Z}_d$ apart from 0 has an inverse. Then eq.(4.19) can be expressed in the form eq.(4.32).
However for $a_{\mathbf{j}} = 0$, $L(0, \bar{b}_{\mathbf{j}}) = L(0,1) = \Gamma(-1)$

$$L(0,1) = \Gamma(-1, ..., -1) \tag{4.35}$$

□

More examples are provided later in this chapter in tabular form. We use the notations $L(a,b)$ and $\Gamma(\Theta_1, ..., \Theta_{\mathbf{k}})$ to show a duality between lines in $\mathcal{G}_d$ and weak mutually unbiased bases in $\mathcal{H}_d$.
Suppose we define two maximal lines,

$$\begin{aligned} L(a_1, \bar{b}_1) \times ... \times L(a_{\mathbf{k}}, \bar{b}_{\mathbf{k}}) &= \Gamma(\Theta_1, ..., \Theta_{\mathbf{k}}) \\ L(a'_1, \bar{b}'_1) \times ... \times L(a'_{\mathbf{k}}, \bar{b}'_{\mathbf{k}}) &= \Gamma(\Theta'_1, ..., \Theta'_{\mathbf{k}}) \end{aligned} \tag{4.36}$$

Let $\mathcal{I}_1 \subseteq \mathcal{I}$, where $\mathcal{I} = \{1, 2, ..., \mathcal{N}\}$ as set of indices, such that $\Theta = \Theta'_{\mathbf{j}}\ \forall\ \mathbf{j} \in \mathcal{I}_1$.

Then the line $L(\mathbf{r_1}, \bar{\mathbf{s}}_1) \times ... \times L(\mathbf{r_k}, \bar{\mathbf{s}}_\mathbf{k})$ with

$$\mathbf{r_j} = a_\mathbf{j}; \quad \bar{\mathbf{s}}_\mathbf{j} = \bar{b}_\mathbf{j}; \quad \text{if } \mathbf{j} \in \mathcal{I}_1$$
$$\mathbf{r_j} = \bar{\mathbf{s}}_\mathbf{j} = 0; \quad \text{if } \mathbf{j} \in \mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1 \tag{4.37}$$

is a subline that are present in the two lines eq.(4.36).

In this chapter, we express all maximal lines in $\mathcal{G}_{42}$ in terms of prime factor lines discussed earlier in eq.(4.33) as follows. As an illustration, factorizing line $L(1, 2)$ in eq.(4.22) further using the new notation we have,

$$L(1, 2) = L(1, 0) \times L(1, 1) \times L(1, 5) \equiv \Gamma(0, 1, 5). \tag{4.38}$$

The following propositions describe the concept of partial ordered relation between smaller geometries and bigger geometries and embedding of smaller geometries into bigger geometries.

**Proposition 2.** *(i) Suppose we define* $\mathcal{I} = \{1, ..., \mathcal{N}\}$ *as set of indices, where* $\mathcal{I}_1 \subseteq \mathcal{I}$ *and* $\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1$. *Suppose we express d as the product of its prime,* $\mathbf{p_1} \times ... \times \mathbf{p_k}$*, and*

$$q = \prod_{\mathbf{j} \in \mathcal{I}_1} \mathbf{p_j}; \ \frac{d}{q} = \prod_{\mathbf{j} \in \mathcal{I}_2} \mathbf{p_j}, \tag{4.39}$$

*From eq.(4.19) if* $L(a_\mathbf{j}, \bar{b}_\mathbf{j}) = L(0, 0)\ \forall \mathbf{j} \in \mathcal{I}_1$, *this implies that* $L(a, b)$ *is a subline in the subgeometry* $\mathcal{G}_{\frac{d}{q}}$

*(ii) There exists $\psi(q)$ maximal lines in geometry $\mathcal{G}_d$ each with $(\frac{d}{q})$ points in common which make a subline.*

*Proof.* (i) Recall eqs.(3.52) and (3.53), suppose $a_{\mathbf{j}} = 0, \forall\ \mathbf{j}\ \in \mathcal{I}_1$ then $a \in q\mathcal{Z}_{\frac{d}{q}}$. In converse, suppose $a \in q\mathcal{Z}_{\frac{d}{q}}$, then $a_{\mathbf{j}} = \bar{a}_{\mathbf{j}} = 0$. Since $L(a_{\mathbf{j}}, \bar{b}_{\mathbf{j}}) = L(0,0)\ \forall \mathbf{j} \in \mathcal{I}_1$, this means that $a_{\mathbf{j}} = \bar{a}_{\mathbf{j}} = 0$. Hence the points of the line $L(a,b)$ belong to $q\mathcal{Z}_{\frac{d}{q}} \times q\mathcal{Z}_{\frac{d}{q}}$. That is, the line $L(a,b)$ is a subline in the subgeometry $\mathcal{G}_{\frac{d}{q}}$.
(ii) We recall that the line $\Gamma(\Theta_1, ..., \Theta_{\mathbf{k}})$, have all $\Theta_{\mathbf{j}}$ in common with $\mathbf{j} \in \mathcal{I}_2$. This is not the same in $\Theta_{\mathbf{i}}$ with $\mathbf{i} \in \mathcal{I}_1$.
There exists

$$\prod_{\mathbf{j}\in\mathcal{I}_1} (\mathbf{p_j} + 1) = \psi(q) \tag{4.40}$$

of such lines. As discussed initially, to get the common sublines, we substitute the line $L(a_{\mathbf{j}}, \bar{b}_{\mathbf{j}})$ with $L(0,0)\ \ \forall\ \mathbf{j} \in \mathcal{I}_1$. Hence, the common subline has

$$\prod_{\mathbf{j}\in\mathcal{I}_2} \mathbf{p_j} = \frac{d}{q} \tag{4.41}$$

points. □

The table below shows the detail of the $\psi(d)$ maximal lines in geometry $\mathcal{G}_d$ for $d = 42$ using two different notations $L(a,b)$ and $\Gamma(\Theta_1, ..., \Theta_{\mathbf{k}})$ for the natural notations of lines and its duality with weak mutually unbiased bases respectively.

### 4.5.2 Example

Suppose $d = 42, \mathbf{p_1} = 2, \mathbf{p_2} = 3, \mathbf{p_3} = 7, \mathbf{t_1} = 1, \mathbf{t_2} = 2, \mathbf{t_3} = 6, \mathbf{r_1} = 21, \mathbf{r_2} = 14$, and $\mathbf{r_3} = 6$.

Table 4.1: Maximal lines in finite geometry $\mathcal{G}_{42}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{42}$ | $L(0,1) = \Gamma(-1,-1,-1)$, | $L(1,0) = \Gamma(0,0,0)$, | $L(1,1) = \Gamma(1,2,6)$, |
| | $L(1,2) = \Gamma(0,1,5)$, | $L(1,3) = \Gamma(1,0,4)$, | $L(1,4) = \Gamma(0,2,3)$, |
| | $L(1,5) = \Gamma(1,1,2)$, | $L(1,6) = \Gamma(0,0,1)$, | $L(1,7) = \Gamma(1,2,0)$, |
| | $L(1,8) = \Gamma(0,1,6)$, | $L(1,9) = \Gamma(1,0,5)$, | $L(1,10) = \Gamma(0,2,4)$, |
| | $L(1,11) = \Gamma(1,1,3)$, | $L(1,12) = \Gamma(0,0,2)$, | $L(1,13) = \Gamma(1,2,1)$, |
| | $L(1,14) = \Gamma(0,1,0)$, | $L(1,15) = \Gamma(1,0,6)$, | $L(1,16) = \Gamma(0,2,5)$, |
| | $L(1,17) = \Gamma(1,1,4)$, | $L(1,18) = \Gamma(0,0,3)$, | $L(1,19) = \Gamma(1,2,2)$, |
| | $L(1,20) = \Gamma(0,1,1)$, | $L(1,21) = \Gamma(1,0,0)$, | $L(1,22) = \Gamma(0,2,6)$, |
| | $L(1,23) = \Gamma(1,1,5)$, | $L(1,24) = \Gamma(0,0,4)$, | $L(1,25) = \Gamma(1,2,3)$, |
| | $L(1,26) = \Gamma(0,1,2)$, | $L(1,27) = \Gamma(1,0,1)$, | $L(1,28) = \Gamma(0,2,0)$, |
| | $L(1,29) = \Gamma(1,1,6)$, | $L(1,30) = \Gamma(0,0,5)$, | $L(1,31) = \Gamma(1,2,4)$, |
| | $L(1,32) = \Gamma(0,1,3)$, | $L(1,33) = \Gamma(1,0,2)$, | $L(1,34) = \Gamma(0,2,1)$, |
| | $L(1,35) = \Gamma(1,1,0)$, | $L(1,36) = \Gamma(0,0,6)$, | $L(1,37) = \Gamma(1,2,5)$, |
| | $L(1,38) = \Gamma(0,1,4)$, | $L(1,39) = \Gamma(1,0,3)$, | $L(1,40) = \Gamma(0,2,2)$, |
| | $L(1,41) = \Gamma(1,1,1)$, | $L(2,1) = \Gamma(-1,1,3)$, | $L(2,3) = \Gamma(-1,0,2)$, |
| | $L(2,5) = \Gamma(-1,2,1)$, | $L(2,7) = \Gamma(-1,1,0)$, | $L(2,9) = \Gamma(-1,0,6)$, |
| | $L(2,11) = \Gamma(-1,2,5)$, | $L(2,13) = \Gamma(-1,1,4)$, | $L(2,15) = \Gamma(-1,0,3)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{42}$ | $L(2,17) = \Gamma(-1,2,2)$, | $L(2,19) = \Gamma(-1,1,1)$, | $L(2,21) = \Gamma(-1,0,0)$, |
| | $L(2,23) = \Gamma(-1,2,6)$, | $L(2,25) = \Gamma(-1,1,5)$, | $L(2,27) = \Gamma(-1,0,4)$, |
| | $L(2,29) = \Gamma(-1,2,3)$, | $L(2,31) = \Gamma(-1,1,2)$, | $L(2,33) = \Gamma(-1,0,1)$, |
| | $L(2,35) = \Gamma(-1,2,0)$, | $L(2,37) = \Gamma(-1,1,6)$, | $L(2,39) = \Gamma(-1,0,5)$, |
| | $L(2,41) = \Gamma(-1,2,4)$, | $L(3,1) = \Gamma(1,-1,2)$, | $L(3,2) = \Gamma(0,-1,4)$, |
| | $L(3,4) = \Gamma(0,-1,1)$, | $L(3,5) = \Gamma(1,-1,3)$, | $L(3,7) = \Gamma(1,-1,0)$, |
| | $L(3,8) = \Gamma(0,-1,2)$, | $L(3,10) = \Gamma(0,-1,6)$, | $L(3,11) = \Gamma(1,-1,1)$ |
| | $L(3,13) = \Gamma(1,-1,5)$, | $L(3,14) = \Gamma(0,-1,0)$, | $L(3,17) = \Gamma(1,-1,6)$, |
| | $L(3,20) = \Gamma(0,-1,5)$, | $L(3,23) = \Gamma(1,-1,4)$, | $L(3,26) = \Gamma(0,-1,3)$, |
| | $L(6,1) = \Gamma(-1,-1,1)$, | $L(6,5) = \Gamma(-1,-1,5)$, | $L(6,7) = \Gamma(-1,-1,0)$, |
| | $L(6,11) = \Gamma(-1,-1,4)$, | $L(6,13) = \Gamma(-1,-1,6)$, | $L(6,17) = \Gamma(-1,-1,3)$, |
| | $L(6,23) = \Gamma(-1,-1,2)$, | $L(7,1) = \Gamma(1,2,-1)$, | $L(7,2) = \Gamma(0,1,-1)$, |
| | $L(7,3) = \Gamma(1,0,-1)$, | $L(7,4) = \Gamma(0,2,-1)$, | $L(7,5) = \Gamma(1,1,-1)$, |
| | $L(7,6) = \Gamma(0,0,-1)$, | $L(14,1) = \Gamma(-1,1,-1)$, | $L(14,3) = \Gamma(-1,0,-1)$, |
| | $L(14,5) = \Gamma(-1,2,-1)$, | $L(21,1) = \Gamma(1,-1,-1)$, | $L(21,2) = \Gamma(0,-1,-1)$. |

## 4.6 Partial ordering of the set of quantum systems with variables in $\mathcal{Z}_d$

In this section, we focus on finite quantum systems of dimension $d$ with positions and momenta in $\mathcal{Z}_d$ which we represent by $\Lambda(d)$. If $q|d$, $\mathcal{Z}_q$ is a subgroup of $\mathcal{Z}_d$. In this case we say that $\Lambda(q)$ is a subsystem of $\Lambda(d)$.

We define $|\mathbb{X}_d; m\rangle$ and $|\mathbb{P}_d; m\rangle$ to be position and momentum states, where $m \in \mathcal{Z}_d$, $\mathbb{X}_d$ and $\mathbb{P}_d$ represent position and momentum states respectively in a $d-$ dimensional Hilbert space $\mathcal{H}_d$. The Fourier transform is defined earlier in eq.(3.3). Likewise the momentum states is defined earlier in eq.(3.9).

### 4.6.1 Factorization of finite quantum systems as products of its subsystems

In this subsection, we express a system with variables in $\mathcal{Z}_d$ in terms of $\mathbf{k}$ subsystems. This concept was used by [25, 26] to decompose a finite quantum system $\Lambda(d)$ with variables in $\mathcal{Z}_d$ as products of systems $\Lambda(\mathbf{p_1}), .., \Lambda(\mathbf{p_k})$ with variables in $\mathcal{Z}_{\mathbf{p_k}}$ using eqs.(3.52) and (3.53). Here, we call it factorization of $d$ -dimensional system in terms of $\mathbf{k}$ subsystems. There exists a bijection between the Hilbert space $\mathcal{H}_d$ and tensor products $\bigotimes_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{H}_{\mathbf{p_j}}$, ($\mathbf{p_j}$ is a prime), where

$$|\mathbb{X}_d; m\rangle \longleftrightarrow |\mathbb{X}_{\mathbf{p_1}}; \bar{m}_1\rangle \otimes ... \otimes |\mathbb{X}_{\mathbf{p_k}}; \bar{m}_{\mathbf{k}}\rangle \tag{4.42}$$

and

$$|\mathbb{P}_d; m\rangle \longleftrightarrow |\mathbb{P}_{\mathbf{p_1}}; m_1\rangle \otimes ... \otimes |\mathbb{P}_{\mathbf{p_k}}; m_{\mathbf{k}}\rangle \tag{4.43}$$

As an illustration, we factorize a finite Hilbert space of dimension $\mathcal{H}_{42}$ as products of spaces $\mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_7$ thus using eqs.(3.52) and (3.53) for case $d = 42$, the first bijection is, $5 \longleftrightarrow (1, 2, 5)$ and the second bijection is $5 \longleftrightarrow (1, 1, 2)$.

Therefore; position basis in $\mathcal{H}_{42}$ is factorized as;

$$|\mathbb{X}_{42}; 5\rangle \longleftrightarrow |\mathbb{X}_2; 1\rangle \otimes |\mathbb{X}_3; 1\rangle \otimes |\mathbb{X}_7; 2\rangle \tag{4.44}$$

The same analogy is done for momentum basis thus;

$$|\mathbb{P}_{42};5\rangle \longleftrightarrow |\mathbb{P}_2;1\rangle \otimes |\mathbb{P}_3;2\rangle \otimes |\mathbb{P}_7;5\rangle \tag{4.45}$$

### 4.6.2 Embedding of small systems into large systems

In this subsection, we discuss embedding of a small system into a large system. We define a $d-$ dimensional finite quantum system $\Lambda(d)$. Also, we consider an orthonormal basis $|\mathbb{X}_d;m\rangle$ where $m \in \mathcal{Z}_d$.
If $q|d$ then $\mathcal{Z}_q \subset \mathcal{Z}_d$, this also means that $\Lambda(q)$ is a subsystem of $\Lambda(d)$.
Suppose we define a quantum subsystem $\Lambda(q)$ embedded into $\Lambda(d)$, we define a $1-1$ map with respect to position states as;

$$\sum_{m=0}^{q-1} \mathcal{S}_m \Big|\mathbb{X}_q;m\rangle \to \sum_{m=0}^{q-1} \mathcal{S}_m \Big|\mathbb{X}_d;\frac{dm}{q}\Big\rangle \tag{4.46}$$

Example: for $d=6, q=2,3$; the subgroup of $\mathcal{Z}_6$ are

$$\mathcal{Z}_2 = \{0,1\} \text{ and } \mathcal{Z}_3 = \{0,1,2\}. \tag{4.47}$$

The finite quantum system $\Lambda(6)$ is expressed below as,

$$\Lambda(6) = |\mathbb{X}_6;m\rangle = \{|\mathbb{X}_6;0\rangle, |\mathbb{X}_6;1\rangle, |\mathbb{X}_6;2\rangle, |\mathbb{X}_6;3\rangle, |\mathbb{X}_6;4\rangle, |\mathbb{X}_6;5\rangle\}. \tag{4.48}$$

Its subsystems are

$$|\mathbb{X}_3; 2m\rangle = \{|\mathbb{X}_3; 0\rangle, |\mathbb{X}_3; 2\rangle, |\mathbb{X}_3; 4\rangle\}. \tag{4.49}$$

$$|\mathbb{X}_2; 3m\rangle = \{|\mathbb{X}_2; 0\rangle, |\mathbb{X}_2; 3\rangle\}. \tag{4.50}$$

This implies that the variables of $\Lambda(q)$ takes values from $\mathcal{Z}_q$ of $\mathcal{Z}_d$ of the variables $\Lambda(d)$.

It is observed above that eq.(4.49) is embedded in eq.(4.48).

In addition, there exists an injection between $\Lambda(3)$ and $\Lambda(6)$ and it means that the quantum states of $\Lambda(q)$ are embedded into $\Lambda(d)$ as shown below.

$$\begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{S}_0 \\ 0 \\ \mathcal{S}_1 \\ 0 \\ \mathcal{S}_2 \\ 0 \end{pmatrix} \tag{4.51}$$

The above relation in eq.(4.46) is expressed in terms of momentum states as;

$$\sum_{m=0}^{q-1} \mathcal{T}_m \Big|\mathbb{P}_q; m\Big\rangle \to \sum_{m=0}^{q-1} \mathcal{T}_m \Big|\mathbb{P}_d; \frac{dm}{q}\Big\rangle \tag{4.52}$$

for $d = 6$ and $q = 2$ the left hand side (LHS) of eq.(4.52) yields,

$$2^{-\frac{1}{2}} \begin{pmatrix} \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1\omega \end{pmatrix} \tag{4.53}$$

For the right hand side ($RHS$) of eq.(4.52) we have

$$\sum_{m=0}^{q-1} \mathcal{T}_m \left| \mathbb{P}_d ; \frac{dm}{q} \right\rangle = 6^{-\frac{1}{2}} \begin{pmatrix} \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \\ \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \\ \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \end{pmatrix} \tag{4.54}$$

There exists an injection between eqs.(4.53) and (4.54). This implies that eq.(4.53) is embedded in eq.(4.54) confirming eq.(4.52). That is

$$2^{-\frac{1}{2}} \begin{pmatrix} \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \end{pmatrix} \rightarrow 6^{-\frac{1}{2}} \begin{pmatrix} \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \\ \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \\ \mathcal{T}_0 + \mathcal{T}_1 \\ \mathcal{T}_0 + \mathcal{T}_1 \omega \end{pmatrix} \tag{4.55}$$

Hence, there exists a partial order subsystem in the set $\{\Upsilon(d)\}$ of all subsystems of $\Lambda(d)$. Likewise, there exists a partial order subspace in the set $\{\mathbf{h}_d\}$ of their Hilbert space. This thereby strengthen more, the concept of duality between geometries and quantum systems.

## 4.7 Mutually unbiased bases

Over the years, the study of mutually unbiased bases has received great interest in physics and mathematics in relation to quantum information, entanglement, tomography and cryptography. This is due to its wide area of application. For example, in quantum cryptography, mutually unbiased bases play a significant role in the implementation of BB84 key distribution protocol [82-85].

We define $\{|\mathcal{B}(\Theta_{\mathbf{i}}); n\rangle\}$ as a set of $\mathbf{g}$ orthonormal bases in $\mathcal{H}_d$ where $n \in \mathcal{Z}_d$. Two orthonormal bases $|\mathcal{B}(\Theta_{\mathbf{i}}); n\rangle$ and $|\mathcal{B}(\Theta_{\mathbf{j}}); m\rangle$ are mutually unbiased if and only if

$$|\langle \mathcal{X}_{\Theta_{\mathbf{i}}}; n|\mathcal{X}_{\Theta_{\mathbf{j}}}; m\rangle| = \frac{1}{\sqrt{d}}\ , \ \forall |\mathcal{X}_{\Theta_{\mathbf{i}}}; n\rangle \in |\mathcal{B}(\Theta_{\mathbf{i}}); n\rangle \text{ and } |\mathcal{X}_{\Theta_{\mathbf{j}}}; m\rangle \in |\mathcal{B}(\Theta_{\mathbf{j}}); m\rangle, \tag{4.56}$$

for $\Theta_{\mathbf{i}} \neq \Theta_{\mathbf{j}}$.

Due to a known disparity between a finite quantum system with odd and even dimension [37-39], we focus our work on finite systems with odd dimension only (that is, lack of inverse of integer 2 in even dimensional Hilbert space).

The displacement operators is defined earlier in eq.(3.18), it forms a representation of Heisenberg-Weyl group. Symplectic transformation has been studied in [35, 36]. It satisfies the conditions;

$$[\mathbf{m}(z, w|y, x)]X_d[\mathbf{m}(z, w|y, x)]^\dagger = D(w, z)$$

$$[\mathbf{m}(z, w|y, x)]Z_d[\mathbf{m}(z, w|y, x)]^\dagger = D(x, y)$$

$$\mathbf{m}(zx - wy) = 1(mod\ (d)), z, w, y, x \in \mathcal{Z}_d \tag{4.57}$$

Note that the expressions, $\mathcal{M}(zx - wy)$ defined in eq.(4.24) and $\mathbf{m}(zx - wy)$ in eq.(4.57) do not belong to the same representation. Hence in this work, we define the Fourier transform

$$\mathcal{F}_d = \mathbf{m}(0, 1| - 1, 0). \tag{4.58}$$

Mutually unbiased bases for finite quantum systems with prime dimension $\mathbf{p}$, is expressed in this work as follows;
Let

$$\begin{aligned} \Theta = -1 \to |\mathcal{X}_{-1}; m\rangle &= \mathbf{m}(1, 0|0, 1)|\mathbb{X}_{\mathbf{p}}; m\rangle \\ \Theta = 0, ..., \mathbf{p} - 1 \to |\mathcal{X}_{\Theta}; m\rangle &= \mathbf{m}(0, 1| - 1, \Theta)|\mathbb{X}_{\mathbf{p}}; m\rangle \end{aligned} \tag{4.59}$$

for $\Theta = 0, |\mathcal{X}_0; m\rangle = |\mathbb{P}_{\mathbf{p}}; m\rangle$.
If we take the absolute value of the overlap of any two states each from different bases in eq.(4.59). The result satisfies eq.(4.56).
In this case, there exists $\psi(\mathbf{p})$ mutually unbiased bases. We label the bases in a way similar to that of lines in eq.(4.33) that is

$$\mathcal{B}(\Theta) = \{|\mathcal{X}_{\Theta}; m\rangle\}; \quad \Theta = -1, ..., \mathbf{p} - 1 \tag{4.60}$$

In the example below, we demonstrate how to construct mutually unbiased bases for prime dimension, $\mathbf{p} = 3$.

We define the standard bases labelled here as

$$|\mathcal{B}(-1); m\rangle = \{|\mathcal{X}_{-1}(1,0|0,1); m\rangle\}, m \in \mathcal{Z}_3 \tag{4.61}$$

Note that $|\mathcal{X}_{-1}(1,0|0,1); m\rangle \equiv \mathbf{m}(1,0|0,1)|\mathbb{X}_3; m\rangle$.

The remaining bases are obtained by transforming $|\mathcal{X}_{-1}; m\rangle$ symplectically into new one thus;

$\mathbf{m}(0,1|-1,\Theta)|\mathbb{X}_3; m\rangle = |\mathcal{X}_\Theta(0,1|-1,\Theta); m\rangle$ where $\Theta = 0,..,\mathbf{p}-1$, $m = 0,...,\mathbf{p}-1$

$$\begin{aligned}
&\text{(i) For } \Theta = 0;\ |\mathcal{B}(0); m\rangle = \{|\mathcal{X}_0(0,1|-1,0); m\rangle\};\\
&\text{(ii) For } \Theta = 1;\ |\mathcal{B}(1); m\rangle = \{|\mathcal{X}_1(0,1|-1,1); m\rangle\};\\
&\text{(iii) For } \Theta = 2; |\mathcal{B}(2); m\rangle = \{|\mathcal{X}_2(0,1|-1,2); m\rangle\}.
\end{aligned} \tag{4.62}$$

If we take the absolute value of the inner product of any two states each belonging to different bases, the result gives $\mathbf{p}^{-\frac{1}{2}}$.

## 4.8 Duality between weak mutually unbiased bases in $\mathcal{H}_d$ and lines in $\mathcal{G}_d$

In this section, we discuss the concept of duality that we discover which exists between $\mathcal{H}_d$ and $\mathcal{G}_d$. We begin by explaining in brief weak mutually unbiased bases. The formalism was treated in detail in [23, 24]. It is a product of mutually unbiased bases in each of prime dimensional Hilbert space $\mathcal{H}_{\mathbf{p_k}}$. It is obtained by taking the tensor products of distinct states each belonging to different bases in $\mathcal{H}_d$.

### 4.8.1 Weak mutually unbiased bases ($\mathcal{WMUB}$)

We define a set $\{|\mathcal{B}(\mathbf{i}); n\rangle\}$ of $\mathbf{g}$ orthonormal bases in the Hilbert spaces $\mathcal{H}_d$ where $n \in \mathcal{Z}_d$ and $\mathbf{i} = 1, 2, ..., \mathbf{g}$. We consider the expression

$$g_{\mathbf{ji}}(n, m) = |\langle \mathcal{B}(\mathbf{j}); n|\mathcal{B}(\mathbf{i}); m\rangle|; \quad g_{\mathbf{ij}}(m, n) = g_{\mathbf{ji}}(n, m) \tag{4.63}$$

It is called a weak mutually unbiased bases if for any pair of distinct bases ($\mathbf{i} \neq \mathbf{j}$), one of the following conditions is satisfied:

$$g_{\mathbf{ji}}(n, m) = q^{-\frac{1}{2}} \text{ or } 0. \tag{4.64}$$

For $q|d$, any set of weak mutually unbiased bases in $\mathcal{H}_d$ can be expressed in the form $|\mathcal{X}_{\Theta_1}; \bar{m}_1\rangle \otimes ... \otimes |\mathcal{X}_{\Theta_\mathbf{k}}; \bar{m}_\mathbf{k}\rangle$ where $\{|\mathcal{X}_{\Theta_1}; \bar{m}_1\rangle\}$ is a set of mutually unbiased bases in Hilbert subspace $\mathcal{H}_{\mathbf{p_1}}, \{|\mathcal{X}_{\Theta_2}; \bar{m}_2\rangle\}$ is a set of mutually unbiased bases in Hilbert subspace $\mathcal{H}_{\mathbf{p_2}}, ..., \{|\mathcal{X}_{\Theta_\mathbf{k}}; \bar{m}_\mathbf{k}\}\rangle$ is a set of mutually unbiased bases in Hilbert subspace $\mathcal{H}_{\mathbf{p_k}}$,

The weak mutually unbiased bases is expressed in our work as

$$|\mathcal{X}_{\Theta_1,...,\Theta_\mathbf{k}}; \bar{m}_1, ..., \bar{m}_\mathbf{k}\rangle = |\mathcal{X}_{1,\Theta_1}; \bar{m}_1\rangle \otimes ... \otimes |\mathcal{X}_{\mathbf{k},\Theta_\mathbf{k}}; \bar{m}_\mathbf{k}\rangle \tag{4.65}$$

where $\bar{m}_\mathbf{j} \in \mathcal{Z}_{\mathbf{p_j}}$ and $-1 \leq \Theta_\mathbf{j} \leq \mathbf{p_j} - 1$.

In particular, suppose

$$\Theta_1 = ... = \Theta_\mathbf{k} = -1, \tag{4.66}$$

then

$$\begin{aligned}|\mathcal{X}_{-1,...,-1};\bar{m}_1,...,\bar{m}_{\mathbf{k}}\rangle &= |\mathcal{X}_{1,-1};\bar{m}_1\rangle\otimes...\otimes|\mathcal{X}_{\mathbf{k},-1};\bar{m}_{\mathbf{k}}\rangle\\ &= |\mathbb{X}_1;\bar{m}_1\rangle\otimes...\otimes|\mathbb{X}_{\mathbf{k}};\bar{m}_{\mathbf{k}}\rangle \qquad (4.67)\end{aligned}$$

Suppose

$$\Theta_1 = ... = \Theta_{\mathbf{k}} = 0, \qquad (4.68)$$

then

$$\begin{aligned}|\mathcal{X}_{0,...,0};\bar{m}_1,...,\bar{m}_{\mathbf{k}}\rangle &= |\mathcal{X}_{1,0};\bar{m}_1\rangle\otimes...\otimes|\mathcal{X}_{\mathbf{k},0};\bar{m}_{\mathbf{k}}\rangle\\ &= |\mathbb{P}_1;m_1\rangle\otimes...\otimes|\mathbb{P}_{\mathbf{k}};m_{\mathbf{k}}\rangle \qquad (4.69)\end{aligned}$$

taking the absolute value of the dot product of any two states each belonging to different bases in eqs.(4.67) and (4.69), it satisfies the relation;

$$|\langle\mathcal{X}_{\Theta_1,...,\Theta_{\mathbf{k}}};\bar{n}_1,...,\bar{n}_{\mathbf{k}}|\mathcal{X}_{\Theta_1,...,\Theta_{\mathbf{k}}};\bar{m}_1,...,\bar{m}_{\mathbf{k}}\rangle| = \frac{1}{\sqrt{\mathbf{w}}}\ \mathit{or}\ 0,\ \ \mathbf{w}|d. \qquad (4.70)$$

There exists

$$\psi(d) = (\mathbf{p_1}+1)(\mathbf{p_2}+1)...(\mathbf{p_k}+1) \qquad (4.71)$$

maximum number of weak unbiased bases in Hilbert space $\mathcal{H}_d$.

Below we label weak mutually unbiased bases in $\mathcal{H}_d$ in a format similar to that of lines in $\mathcal{G}_d$ discussed earlier. That is;

$$\mathcal{B}(\Theta_1, ..., \Theta_{\mathbf{k}}) = \{|\mathcal{X}_{\Theta_1,...,\Theta_{\mathbf{k}}}; \bar{m}_1, ..., \bar{m}_{\mathbf{k}}\rangle\} \tag{4.72}$$

There exists a duality between lines in eq.(4.33) and weak mutually unbiased bases: The duality between the lines in $\mathcal{G}_d$ which we factorized as prime factor lines and weak mutually unbiased bases in Hilbert space $\mathcal{H}_d$ expressed as products of mutually unbiased bases is obvious, having established a bijection between the two concepts.

Thus, the maximal lines in $\mathcal{G}_d$ corresponds to weak mutually unbiased bases in $\mathcal{H}_d$. The $\psi(d)$ maximal lines in $\mathcal{G}_d$ conforms to $\psi(d)$ weak mutually unbiased bases in $\mathcal{H}_d$.

Each maximal lines has $d$ points, also there are $d$ orthogonal vectors in each of $\mathcal{WMUB}$ in $\mathcal{H}_d$.

For $q|d$, the subgeometries $\mathcal{G}_q$ of $\mathcal{G}_d$ corresponds to the subsystems $\Lambda(q)$ of $\Lambda(d)$.

There are $\sigma_0(d)$ subgeometries $\mathcal{G}_q$ of $\mathcal{G}_d$ and likewise there are $\sigma_0(d)$ subsystems $\Lambda(q)$ of $\Lambda(d)$.

The weak mutually unbiased bases $|\mathcal{B}(\Theta_1, \Theta_2, \Theta_3); m\rangle$ in $\mathcal{H}_{42}$ and their factorization in terms of their mutually unbiased bases $\{|\mathcal{X}_{\Theta_1}; \bar{m}_1\rangle\}$ in $\mathcal{H}_2$, $\{|\mathcal{X}_{\Theta_2}; \bar{m}_2\rangle\}$ in $\mathcal{H}_3$, and $\{|\mathcal{X}_{\Theta_3}; \bar{m}_3\rangle\}$ in $\mathcal{H}_7$ are summarized in the table below.

Example: suppose $d = 42 = 2 \times 3 \times 7$. There are 96 maximal number of weak mutually unbiased bases altogether thus where $m \in \mathcal{Z}_d$, and $\bar{m}_{\mathbf{j}} \in \mathcal{Z}_{\mathbf{p_j}}$.

Table 4.3: Weak mutually unbiased bases for $\mathcal{H}_{42} = \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_7$ .

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(-1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,5);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(0,1,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,3);m\rangle$ | $\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,0);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,0);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\lvert\mathcal{B}(1,1,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert -1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert -1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert -1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert -1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert -1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert -1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert -1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert -1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert -1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert -1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert -1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert -1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert -1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,0);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert -1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,1);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert -1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,2);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert -1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,3);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert -1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,4);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert -1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,5);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert -1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,6);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert -1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert -1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,1,0);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert -1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,1,1);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert -1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,1,2);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert -1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,1,3);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert -1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,1,4);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert -1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert -1,4);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(-1,1,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,5);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,6);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6;\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(0,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |

**Proposition 3.** *There exists a duality between maximal lines in $\mathcal{G}_d$ and weak mutually unbiased bases ($\mathcal{WMUB}$) in Hilbert space $\mathcal{H}_d$:*

$$\Gamma(\Theta_1,...,\Theta_{\mathbf{k}}) \longleftrightarrow \mathcal{B}(\Theta_1,...,\Theta_{\mathbf{k}}) \tag{4.73}$$

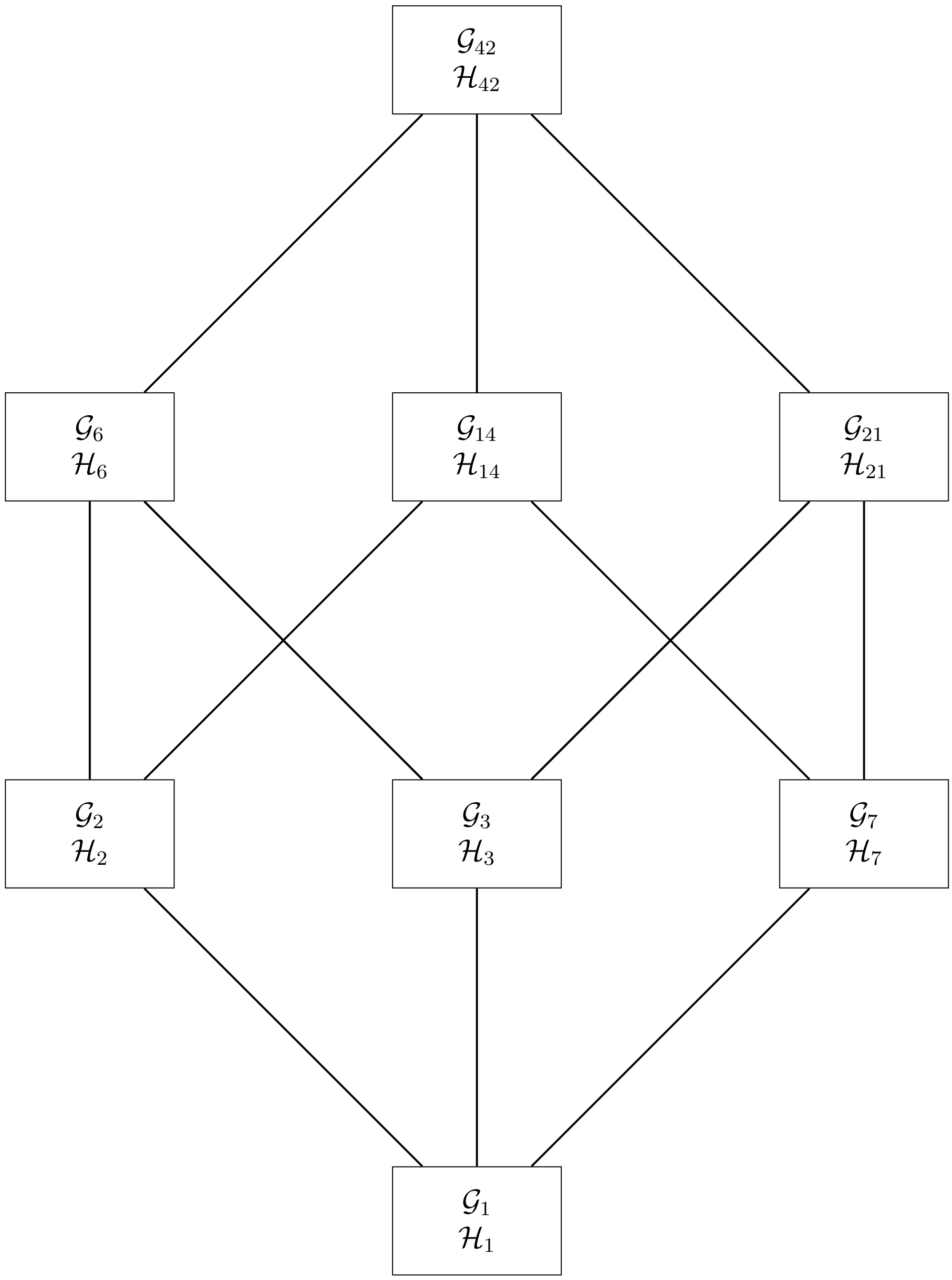
*Proof.* From the labelling system it is obvious that each maximal lines in $\mathcal{G}_d$ has a unique image in weak mutually unbiased bases in a finite dimensional Hilbert space $\mathcal{H}_d$ and vice versa. Since for every maximal line in $\mathcal{G}_d$ there exists $d-$ points, this corresponds to $d-$ orthogonal vectors in each $\mathcal{WMUB}$ in Hilbert space $\mathcal{H}_d$.

The small geometries that is the subgeometries $\mathcal{G}_q$ of $\mathcal{G}_d$ corresponds to the subsystems $\Lambda(q)$ of a quantum system $\Lambda(d)$. Since there exists $\sigma_0(d)$ divisors of $d$. Hence there are $\sigma_0(d)$ subgeometries of $\mathcal{G}_d$. Likewise, there are $\sigma_0(d)$ subsystems of $\Lambda(d)$. □

Table 4.8: The table below shows in brief the relation between the results of our work.

| Finite geometry, $\mathcal{G}_d$ | Finite dimensional Hilbert space, $\mathcal{H}_d$ |
|---|---|
| $\Gamma(\Theta_1, ..., \Theta_{\mathbf{k}})$ (Equation (4.33))<br>$\psi(d)$ maximal lines | $\mathcal{B}(\Theta_1, ..., \Theta_{\mathbf{k}})$(Equation (4.72))<br>$\psi(d)(\mathcal{WMUB})$ |
| $d$ points<br>in each maximal line | $d$ orthogonal vectors<br>in each ($\mathcal{WMUB}$) |
| subgeometries $\mathcal{G}_q$ $(q\|d)$ | subsytems $\Lambda(q)$ with Hilbert space $\mathcal{H}_q$ $(q\|d)$ |

Figure 4.1: The Hasse diagram showing the geometry $\mathcal{G}_{42}$ and its subgeometries, and along with Hilbert spaces $\mathcal{H}_{42}$ of the subsystems of $\Lambda(42)$



## 4.9 Conclusion

In this chapter, we discuss the existence of partial ordered relation between:

(i) subgeometry in the set $\{\mathbf{G}_d\}$ of all subgeometries of $\mathcal{G}_d$.

(ii) subsystem in the set $\{\Upsilon(d)\}$ of all subsystems of $\Lambda(d)$.

(iii) subspace in the set $\{\mathbf{h}_d\}$ of all subspace of $\mathcal{H}_d$.

Here, each maximal lines in $\mathcal{G}_d$ were expressed as products of their prime factor lines. Likewise each weak mutually unbiased bases in $\mathcal{H}_d$ were expressed as products of mutually unbiased bases in $\bigotimes_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{H}_{\mathbf{p_j}}$.This concept is similar to factorization of integers as product of their prime factors. Thereafter, we show a duality between lines in $\mathcal{G}_d$ and weak mutually unbiased bases in $\mathcal{H}_d$.

# Chapter 5

# Numerical Examples

## 5.1 Introduction

In this chapter, we demostrate how a small geometry is embedded in large geometry via partial ordering. Also, we establish a bijection between subgeometries of a finite geometry and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$. Lines with $d$ points are called maximal lines (lines). Furthermore, lines with $q \leq d$ points are called sublines. Here, we demonstrate partial ordered relation between $\mathcal{G}_{210}$ and its subgeometries as follows.

A finite geometry $\mathcal{G}_d$ and a line through the origin in $\mathcal{G}_d$ are defined earlier in eqs.(4.3) and (4.2) respectively.

From eq.(4.2), we confirm the following results in $\mathcal{G}_{210}$:

(i) There exists $\psi(d)$ maximal line in finite geometry $\mathcal{G}_d$.

Example: $\mathcal{G}_{210}$ has 576 maximal lines.

(ii) Suppose $b \in \mathcal{Z}_d^*$, then $L(bx, by) = L(x, y)$.

Example: $11 \in \mathcal{Z}^*_{210}$, therefore $L(1,7) = L(11,77)$ in $\mathcal{G}_{210}$.

However, if $b \in \mathcal{Z}_d - \mathcal{Z}^*_d$ , then $L(bx, by) \subset L(x,y)$ this implies $L(bx, by) \prec L(x,y)$.

Example: $3 \in \mathcal{Z}_{210} - \mathcal{Z}^*_{210}$, it implies $L(3,21) \prec L(1,7)$ in $\mathcal{G}_{210}$. Hence from $\mathcal{G}_{210}$ we deduce the following partial ordered relation between line $L(1,1)$ and its sublines with subline as partial order:

(a) $L(105,105) \prec L(35,35) \prec L(5,5) \prec L(1,1)$

(b) $L(105,105) \prec L(35,35) \prec L7,7) \prec L(1,1)$

(c) $L(105,105) \prec L(21,21) \prec L(3,3) \prec L(1,1)$

(d) $L(105,105) \prec L(21,21) \prec L(7,7) \prec L(1,1)$

(e) $L(105,105) \prec L(15,15) \prec L(3,3) \prec L(1,1)$

(f) $L(105,105) \prec L(15,15) \prec L(7,7) \prec L(1,1)$

(g) $L(70,70) \prec L(35,35) \prec L(5,5) \prec L(1,1)$

(h) $L(70,70) \prec L(35,35) \prec L(7,7) \prec L(1,1)$

(i) $L(70,70) \prec L(14,14) \prec L(2,2) \prec L(1,1)$

(j) $L(70,70) \prec L(14,14) \prec L(7,7) \prec L(1,1)$

(k) $L(70,70) \prec L(10,10) \prec L(2,2) \prec L(1,1)$

(l) $L(70,70) \prec L(10,10) \prec L(5,5) \prec L(1,1)$

(m) $L(42,42) \prec L(21,21) \prec L(3,3) \prec L(1,1)$

(n) $L(42, 42) \prec L(21, 21) \prec L(7, 7) \prec L(1, 1)$

(o) $L(42, 42) \prec L(14, 14) \prec L(2, 2) \prec L(1, 1)$

(p) $L(42, 42) \prec L(14, 14) \prec L(7, 7) \prec L(1, 1)$

(q) $L(42, 42) \prec L(6, 6) \prec L(2, 2) \prec L(1, 1)$

(r) $L(42, 42) \prec L(6, 6) \prec L(3, 3) \prec L(1, 1)$

(s) $L(30, 30) \prec L(15, 15) \prec L(3, 3) \prec L(1, 1)$

(t) $L(30, 30) \prec L(15, 15) \prec L(5, 5) \prec L(1, 1)$

(u) $L(30, 30) \prec L(10, 10) \prec L(2, 2) \prec L(1, 1)$

(v) $L(30, 30) \prec L(10, 10) \prec L(5, 5) \prec L(1, 1)$

(w) $L(30, 30) \prec L(6, 6) \prec L(2, 2) \prec L(1, 1)$

(x) $L(30, 30) \prec L(6, 6) \prec L(3, 3) \prec L(1, 1)$

(iii) The maximal line, $L(x, y)$ in $\mathcal{G}_d = L(\mathbf{w}x, \mathbf{w}y)$ in $\mathcal{G}_{\mathbf{w}d}$. Also, $L(\mathbf{w}x, \mathbf{w}y)$ in $\mathcal{G}_{\mathbf{w}d}$ is a subline of $L(x, y)$ in a larger geometry $\mathcal{G}_d$.

Examples:

(1) the maximal line $L(1, 1) = \{(0, 0)(1, 1)\}$ in $\mathcal{G}_2$ is the same as the following sublines:

(a) $L(3, 3) = \{(0, 0)(3, 3)\}$ in $\mathcal{G}_6$,

(b) $L(5, 5) = \{(0, 0)(5, 5)\}$ in $\mathcal{G}_{10}$,

(c) $L(7, 7) = \{(0, 0)(7, 7)\}$ in $\mathcal{G}_{14}$,

(d) $L(15,15) = \{(0,0)(15,15)\}$ in $\mathcal{G}_{30}$,

(e) $L(21,21) = \{(0,0)(21,21)\}$ in $\mathcal{G}_{42}$,

(f) $L(35,35) = \{(0,0)(35,35)\}$ in $\mathcal{G}_{70}$,

(g) $L(105,105) = \{(0,0)(105,105)\}$ in $\mathcal{G}_{210}$ and

finally, (a)- (g) are sublines of maximal line $L(1,1)$ in $\mathcal{G}_{210}$.

(2) The maximal line $L(1,1)$ in the subgeometry $\mathcal{G}_2$ is the subline $L(3,3)$ of $L(1,1)$ in $\mathcal{G}_6$, where in $\mathcal{G}_6$ :

(a) $L(1,1) = \{(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)\}$. Therefore $L(3,3) \prec L(1,1)$, this implies $\mathcal{G}_2 \prec \mathcal{G}_6$.

(3) The maximal line $L(1,1)$ in small geometry $\mathcal{G}_6$, is the subline $L(7,7)$ of the maximal line $L(1,1)$ in the big geometry $\mathcal{G}_{42}$, where in $\mathcal{G}_{42}$:

(a) $L(1,1) = \{(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)(7,7)(8,8)(9,9)$
$(10,10)(11,11)(12,12)(13,13)(14,14)(15,15)(16,16)(17,17)(18,18)(19,19)$
$(20,20)(21,21)(22,22)(23,23)(24,24)(25,25)(26,26)(27,27)(28,28)(29,29)$
$(30,30)(31,31)(32,32)(33,33)(34,34)(35,35)(36,36)(37,37)(38,38)(39,39)$
$(40,40)(41,41)\}$.

(b) $L(7,7) = \{(0,0)(7,7)(14,14)(21,21)(28,28)(35,35)\}$ and

(c) $L(21,21) = \{(0,0)(21,21)\}$.
Hence $L(21,21) \prec L(7,7) \prec L(1,1)$ and $\mathcal{G}_2 \prec \mathcal{G}_6 \prec \mathcal{G}_{42}$. Also,

(4) The maximal line $L(1,1)$ in small geometry $\mathcal{G}_{42}$ is the subline $L(5,5)$ of the maximal line $L(1,1)$ in the big geometry $\mathcal{G}_{210}$ where:

(a) $L(1,1) = \{(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)(7,7)(8,8)(9,9)$

$(10,10)(11,11)(12,12)(13,13)(14,14)(15,15)(16,16)(17,17)(18,18)(19,19)$

$(20,20)(21,21)(22,22)(23,23)(24,24)(25,25)(26,26)(27,27)(28,28)(29,29)$

$(30,30)(31,31)(32,32)(33,33)(34,34)(35,35)(36,36)(37,37)(38,38)(39,39)$

$(40,40)(41,41)(42,42)(43,43)(44,44)(45,45)(46,46)(47,47)(48,48)(49,49)$

$(50,50)(51,51)(52,52)(53,53)(54,54)(55,55)(56,56)(57,57)(58,58)(59,59)$

$(60,60)(61,61)(62,62)(63,63)(64,64)(65,65)(66,66)(67,67)(68,68)(69,69)$

$(70,70)(71,71)(72,72)(73,73)(74,74)(75,75)(76,76)(77,77)(78,78)(79,79)$

$(80,80)(81,81)(82,82)(83,83)(84,84)(85,85)(86,86)(87,87)(88,88)(89,89)$

$(90,90)(91,91)(92,92)(93,93)(94,94)(95,95)(96,96)(97,97)(98,98)(99,99)$

$(100,100)(101,101)(102,102)(103,103)(104,104)(105,105)(106,106)$

$(107,107)(108,108)(109,109)(110,110)(111,111)(112,112)(113,113)$

$(114,114)(115,115)(116,116)(117,117)(118,118)(119,119)(120,120)$

$(121,121)(122,122)(123,123)(124,124)(125,125)(126,126)(127,127)$

$(128,128)(129,129)(130,130)(131,131)(132,132)(133,133)(134,134)$

$(135,135)(136,136)(137,137)(138,138)(139,139)(140,140)(141,141)$

$(142,142)(143,143)(144,144)(145,145)(146,146)(147,147)(148,148)$

$(149,149)(150,150)(151,151)(152,152)(153,153)(154,154)(155,155)$

$(156,156)(157,157)(158,158)(159,159)(160,160)(161,161)(162,162)$

$(163,163)(164,164)(165,165)(166,166)(167,167)(168,168)(169,169)$

$(170,170)(171,171)(172,172)(173,173)(174,174)(175,175)(176,176)$

$(177,177)(178,178)(179,179)(180,180)(181,181)(182,182)(183,183)$

$(184,184)(185,185)(186,186)(187,187)(188,188)(189,189)(190,190)$

$(191,191)(192,192)(193,193)(194,194)(195,195)(196,196)(197,197)$

$(198,198)(199,199)(200,200)(201,201)(202,202)(203,203)(204,204)$

$(205,205)(206,206)(207,207)(208,208)(209,209)\}$.

The line $L(1,1)$ in $\mathcal{G}_{70} \cong L(3,3)$ in $\mathcal{G}_{210}$ where

(b) $L(3,3) = \{(0,0)(3,3)(6,6)(9,9)(12,12)(15,15)(18,18)(21,21)(24,24)(27,27)$
$(30,30)(33,33)(36,36)(39,39)(42,42)(45,45)(48,48)(51,51)(54,54)(57,57)$
$(60,60)(63,63)(66,66)(69,69)(72,72)(75,75)(78,78)(81,81)(84,84)(87,87)$
$(90,90)(93,93)(96,96)(99,99)(102,102)(105,105)(108,108)(111,111)$
$(114,114)(117,117)(120,120)(123,123)(126,126)(129,129)(132,132)$
$(135,135)(138,138)(141,141)(144,144)(147,147)(150,150)(153,153)$
$(156,156)(159,159)(162,162)(165,165)(168,168)(171,171)(174,174)$
$(177,177)(180,180)(183,183)(186,186)(189,189)(192,192)(195,195)$
$(198,198)(201,201)(204,204)(207,207)\}$.

The line $L(1,1)$ in $\mathcal{G}_{42} \cong L(5,5)$ in $\mathcal{G}_{210}$ where

(c) $L(5,5) = \{(0,0)(5,5)(10,10)(15,15)(20,20)(25,25)(30,30)(35,35)(40,40)$
$(45,45)(50,50)(55,55)(60,60)(65,65)(70,70)(75,75)(80,80)(85,85)(90,90)$
$(95,95)(100,100)(105,105)(110,110)(115,115)(120,120)(125,125)(130,130)$
$(135,135)(140,140)(145,145)(150,150)(155,155)(160,160)(165,165)(170,170)$
$(175,175)(180,180)(185,185)(190,190)(195,195)(200,200)(205,205)\}$.

The line $L(1,1)$ in $\mathcal{G}_{30} \cong L(7,7)$ in $\mathcal{G}_{210}$ where

(d) $L(7,7) = \{(0,0)(7,7)(14,14)(21,21)(28,28)(35,35)(42,42)(49,49)(56,56)$
$(63,63)(70,70)(77,77)(84,84)(91,91)(98,98)(105,105)(112,112)(119,119)$
$(126,126)(133,133)(140,140)(147,147)(154,154)(161,161)(168,168)$
$(175,175)(182,182)(189,189)(196,196)(203,203)\}$.

The line $L(1,1)$ in $\mathcal{G}_{14} \cong L(15,15)$ in $\mathcal{G}_{210}$ where

(e) $L(15, 15) = \{(0, 0)(15, 15)(30, 30)(45, 45)(60, 60)(75, 75)(90, 90)(105, 105)$ (120,120)(135,135)(150,150)(165,165)(180,180)(195,195)$\}$.

The line $L(1, 1)$ in $\mathcal{G}_{10} \cong L(21, 21)$ in $\mathcal{G}_{210}$ where

(f) $L(21, 21) = \{(0, 0)(21, 21)(42, 42)(63, 63)(84, 84)(105, 105)(126, 126)(147, 147)$ (168,168)(189,189)$\}$.

The line $L(1, 1)$ in $\mathcal{G}_6 \cong L(35, 35)$ in $\mathcal{G}_{210}$ where

(g) $L(35, 35) = \{(0, 0)(35, 35)(70, 70)(105, 105)(140, 140)(175, 175)\}$.

The line $L(1, 1)$ in $\mathcal{G}_2 \cong L(105, 105)$ in $\mathcal{G}_{210}$ where

(h) $L(105, 105) = \{(0, 0)(105, 105)\}$.

Therefore from proposition (iii) above, we deduce the following from $\mathcal{G}_{210}$:

(a) $\mathcal{G}_2 \prec \mathcal{G}_6 \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1, 1) \cong L(3, 3) \cong L(21, 21) \cong L(105, 105) \prec L(1, 1)$

(b) $\mathcal{G}_2 \prec \mathcal{G}_6 \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1, 1) \cong L(3, 3) \cong L(15, 15) \cong L(105, 105) \prec L(1, 1)$

(c) $\mathcal{G}_2 \prec \mathcal{G}_{10} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1, 1) \cong L(5, 5) \cong L(35, 35) \cong L(105, 105) \prec L(1, 1)$

(d) $\mathcal{G}_2 \prec \mathcal{G}_{10} \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1, 1) \cong L(5, 5) \cong L(15, 15) \cong L(105, 105) \prec L(1, 1)$

(e) $\mathcal{G}_2 \prec \mathcal{G}_{14} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1, 1) \cong L(7, 7) \cong L(35, 35) \cong L(105, 105) \prec L(1, 1)$

(f) $\mathcal{G}_2 \prec \mathcal{G}_{14} \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(7,7) \cong L(21,21) \cong L(105,105) \prec L(1,1)$

(g) $\mathcal{G}_3 \prec \mathcal{G}_6 \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(14,14) \cong L(70,70) \prec L(1,1)$

(h) $\mathcal{G}_3 \prec \mathcal{G}_6 \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(10,10) \cong L(70,70) \prec L(1,1)$

(i) $\mathcal{G}_3 \prec \mathcal{G}_{15} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(5,5) \cong L(35,35) \cong L(70,70) \prec L(1,1)$

(j) $\mathcal{G}_3 \prec \mathcal{G}_{15} \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(5,5) \cong L(10,10) \cong L(70,70) \prec L(1,1)$

(k) $\mathcal{G}_3 \prec \mathcal{G}_{21} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(7,7) \cong L(10,10) \cong L(70,70) \prec L(1,1)$

(l) $\mathcal{G}_3 \prec \mathcal{G}_{21} \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(7,7) \cong L(14,14) \cong L(70,70) \prec L(1,1)$

(m) $\mathcal{G}_5 \prec \mathcal{G}_{10} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(14,14) \cong L(42,42) \prec L(1,1)$

(n) $\mathcal{G}_5 \prec \mathcal{G}_{10} \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(6,6) \cong L(42,42) \prec L(1,1)$

(o) $\mathcal{G}_5 \prec \mathcal{G}_{15} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(3,3) \cong L(21,21) \cong L(42,42) \prec L(1,1)$

(p) $\mathcal{G}_5 \prec \mathcal{G}_{15} \prec \mathcal{G}_{30} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(3,3) \cong L(6,6) \cong L(42,42) \prec L(1,1)$

(q) $\mathcal{G}_5 \prec \mathcal{G}_{35} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(7,7) \cong L(21,21) \cong L(42,42) \prec L(1,1)$

(r) $\mathcal{G}_5 \prec \mathcal{G}_{35} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(7,7) \cong L(14,14) \cong L(42,42) \prec L(1,1)$

(s) $\mathcal{G}_7 \prec \mathcal{G}_{14} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(10,10) \cong L(30,30) \prec L(1,1)$

(t) $\mathcal{G}_7 \prec \mathcal{G}_{14} \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(2,2) \cong L(6,6) \cong L(30,30) \prec L(1,1)$

(u) $\mathcal{G}_7 \prec \mathcal{G}_{21} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(3,3) \cong L(15,15) \cong L(30,30) \prec L(1,1)$

(v) $\mathcal{G}_7 \prec \mathcal{G}_{21} \prec \mathcal{G}_{42} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(3,3) \cong L(6,6) \cong L(30,30) \prec L(1,1)$

(w) $\mathcal{G}_7 \prec \mathcal{G}_{35} \prec \mathcal{G}_{105} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(5,5) \cong L(15,15) \cong L(30,30) \prec L(1,1)$

(x) $\mathcal{G}_7 \prec \mathcal{G}_{35} \prec \mathcal{G}_{70} \prec \mathcal{G}_{210}$ and $L(1,1) \cong L(5,5) \cong L(10,10) \cong L(30,30) \prec L(1,1)$

Note: (a) above means that $\mathcal{G}_2$ is a subgeometry of $\mathcal{G}_6$, $\mathcal{G}_6$ is a subgeometry of $\mathcal{G}_{42}$, $\mathcal{G}_{42}$ is a subgometry of $\mathcal{G}_{210}$ and $L(1,1)$ in $\mathcal{G}_2$ is the same as $L(3,3)$ in $\mathcal{G}_6$, it is the same as $L(21,21)$ in $\mathcal{G}_{42}$, is the same as $L(105,105)$ while $L(105,105)$

in $\mathcal{G}_{210}$ is a subline of maximal line $L(1,1)$ in $\mathcal{G}_{210}$, (b)-(x) is interpreted the same way.

(iv) In a non-near-linear geometry $\mathcal{G}_d$, two maximal lines have $q$ points in common, for $q|d$. The $q$ points gives a subline $L(z,w)$ where $z, w \in \frac{d}{q}\mathcal{Z}_q$. and $\frac{d}{q}\mathcal{Z}_q$ is defined in eq.(4.15). Example: In $\mathcal{G}_{210}$ :

(a) $L(1,2) = \{(0,0)(1,2)(2,4)(3,6)(4,8)(5,10)(6,12)(7,14)(8,16)$
$(9,18)(10,20)(11,22)(12,24)(13,26)(14,28)(15,30)(16,32)$
$(17,34)(18,36)(19,38)(20,40)(21,42)(22,44)(23,46)(24,48)$
$(25,50)(26,52)(27,54)(28,56)(29,58)(30,60)(31,62)(32,64)$
$(33,66)(34,68)(35,70)(36,72)(37,74)(38,76)(39,78)(40,80)$
$(41,82)(42,84)(43,86)(44,88)(45,90)(46,92)(47,94)(48,96)$
$(49,98)(50,100)(51,102)(52,104)(53,106)(54,108)(55,110)$
$(56,112)(57,114)(58,116)(59,118)(60,120)(61,122)(62,124)$
$(63,126)(64,128)(65,130)(66,132)(67,134)(68,136)(69,138)$
$(70,140)(71,142)(72,144)(73,146)(74,148)(75,150)(76,152)$
$(77,154)(78,156)(79,158)(80,160)(81,162)(82,164)(83,166)$
$(84,168)(85,170)(86,172)(87,174)(88,176)(89,178)(90,180)$
$(91,182)(92,184)(93,186)(94,188)(95,190)(96,192)(97,194)$
$(98,196)(99,198)(100,200)(101,202)(102,204)(103,206)$
$(104,208)(105,0)(106,2)(107,4)(108,6)(109,8)(110,10)$
$(111,12)(112,14)(113,16)(114,18)(115,20)(116,22)(117,24)$
$(118,26)(119,28)(120,30)(121,32)(122,34)(123,36)(124,38)$
$(125,40)(126,42)(127,44)(128,46)(129,48)(130,50)(131,52)$

$(132, 54)(133, 56)(134, 58)(135, 60)(136, 62)(137, 64)(138, 66)$
$(139, 68)(140, 70)(141, 72)(142, 74)(143, 76)(144, 78)(145, 80)$
$(146, 82)(147, 84)(148, 86)(149, 88)(150, 90)(151, 92)(152, 94)$
$(153, 96)(154, 98)(155, 100)(156, 102)(157, 104)(158, 106)$
$(159, 108)(160, 110)(161, 112)(162, 114)(163, 116)(164, 118)$
$(165, 120)(166, 122)(167, 124)(168, 126)(169, 128)(170, 130)$
$(171, 132)(172, 134)(173, 136)(174, 138)(175, 140)(176, 142)$
$(177, 144)(178, 146)(179, 148)(180, 150)(181, 152)(182, 154)$
$(183, 156)(184, 158)(185, 160)(186, 162)(187, 164)(188, 166)$
$(189, 168)(190, 170)(191, 172)(192, 174)(193, 176)(194, 178)$
$(195, 180)(196, 182)(197, 184)(198, 186)(199, 188)(200, 190)$
$(201, 192)(202, 194)(203, 196)(204, 198)(205, 200)(206, 202)$
$(207, 204)(208, 206)(209, 208)\}.$

(b) $L(1, 4) = \{(0, 0)(1, 4)(2, 8)(3, 12)(4, 16)(5, 20)(6, 24)(7, 28)(8, 32)$
$(9, 36)(10, 40)(11, 44)(12, 48)(13, 52)(14, 56)(15, 60)(16, 64)(17, 68)$
$(18, 72)(19, 76)(20, 80)(21, 84)(22, 88)(23, 92)(24, 96)(25, 100)$
$(26, 104)(27, 108)(28, 112)(29, 116)(30, 120)(31, 124)(32, 128)$
$(33, 132)(34, 136)(35, 140)(36, 144)(37, 148)(38, 152)(39, 156)$
$(40, 160)(41, 164)(42, 168)(43, 172)(44, 176)(45, 180)(46, 184)$
$(47, 188)(48, 192)(49, 196)(50, 200)(51, 204)(52, 208)(53, 2)(54, 6)$
$(55, 10)(56, 14)(57, 18)(58, 22)(59, 26)(60, 30)(61, 34)(62, 38)$
$(63, 42)(64, 46)(65, 50)(66, 54)(67, 58)(68, 62)(69, 66)(70, 70)$
$(71, 74)(72, 78)(73, 82)(74, 86)(75, 90)(76, 94)(77, 98)(78, 102)$
$(79, 106)(80, 110)(81, 114)(82, 118)(83, 122)(84, 126)(85, 130)$

$(86,134)(87,138)(88,142)(89,146)(90,150)(91,154)(92,158)$

$(93,162)(94,166)(95,170)(96,174)(97,178)(98,182)(99,186)$

$(100,190)(101,194)(102,198)(103,202)(104,206)(105,0)$

$(106,4)(107,8)(108,12)(109,16)(110,20)(111,24)(112,28)$

$(113,32)(114,36)(115,40)(116,44)(117,48)(118,52)(119,56)$

$(120,60)(121,64)(122,68)(123,72)(124,76)(125,80)(126,84)$

$(127,88)(128,92)(129,96)(130,100)(131,104)(132,108)(133,112)$

$(134,116)(135,120)(136,124)(137,128)(138,132)(139,136)$

$(140,140)(141,144)(142,148)(143,152)(144,156)(145,160)$

$(146,164)(147,168)(148,172)(149,176)(150,180)(151,184)$

$(152,188)(153,192)(154,196)(155,200)(156,204)(157,208)$

$(158,2)(159,6)(160,10)(161,14)(162,18)(163,22)(164,26)$

$(165,30)(166,34)(167,38)(168,42)(169,46)(170,50)(171,54)$

$(172,58)(173,62)(174,66)(175,70)(176,74)(177,78)(178,82)$

$(179,86)(180,90)(181,94)(182,98)(183,102)(184,106)(185,110)$

$(186,114)(187,118)(188,122)(189,126)(190,130)(191,134)$

$(192,138)(193,142)(194,146)(195,150)(196,154)(197,158)$

$(198,162)(199,166)(200,170)(201,174)(202,178)(203,182)$

$(204,186)(205,190)(206,194)(207,198)(208,202)(209,206)\}.$

$$L(1,2) \cap L(1,4) = L(105,0) \tag{5.1}$$

where

$L(105,0) = \{(0,0)(105,0)\}$ has 2 points.

These points form a subline $L(105,0) \cong L(1,0)$ in $\mathcal{G}_2$.

If we consider the subgeometry $\mathcal{G}_q$, the subline $L(z, w)$ in $\mathcal{G}_d$ is a maximal line in $\mathcal{G}_q$. There exists $\psi(q)$ maximal lines in subgeometry $\mathcal{G}_q$ of $\mathcal{G}_d$.
Example: $\mathcal{G}_2$ has 3 maximal lines. They are $L(0,1), L(1,0), L(1,1)$.
Below,we present examples which shows the following concepts:

(i) Partial ordered relation between $\mathcal{G}_{210}$ and its subgeometries with subgeometry as partial order.

(ii) Isomorphism between the subgeometries of $\mathcal{G}_{210}$ and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$ where $d = 210$ using tables and Hasse diagram.

Table 5.1: A table of maximal lines in finite geometry $\mathcal{G}_{210}$ and its subgeometries.

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|---|---|
| $\mathcal{G}_{210}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6),$ $L(1,7), L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14),$ $L(1,15), L(1,16), L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22),$ $L(1,23), L(1,24), L(1,25), L(1,26), L(1,27), L(1,28), L(1,29), L(1,30),$ $L(1,35), L(1,36), L(1,37), L(1,38), L(1,39), L(1,40), L(1,41), L(1,42),$ $L(1,43), L(1,31), L(1,32), L(1,33), L(1,34), L(1,44), L(1,45), L(1,46),$ $L(1,47), L(1,48), L(1,49), L(1,50), L(1,51), L(1,52), L(105,1), L(105,2),$ $L(1,53), L(1,54), L(1,55), L(1,56), L(1,57), L(1,58), L(1,59), L(1,60),$ $L(1,61), L(1,62), L(1,63), L(1,64), L(1,65), L(1,66), L(1,67), L(1,68)$ $L(1,69), L(1,70), L(1,71), L(1,72), L(1,73), L(1,74), L(1,75), L(1,76),$ $L(1,77), L(1,78), L(1,79), L(1,80), L(1,81), L(1,82), L(1,83), L(1,84),$ $L(1,85), L(1,86), L(1,87), L(1,88), L(1,89), L(1,90), L(1,91), L(1,92),$ |

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| $\mathcal{G}_{210}$ | $L(1,93), L(1,94), L(1,95), L(1,96), L(1,97), L(1,98), L(1,99), L(1,100),$ $L(1,109), L(1,110), L(1,111), L(1,112), L(1,113), L(1,101), L(1,102), L(1,103),$ $L(1,104), L(1,105), L(1,106), L(1,107), L(1,108), L(1,114), L(1,115), L(1,116),$ $L(1,117), L(1,118), L(1,119), L(1,120), L(1,121), L(1,122), L(1,123), L(1,124),$ $L(1,125), L(1,126), L(1,127), L(1,128), L(1,129), L(1,130), L(1,131), L(1,132),$ $L(1,133), L(1,134), L(1,135), L(1,136), L(1,137), L(1,138), L(1,139), L(1,140),$ $L(1,141), L(1,142), L(1,143), L(1,144), L(1,145), L(1,146), L(1,147), L(1,148),$ $L(1,150), L(1,151), L(1,152), L(1,153), L(1,154), L(1,155), L(1,156), L(1,157),$ $L(1,158), L(1,160), L(1,161), L(1,162), L(1,163), L(1,164), L(1,165), L(1,166),$ $L(1,167), L(1,168), L(1,169), L(1,170), L(1,171), L(1,172), L(1,173), L(1,174),$ $L(1,175), L(1,176), L(1,177)L(1,178), L(1,179), L(1,180), L(1,181), L(1,182),$ $L(1,183), L(1,184), L(1,185), L(1,186), L(1,187), L(1,188), L(1,189), L(1,190),$ $L(1,191), L(1,192), L(1,193), L(1,194), L(1,195)L(1,196), L(1,197), L(1,198),$ $L(1,199), L(1,200), L(1,201), L(1,202), L(1,203), L(1,204)L(1,205), L(1,206),$ $L(1,207), L(1,208), L(1,209), L(2,1), L(2,23), L(2,3), L(2,5), L(2,7),$ $L(2,9), L(2,11), L(2,13), L(2,15), L(2,17), L(2,19), L(2,21), L(2,25),$ $L(2,27), L(2,29), L(2,31), L(2,33), L(2,35), L(2,37), L(2,39), L(2,41),$ $L(2,43), L(2,45), L(2,47), L(2,49), L(2,51), L(2,53), L(2,55), L(2,57),$ $L(2,59), L(2,61), L(2,63), L(2,65), L(2,67), L(2,69), L(2,71), L(2,73),$ $L(2,75), L(2,77), L(2,79)L(2,81), L(2,83), L(2,85), L(2,87), L(2,89),$ $L(2,91), L(2,93), L(2,95), L(2,97), L(2,99), L(2,101), L(2,103), L(2,105),$ $L(2,107), L(2,109), L(2,111), L(2,113), L(2,115), L(2,117), L(2,119), L(2,121),$ $L(2,123), L(2,125), L(2,127), L(2,129), L(2,131), L(2,133), L(2,135), L(2,137),$ |

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| $\mathcal{G}_{210}$ | $L(2,139), L(2,141), L(2,143), L(2,145), L(2,147), L(2,149), L(2,151), L(2,153),$<br>$L(2,155), L(2,157), L(2,159), L(2,161), L(2,163), L(2,165), L(2,167), L(2,169),$<br>$L(2,171), L(2,173), L(2,175), L(2,177), L(2,179), L(2,181), L(2,183), L(2,185),$<br>$L(2,187), L(2,189), L(2,191), L(2,193), L(2,195), L(2,197), L(2,199), L(2,201),$<br>$L(2,203), L(2,205), L(2,207), L(2,209), L(3,1), L(3,2), L(3,4), L(3,5),$<br>$L(3,7), L(3,8), L(3,10), L(3,11), L(3,13), L(3,14), L(3,16), L(3,17),$<br>$L(3,19), L(3,20), L(3,22), L(3,23), L(3,25), L(3,26), L(3,28), L(3,29),$<br>$L(3,31), L(3,32), L(3,34), L(3,35), L(3,37), L(3,38), L(3,40), L(3,41),$<br>$L(3,43), L(3,44), L(3,46), L(3,47), L(3,49), L(3,50), L(1,159), L(3,52),$<br>$L(3,53), L(3,55), L(3,56), L(3,58), L(3,59), L(3,61), L(3,62), L(3,64),$<br>$L(3,65), L(3,67), L(3,68), L(3,70), L(3,73), L(3,76), L(3,79), L(3,82),$<br>$L(3,85), L(3,88), L(3,91), L(3,94), L(3,97), L(3,100), L(3,103), L(3,106),$<br>$L(3,109), L(3,112), L(3,115), L(3,118), L(3,121), L(3,124), L(3,127), L(3,130),$<br>$L(3,133), L(3,136), L(3,139), L(5,1), L(5,2), L(5,3), L(5,4), L(5,6),$<br>$L(5,7), L(5,8)L(5,9), L(5,11), L(5,12), L(5,13), L(5,14), L(5,16),$<br>$L(5,17), L(5,18), L(5,19), L(5,21), L(5,22), L(5,23), L(5,24), L(5,26),$<br>$L(5,27), L(5,28), L(5,29), L(5,31), L(5,32), L(5,33), L(5,34), L(5,36),$<br>$L(5,37), L(5,38), L(5,39), L(5,41), L(5,42), L(5,47), L(5,52), L(5,57),$<br>$L(5,62), L(5,67), L(5,72), L(5,77), L(5,82), L(6,1), L(10,29), L(6,5),$<br>$L(6,7), L(6,11), L(6,13), L(6,17), L(6,19), L(6,23), L(6,25), L(6,29),$<br>$L(6,31), L(6,35), L(6,37), L(6,41), L(6,43), L(6,47), L(6,49), L(6,53),$<br>$L(6,55), L(6,59), L(6,61), L(6,65), L(6,67), L(6,73), L(6,79), L(6,85),$<br>$L(6,91), L(6,97), L(6,103), L(6,109), L(6,115), L(6,121), L(6,127), L(6,133),$ |

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| $\mathcal{G}_{210}$ | $L(6,139), L(7,1), L(7,2), L(7,3), L(7,4), L(7,5), L(7,6), L(7,8),$ $L(7,9), L(7,10), L(7,11), L(7,12), L(7,13), L(7,15), L(7,16), L(7,17),$ $L(7,18), L(7,19), L(7,20), L(7,22), L(7,23), L(7,24), L(7,25), L(7,26),$ $L(7,27), L(7,29), L(7,30), L(7,37), L(7,44), L(7,51), L(7,58), L(10,1),$ $L(10,3), L(10,7), L(10,9), L(10,11), L(10,13), L(10,17), L(10,19), L(10,21),$ $L(10,23), L(10,27), L(10,31), L(10,33), L(10,37), L(10,39), L(10,41), L(10,47),$ $L(10,57), L(10,67), L(10,77), L(14,1), L(14,3), L(14,5), L(14,9), L(14,11),$ $L(14,13), L(14,15), L(14,17), L(14,19), L(14,23), L(14,25), L(14,27), L(14,29),$ $L(14,37), L(14,51), L(15,1), L(15,2), L(15,4), L(15,7), L(15,8), L(15,11),$ $L(15,13), L(15,14), L(15,17), L(15,19), L(15,23), L(15,26), L(15,34), L(15,38),$ $L(21,1), L(21,2), L(21,4), L(21,5), L(21,8), L(21,10), L(21,13), L(21,16),$ $L(21,17), L(21,19), L(30,1), L(30,7), L(30,11), L(30,13), L(30,17), L(30,19),$ $L(30,23), L(35,1), L(35,2), L(35,3), L(35,4), L(35,6), L(35,11), L(42,1),$ $L(42,5), L(42,13), L(42,17), L(42,19), L(70,1), L(70,3), L(70,11), L(1,149).$ |

Likewise, the sublines of finite geometry $\mathcal{G}_{210}$ which is also known as lines in subgeometry $\mathcal{G}_q$ where $q$ is a divisor of $d$ are obtained as follows,

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| $\mathcal{G}_{105}$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8), L(2,10), L(2,12),$ $L(2,14), L(2,16), L(2,18), L(2,20), L(2,22), L(2,24), L(2,26), L(2,28),$ $L(2,30), L(2,32), L(2,34), L(2,36), L(2,38), L(2,40), L(2,42), L(2,44),$ $L(2,46), L(2,48), L(2,50), L(2,52), L(2,54), L(2,56), L(2,58), L(2,60),$ $L(2,62), L(2,64), L(2,66), L(2,68), L(2,70), L(2,72), L(2,74), L(2,76),$ |

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| $\mathcal{G}_{105}$ | $L(2,78), L(2,80), L(2,82), L(2,84), L(2,86), L(2,88), L(2,90), L(2,92),$ $L(2,94), L(2,96), L(2,98), L(2,100), L(2,102), L(2,104), L(2,106), L(2,108),$ $L(2,110), L(2,112), L(2,114), L(2,116), L(2,118), L(30,38), L(42,2), L(2,120),$ $L(2,122), L(2,124), L(2,126), L(2,128), L(2,130), L(2,132), L(2,134), L(2,136),$ $L(2,138), L(2,140), L(2,142), L(2,144), L(2,146), L(2,148), L(2,150), L(2,152),$ $L(2,154), L(2,156), L(2,158), L(2,160), L(2,162), L(2,164), L(2,166), L(2,168),$ $L(2,170), L(2,172), L(2,174), L(2,176), L(2,178), L(2,180), L(2,182), L(2,184),$ $L(2,186), L(2,188), L(2,190), L(2,192), L(2,194), L(2,196), L(2,198), L(2,200),$ $L(2,202), L(2,204), L(2,206), L(2,208), L(6,2), L(6,4), L(6,8), L(6,10),$ $L(6,14), L(6,16), L(6,20), L(14,58), L(42,4), L(6,22), L(6,26), L(6,28),$ $L(6,32), L(6,34), L(6,38), L(6,40), L(6,44), L(6,46), L(6,50), L(6,52),$ $L(6,56), L(6,58), L(6,62), L(6,64), L(6,68), L(6,70), L(6,76), L(6,82),$ $L(6,88), L(6,94), L(6,100), L(6,106), L(6,112), L(6,118), L(6,124), L(6,130),$ $L(6,136), L(10,2), L(10,4), L(10,6), L(10,8), L(10,12), L(10,14), (10,16),$ $L(10,18), L(10,22), L(10,24), L(10,26), L(10,28), L(10,32), L(10,34), L(10,36),$ $L(10,38), L(10,42), L(10,52), L(10,62), L(10,72), L(10,82), L(14,2), L(14,4),$ $L(14,6), L(14,8), L(14,10), L(14,12), L(14,16), L(14,18), L(14,20), L(14,22),$ $L(14,24), L(14,26), L(14,30), L(14,44), L(30,2), L(30,4), L(30,8), L(30,14),$ $L(30,26), L(30,34), L(70,2), L(70,4), L(70,6), L(42,8), L(42,10), L(42,16).$ |
| $\mathcal{G}_{70}$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9), L(3,12), L(3,15), L(3,18),$ $L(3,21), L(3,24), L(3,27), L(3,30), L(3,33), L(3,36)L(3,39), L(3,42),$ $L(3,45), L(3,48), L(3,51), L(3,54), L(3,57), L(3,60), L(3,63), L(3,66),$ $L(3,69), L(3,72), L(3,75), L(3,78), L(3,81), L(3,84), L(3,87), L(3,90),$ |

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| $\mathcal{G}_{70}$ | $L(3,93), L(3,96), L(3,99), L(3,102), L(3,105), L(3,108), L(3,111), L(3,114),$ $L(3,117), L(3,120), L(3,123), L(3,126), L(3,129), L(3,132), L(3,135), L(3,138),$ $L(3,141), L(3,144), L(3,147), L(3,150), L(3,153), L(3,156), L(3,159), L(3,162),$ $L(3,165), L(3,168), L(3,171), L(3,174), L(3,177), L(105,3), L(105,6)L(3,180),$ $L(3,183), L(3,186), L(3,189), L(3,192), L(3,195), L(3,198), L(3,201), L(3,204),$ $L(3,207), L(6,3), L(6,69), L(6,9), L(6,15), L(15,21), L(15,24), L(15,27),$ $L(42,3), L(6,21), L(6,27), L(6,33), L(6,39), L(6,45), L(6,51), L(6,57),$ $L(6,117), L(6,123), L(6,129), L(6,63), L(6,75), L(6,81), L(6,87), L(6,93),$ $L(6,99), L(6,105), L(6,111), L(6,135), L(6,141), L(6,147), L(6,153), L(6,159),$ $L(6,165), L(6,171), L(6,177), L(6,183), L(6,189), L(6,195), L(6,201), L(6,207),$ $L(15,3), L(15,6), L(15,9), L(15,12), L(15,18), L(15,33), L(15,36), L(15,39),$ $L(15,42), L(15,57), L(15,72), L(21,3), L(21,6), L(42,9), L(21,9), L(21,12),$ $L(21,15), L(21,18), L(21,24), L(21,27), L(21,30), L(21,51), L(42,15), L(30,3),$ $L(30,9), L(30,21), L(30,27), L(30,33), L(30,39), L(30,57), L(42,27), L(42,51).$ |
| $\mathcal{G}_{42}$ | $L(0,5), L(5,0), L(5,5), L(5,10), L(5,15), L(5,20), L(5,25), L(5,30),$ $L(5,35), L(5,40), L(5,45), L(5,50), L(5,55), L(5,60)L(5,65), L(5,70),$ $L(5,75), L(5,80), L(5,85), L(5,90), L(5,95), L(5,100), L(5,105), L(5,110),$ $L(5,115), L(5,120), L(5,125), L(5,130), L(5,135), L(5,140), L(5,145), L(5,150),$ $L(5,155), L(5,160), L(5,165), L(5,170), L(5,175), L(5,180), L(5,185), L(5,190),$ $L(5,195), L(5,200), L(5,205), L(10,5), L(10,115), L(10,35), L(10,45), L(10,55),$ $L(10,65), L(10,75), L(10,85), L(10,95), L(10,105), L(10,125), L(10,135), L(10,145),$ $L(10,155), L(10,165), L(10,175), L(10,185), L(10,195), L(10,205), L(15,5), L(15,10),$ $L(15,20), L(15,25), L(15,35), L(15,40), L(15,50), L(10,15), L(10,25), L(35,15),$ |

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| $\mathcal{G}_{42}$ | $L(15,55), L(15,65), L(15,70), L(15,85), L(15,100), L(15,115), L(15,130), L(30,5),$ $L(35,20), L(30,25), L(30,35), L(30,55), L(30,65), L(30,85), L(30,115), L(35,5),$ $L(35,10), L(35,25), L(35,30), L(70,5), L(70,15), L(70,25), L(105,5), L(105,10).$ |
| $\mathcal{G}_{35}$ | $L(0,6), L(6,0), L(6,6), L(6,12), L(6,18), L(6,24), L(6,30), L(6,36),$ $L(6,42), L(6,48), L(6,54), L(6,60), L(6,66), L(6,72)L(6,78), L(6,84),$ $L(6,90), L(6,96), L(6,102), L(6,108), L(6,114), L(6,120), L(6,126), L(6,132),$ $L(6,138), L(6,144), L(6,150), L(6,156), L(6,162), L(6,168), L(6,174), L(6,180),$ $L(6,186), L(6,192), L(6,198), L(6,204), L(30,6), L(30,12), L(30,18), L(30,24),$ $L(30,36), L(30,42), L(30,72), L(42,6), L(42,12), L(42,18), L(42,24), L(42,30).$ |
| $\mathcal{G}_{30}$ | $L(0,7), L(7,0), L(7,7), L(7,14), L(7,21), L(7,28), L(7,35), L(7,42), L(21,7),$ $L(7,49), L(7,56), L(7,63), L(7,70), L(7,77), L(7,84), L(7,91), L(7,98),$ $L(7,105), L(7,112), L(7,119), L(7,126), L(7,133), L(7,140), L(7,147), L(7,154),$ $L(7,161), L(7,168), L(7,175), L(7,182), L(7,189), L(7,196), L(7,203), L(14,7),$ $L(14,161), L(14,21), L(14,35), L(14,49), L(14,49), L(14,77), L(14,91), L(14,105),$ $L(14,119), L(14,133), L(14,147), L(14,175), L(105,14), L(14,189), L(14,203),$ $L(21,14), L(21,28), L(21,35), L(21,49), L(21,56), L(21,60), L(21,91), L(21,112),$ $L(21,133), L(35,7), L(35,14), L(35,21), L(35,28), L(35,42), L(35,77), L(42,7),$ $L(42,35), L(42,49), L(42,91), L(42,133), L(70,7), L(70,21), L(70,77), L(105,7).$ |
| $\mathcal{G}_{21}$ | $L(0,10), L(10,0), L(10,10), L(10,20), L(10,30), L(10,40), L(10,50), L(10,60),$ $L(10,70), L(10,80), L(10,90), L(10,100), L(10,110), L(10,120), L(10,130),$ $L(10,140), L(10,150), L(10,160), L(10,170), L(10,180), L(10,190), L(10,200),$ $L(30,10), L(30,20), L(30,40), L(30,50), L(30,70), L(30,100), L(30,130),$ $L(70,10), L(70,20), L(70,30).$ |

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| $\mathcal{G}_{15}$ | $L(0,14), L(14,0), L(14,14), L(14,28), L(14,42), L(14,56), L(14,70), L(14,84), L(14,98), L(14,112), L(14,126), L(14,140), L(14,154), L(14,168), L(14,182), L(14,196), L(42,14), L(42,28), L(42,56), L(42,70), L(42,112), L(70,14), L(70,28), L(70,42).$ |
| $\mathcal{G}_{14}$ | $L(0,15), L(15,0), L(15,15), L(15,30), L(15,45), L(15,60), L(15,75), L(15,90), L(15,105), L(15,120), L(15,135), L(15,150), L(15,165), L(15,180), L(15,195), L(30,15), L(30,45), L(30,75), L(30,105), L(30,135), L(30,165), L(30,195), L(105,15), L(105,30).$ |
| $\mathcal{G}_{10}$ | $L(0,21), L(21,0), L(21,21), L(21,42), L(21,63), L(21,84), L(21,105), L(21,126), L(21,147), L(21,168), L(21,189), L(42,21), L(42,63), L(42,105), L(42,147), L(42,189), L(105,21), L(105,42).$ |
| $\mathcal{G}_{6}$ | $L(0,35), L(35,0), L(35,35), L(35,70), L(35,105), L(35,140), L(35,175), L(70,35), L(70,105), L(70,175), L(105,35), L(105,70)$ |
| $\mathcal{G}_{7}$ | $L(0,30), L(30,0), L(30,30), L(30,60), L(30,90), L(30,120), L(30,150), L(30,180)$ |
| $\mathcal{G}_{5}$ | $L(0,42), L(42,0), L(42,42), L(42,84), L(42,126), L(42,168)$ |
| $\mathcal{G}_{3}$ | $L(0,70), L(70,0), L(70,70), L(170,140)$ |
| $\mathcal{G}_{2}$ | $L(0,105), L(105,0), L(105,105)$ |
| $\mathcal{G}_{1}$ | $L(0,0)$ |

Figure 5.1: The Hasse diagram showing the geometry $\mathcal{G}_{210}$ and its subgeometries,

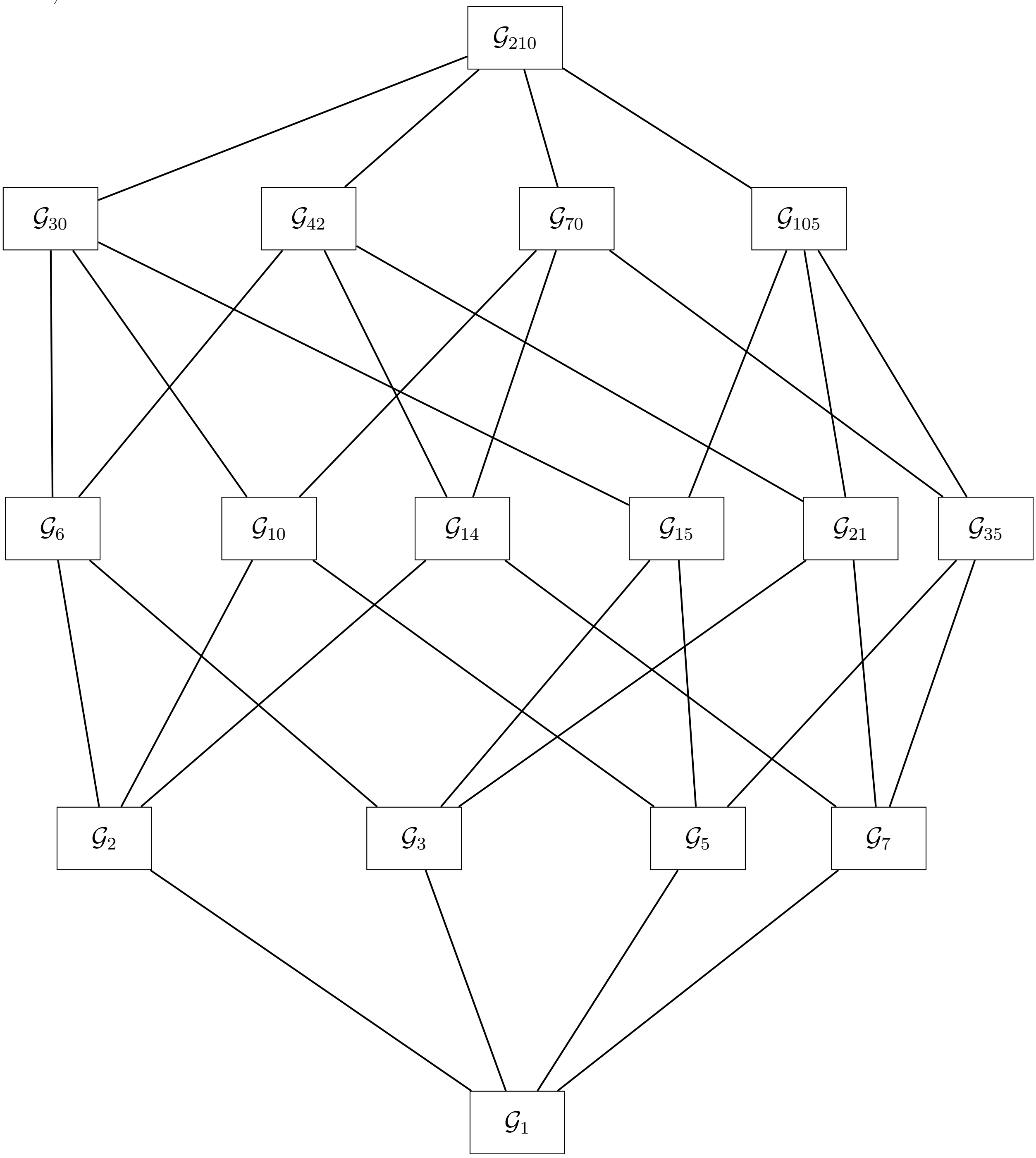

Table 5.10: A table of maximal lines in finite geometry $\mathcal{G}_{105}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{105}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7),$<br>$L(1,8), L(1,9), L(1,10), L(1,11), L(1,12)L(1,13), L(1,14), L(1,15), L(1,16),$<br>$L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22), L(1,23), L(1,24), L(1,25),$<br>$L(1,26), L(1,27), L(1,28), L(1,29), L(1,30), L(1,31), L(1,32), L(1,33), L(1,34),$<br>$L(1,35), L(1,36), L(1,37), L(1,38), L(1,39), L(1,40), L(1,41), L(1,42), L(1,43),$<br>$L(1,44), L(1,45), L(1,46), L(1,47), L(1,48), L(1,49), L(1,50), L(1,51), L(1,52),$<br>$L(1,53), L(1,54), L(1,55), L(1,56), L(1,57), L(1,58), L(1,59), L(15,19), L(21,1),$<br>$L(1,60), L(1,61), L(1,62), L(1,63), L(1,64), L(1,65), L(1,66), L(1,67), L(1,68),$<br>$L(1,69), L(1,70), L(1,71), L(1,72), L(1,73), L(1,74), L(1,75), L(1,76), L(1,77),$<br>$L(1,78), L(1,79), L(1,80), L(1,81), L(1,82), L(1,83), L(1,84), L(1,85), L(1,86),$<br>$L(1,87), L(1,88), L(1,89), L(1,90), L(1,91), L(1,92), L(1,93), L(1,94), L(1,95),$<br>$L(1,96), L(1,97), L(1,98), L(1,99), L(1,100), L(1,101), L(1,102), L(1,103),$<br>$L(3,1), L(3,2), L(3,4), L(3,5), L(3,7), L(3,8), L(3,10), L(7,29), L(21,2),$<br>$L(3,11), L(3,13), L(3,14), L(3,16), L(3,17), L(3,19), L(3,20), L(3,22), L(3,23),$<br>$L(3,25), L(3,26), L(3,28), L(3,29), L(3,31), L(3,32), L(3,34), L(3,35),$<br>$L(3,38), L(3,41), L(3,44), L(3,47), L(3,50), L(3,53), L(3,56), L(3,59), L(3,62),$<br>$L(3,65), L(3,68), L(5,1), L(5,2), L(5,3), L(5,4), L(5,6), L(5,7), L(5,8), L(5,9),$<br>$L(5,11), L(5,12), L(5,13), L(5,14), L(5,16), L(5,17), L(5,18), L(5,19), L(5,21),$<br>$L(5,26), L(5,31), L(5,36), L(5,41), L(7,1), L(7,2), L(7,3), L(7,4), L(7,5),$<br>$L(7,6), L(7,8), L(7,9), L(7,10), L(7,11), L(7,12), L(7,13), L(7,15), L(7,22),$ |

| | |
|---|---|
| $\mathcal{G}_{105}$ | $L(15,1), L(15,2), L(15,4), L(15,7), L(15,13), L(15,17), L(35,1), L(35,2), L(35,3),$ $L(21,4), L(21,5), L(21,8), L(1,104).$ |
| $\mathcal{G}_{35}$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9, L(3,12), L(3,15), L(3,18),$ $L(3,21), L(3,24), L(3,27), L(3,30), L(3,33), L(3,36), L(3,39), L(3,42),$ $L(3,45), L(3,48), L(3,51), L(3,54), L(3,57), L(3,60), L(3,63), L(3,66),$ $L(3,69), L(3,72), L(3,75), L(3,78), L(3,81), L(3,84), L(3,87), L(3,90),$ $L(3,93), L(3,96), L(3,99), L(3,102), L(15,3), L(15,6), L(15,9), L(15,12),$ $L(15,18), L(15,21), L(15,36), L(21,3), L(21,6), L(21,9), L(21,12), L(21,15).$ |
| $\mathcal{G}_{21}$ | $L(0,5), L(5,0), L(5,5), L(5,10), L(5,15), L(5,20), L(5,25), L(5,30), L(5,35),$ $L(5,40), L(5,45), L(5,50), L(5,55), L(5,60), L(5,65), L(5,70), L(5,75), L(5,80),$ $L(5,85), L(5,90), L(5,95), L(5,100), L(15,5), L(15,10), L(15,20), L(15,25), L(15,35),$ $L(15,50), L(15,65), L(35,5), L(35,10), L(35,15).$ |
| $\mathcal{G}_{15}$ | $L(0,7), L(7,0), L(7,7), L(7,14), L(7,21), L(7,28), L(7,35), L(7,42), L(7,49),$ $L(7,56), L(7,63), L(7,70), L(7,77), L(7,84), L(7,91), L(7,98), L(21,7), L(21,14),$ $L(21,28), L(21,35), L(21,56), L(35,7), L(35,14), L(35,21).$ |
| $\mathcal{G}_{7}$ | $L(0,15), L(15,0), L(15,15), L(15,30), L(15,45), L(15,60), L(15,75), L(15,90).$ |
| $\mathcal{G}_{5}$ | $L(0,21), L(21,0), L(21,21), L(21,42), L(21,63), L(21,84).$ |
| $\mathcal{G}_{3}$ | $L(0,35), L(35,0), L(35,35), L(35,70).$ |
| $\mathcal{G}_{1}$ | $L(0,0)$ |

Figure 5.2: The Hasse diagram showing the geometry $\mathcal{G}_{105}$ and its subgeometries

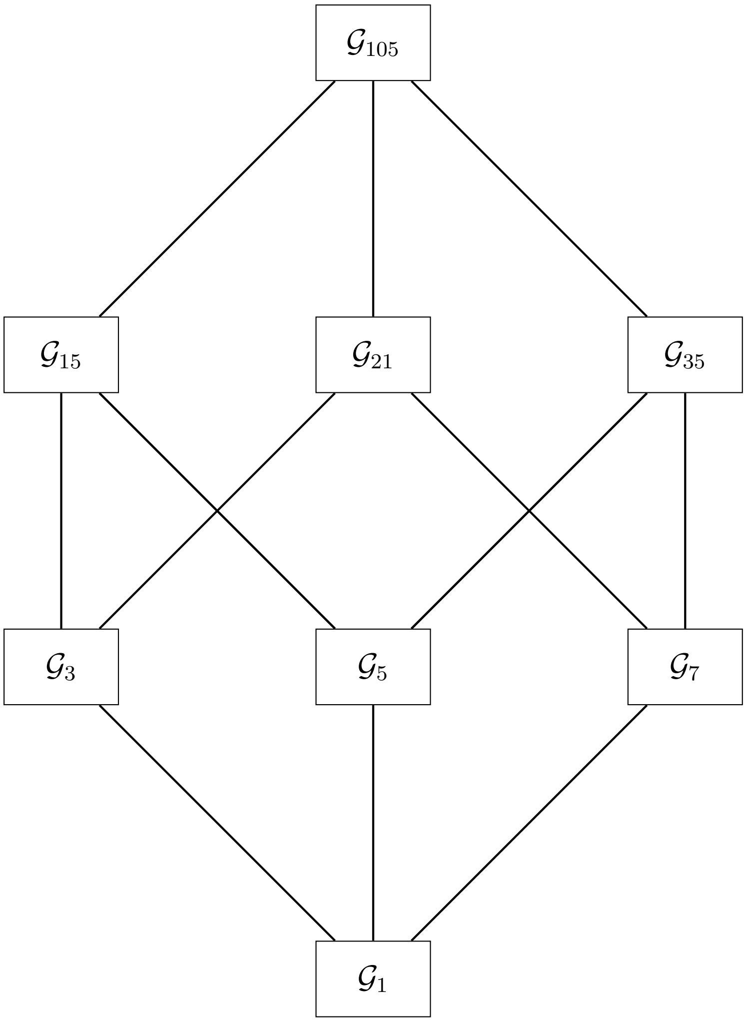

Table 5.12: A table of maximal lines in finite geometry $\mathcal{G}_{70}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{70}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7),$ $L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14), L(1,15), L(1,16),$ $L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22), L(1,23), L(1,24), L(1,25),$ $L(1,26), L(1,27), L(1,28), L(1,29), L(1,30), L(1,31), L(1,32), L(1,33), L(1,34),$ $L(1,35), L(1,36), L(1,37), L(1,38), L(1,39), L(1,40), L(1,41), L(1,42), L(1,43),$ $L(1,44), L(1,45), L(1,46), L(1,47), L(1,48), L(1,49), L(1,50), L(1,51), L(1,52),$ $L(1,53), L(1,54), L(1,55), L(1,56), L(1,57), L(1,58), L(1,59), L(35,1), L(35,2)$ $L(1,60), L(1,61), L(1,62), L(1,63), L(1,64), L(1,65), L(1,66), L(1,67), L(1,68),$ $L(1,69), L(2,1), L(2,23), L(2,3), L(2,5), L(5,7), L(5,8), L(5,9), L(14,1),$ $L(2,7), L(2,9), L(2,11), L(2,13), L(2,15), L(2,17), L(2,19), L(2,21), L(2,25),$ $L(2,27), L(2,29), L(2,31), L(2,33), L(2,35), L(2,37), L(2,39), L(2,41), L(2,43),$ $L(2,45), L(2,47), L(2,49), L(2,51), L(2,53), L(2,55), L(2,57), L(2,59), L(2,61),$ $L(2,63), L(2,65), L(2,67), L(2,69), L(5,1), L(5,2), L(5,3), L(5,4), L(5,6),$ $L(5,11), L(5,12), L(5,13), L(5,14), L(5,19), L(5,24), L(7,1), L(7,2), L(14,3),$ $L(7,3), L(7,4), L(7,5), L(7,6), L(7,8), L(7,9), L(7,10), L(7,17), L(14,5),$ $L(10,1), L(10,3), L(10,7), L(10,9), L(10,11), L(10,13), L(10,19), L(14,9), L(14,17).$ |
| $\mathcal{G}_{35}$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8), L(2,10), L(2,12), L(2,14),$ $L(2,16), L(2,18), L(2,20), L(2,22), L(2,24), L(2,26), L(2,28), L(2,30), L(2,32),$ $L(2,34), L(2,36), L(2,38), L(2,40), L(2,42), L(2,44), L(2,46), L(2,48), L(2,50),$ $L(2,52), L(2,54), L(2,56), L(2,58), L(2,60), L(2,62), L(2,64), L(2,66), L(2,68),$ $L(10,2), L(10,4), L(10,6), L(10,8), L(10,12), L(10,14), L(10,24), L(14,2), L(14,4),$ |

| | |
|---|---|
| $\mathcal{G}_{35}$ | $L(14,6), L(14,8), L(14,10)$. |
| $\mathcal{G}_{14}$ | $L(0,5), L(5,0), L(5,5), L(5,10), L(5,15), L(5,20), L(5,25), L(5,30), L(5,35)$, $L(5,40), L(5,45), L(5,50), L(5,55), L(5,60), L(5,65), L(10,5), L(10,15), L(10,25)$ $L(10,35), L(10,45), L(10,55), L(10,65), L(35,5), L(35,10)$. |
| $\mathcal{G}_{10}$ | $L(0,7), L(7,0), L(7,7), L(7,14), L(7,21), L(7,28), L(7,35), L(7,42), L(7,49)$, $L(7,56), L(7,63), L(14,7), L(14,21), L(14,35), L(14,49), L(14,63), L(35,7), L(35,14)$. |
| $\mathcal{G}_{7}$ | $L(0,10), L(10,0), L(10,10), L(10,20), L(10,30), L(10,40), L(10,50), L(10,60)$ |
| $\mathcal{G}_{5}$ | $L(0,14), L(14,0), L(14,14), L(14,28), L(14,42), L(14,56)$ |
| $\mathcal{G}_{2}$ | $L(0,35), L(35,0), L(35,35)$ |
| $\mathcal{G}_{1}$ | $L(0,0)$ |

Figure 5.3: The Hasse diagram showing the geometry $\mathcal{G}_{70}$ and its subgeometries

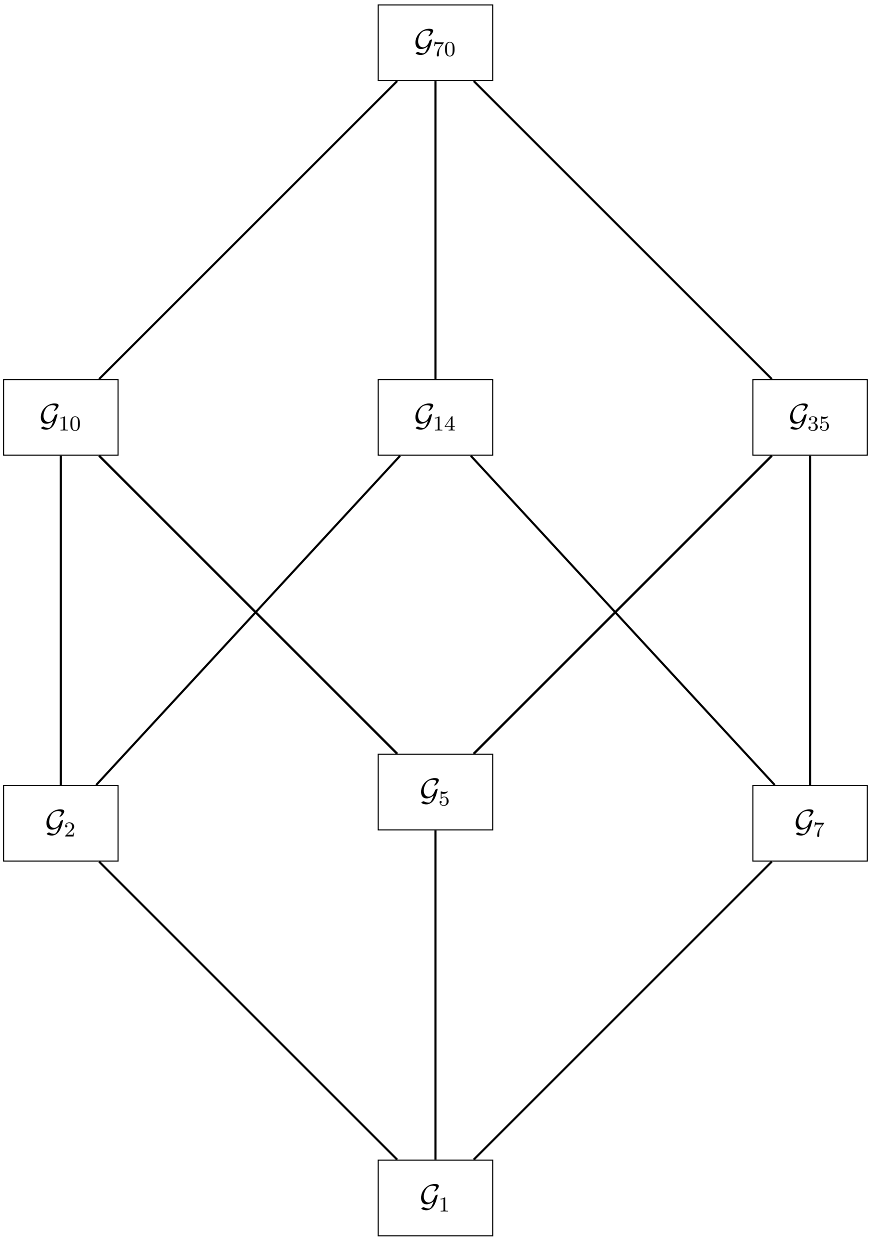

Table 5.14: A table of maximal lines in finite geometry $\mathcal{G}_{42}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{42}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7), L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14), L(1,15), L(1,16), L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22), L(1,23), L(1,24), L(1,25), L(1,26), L(1,27), L(1,28), L(1,29), L(1,30), L(1,31), L(1,32), L(1,33), L(1,34), L(1,35), L(1,36), L(1,37), L(1,38), L(1,39), L(1,40), L(1,41), L(2,1), L(2,23), L(2,7), L(2,9), L(2,11), L(2,13), L(2,15), L(2,17), L(2,19), L(2,21), L(2,25), L(2,27), L(2,29), L(2,31), L(2,33), L(2,35), L(2,37), L(2,39), L(2,41), L(3,1), L(3,2), L(3,4), L(3,5), L(3,7), L(3,8), L(3,10), L(2,3), L(2,5), L(7,3), L(3,11), L(3,13), L(3,14), L(3,17), L(3,20), L(3,23), L(3,26), L(6,1), L(7,4), L(6,5), L(6,7), L(6,11), L(6,13), L(6,17), L(6,23), L(7,1), L(7,2), L(7,5), L(7,6), L(14,1), L(14,3), L(14,5), L(21,1), L(21,2).$ |
| $\mathcal{G}_{21}$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8), L(2,10), L(2,12), L(2,14), L(2,16), L(2,18), L(2,20), L(2,22), L(2,24) L(2,26), L(2,28), L(2,30), L(2,32), L(2,34), L(2,36), L(2,38), L(2,40), L(6,2), L(6,4), L(6,8), L(6,10), L(6,14), L(6,20), L(6,26), L(14,2), L(14,4), L(14,6).$ |
| $\mathcal{G}_{14}$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9), L(3,12), L(3,15), L(3,18), L(3,21), L(3,24), L(3,27), L(3,30), L(3,33), L(3,36), L(3,39), L(6,3), L(6,9), L(6,15), L(6,21), L(6,27), L(6,33), L(6,39), L(21,3), L(21,6)$ |
| $\mathcal{G}_{6}$ | $L(0,7), L(7,0), L(7,7), L(7,14), L(7,21), L(7,28), L(7,35), L(14,7), L(14,21), L(14,35), L(21,7), L(21,14)$ |
| $\mathcal{G}_{7}$ | $L(0,6), L(6,0), L(6,6), L(6,12), L(6,18), L(6,24), L(6,30), L(6,36)$ |

| | |
|---|---|
| $\mathcal{G}_3$ | $L(0,14), L(14,0), L(14,14), L(14,28)$ |
| $\mathcal{G}_2$ | $L(0,21), L(21,0), L(21,21)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Table 5.15: A table of maximal lines in finite geometry $\mathcal{G}_{30}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{30}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7),$ $L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14), L(1,15), L(1,16),$ $L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22), L(1,23), L(1,24), L(1,25),$ |
| | $L(1,26), L(1,27), L(1,28), L(1,29), L(2,1), L(2,23), L(2,3), L(2,5), L(2,7),$ $L(2,9), L(2,11), L(2,13), L(2,15), L(2,17), L(2,19), L(2,21), L(2,25), L(15,2)$ $L(2,27), L(2,29), L(3,1), L(3,2), L(3,4), L(3,5), L(3,7), L(3,8), L(3,10),$ $L(3,13), L(3,16), L(3,19), L(5,1), L(5,2), L(5,3), L(5,4), L(5,6), L(5,11)$ $L(6,1), L(6,5), L(6,7), L(6,13), L(6,19), L(10,1), L(10,3), L(10,11), L(15,1).$ |
| $\mathcal{G}_{15}$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8), L(2,10), L(2,12), L(2,14),$ $L(2,16), L(2,18), L(2,20), L(2,22), L(2,24), L(2,26), L(2,28), L(6,2), L(6,4),$ $L(6,8), L(6,10), L(6,16), L(10,2), L(10,4), L(10,6).$ |
| $\mathcal{G}_{10}$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9), L(3,12), L(3,15), L(3,18), L(3,21),$ $L(3,24), L(3,27), L(6,2), L(6,9), L(6,15)$ $L(6,21), L(6,27), L(15,3), L(15,6).$ |
| $\mathcal{G}_6$ | $L(0,5), L(5,0), L(5,5), L(5,10), L(5,15), L(5,20), L(5,25), L(10,5), L(10,15),$ $L(10,25), L(15,5), L(15,10).$ |
| $\mathcal{G}_5$ | $L(0,6), L(6,0), L(6,6), L(6,12), L(6,18), L(6,24)$ |
| $\mathcal{G}_3$ | $L(0,10), L(10,0), L(10,10), L(10,20)$ |
| $\mathcal{G}_2$ | $L(0,15), L(15,0), L(15,15)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.4: The Hasse diagram showing the geometry $\mathcal{G}_{42}$ and its subgeometries

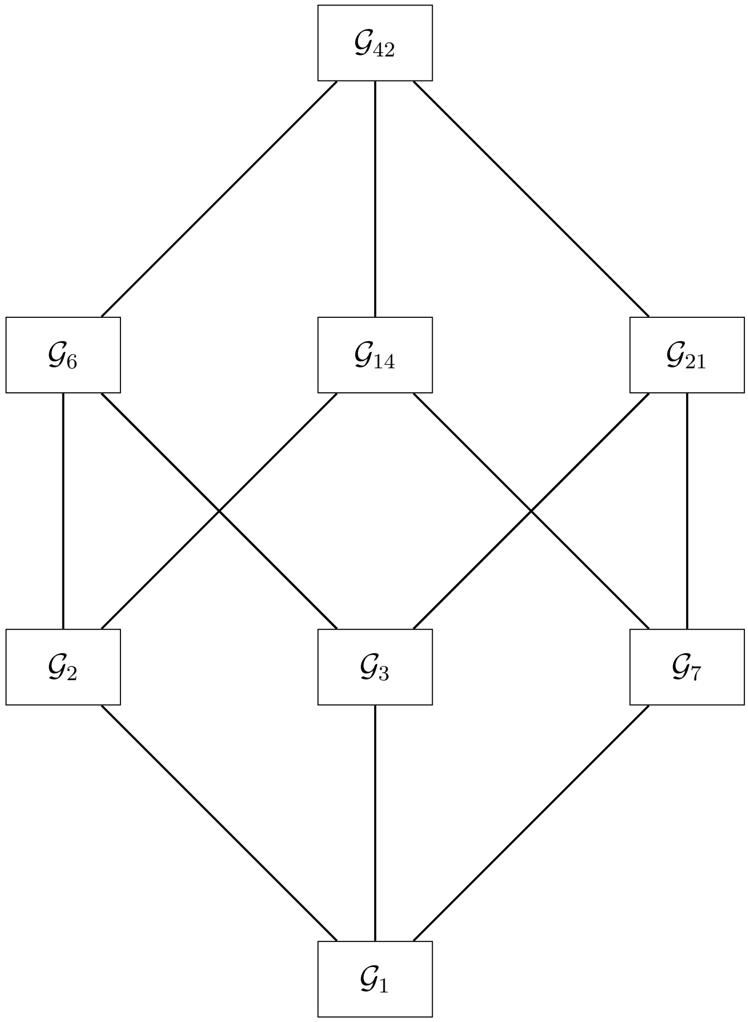

Table 5.16: A table of maximal lines in finite geometry $\mathcal{G}_{35}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{35}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7),$ $L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14), L(1,15), L(1,16),$ $L(1,17), L(1,18), L(1,19), L(1,20), L(1,21), L(1,22), L(1,23), L(1,24), L(1,25),$ $L(1,26), L(1,27), L(1,28), L(1,29), L(1,30), L(1,31), L(1,32), L(1,33), L(1,34),$ $L(5,1), L(5,2), L(5,3), L(5,4), L(5,6), L(5,7), L(5,12), L(7,1), L(7,2),$ $L(7,3), L(7,4), L(7,5).$ |
| $\mathcal{G}_7$ | $L(0,5), L(5,0), L(5,5), L(5,10), L(5,15), L(5,20), L(5,25), L(5,30)$ |
| $\mathcal{G}_5$ | $L(0,7), L(7,0), L(7,7), L(7,14), L(7,21), L(7,28)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.5: The Hasse diagram showing the geometry $\mathcal{G}_{30}$ and its subgeometries

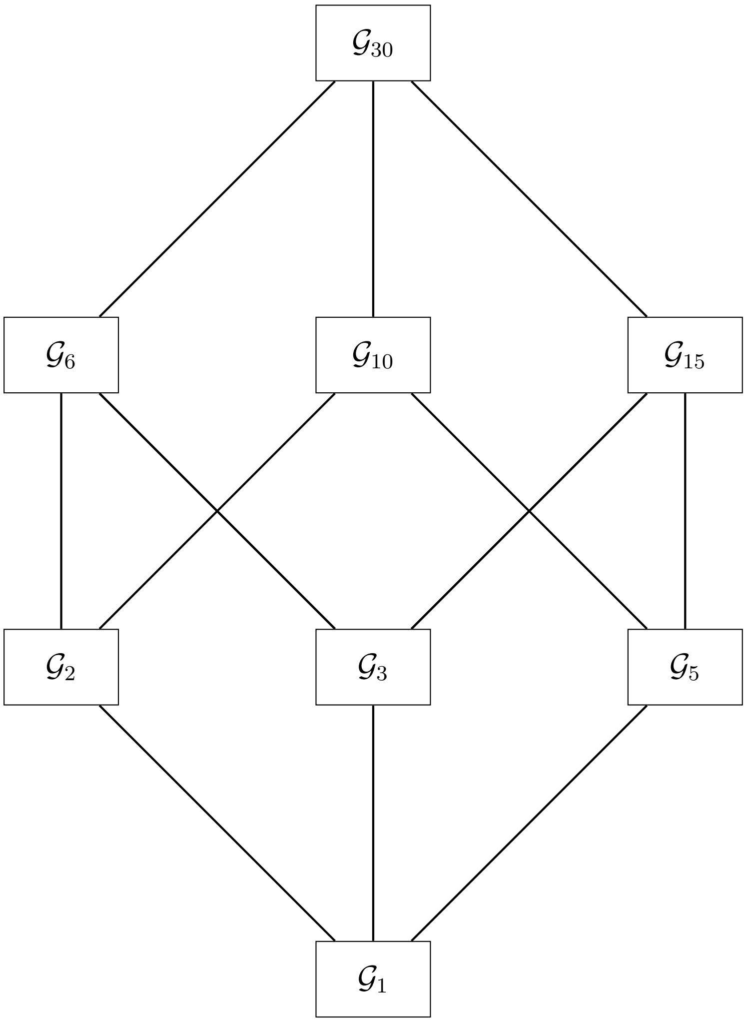

Figure 5.6: The Hasse diagram showing the geometry $\mathcal{G}_{35}$ and its subgeometries

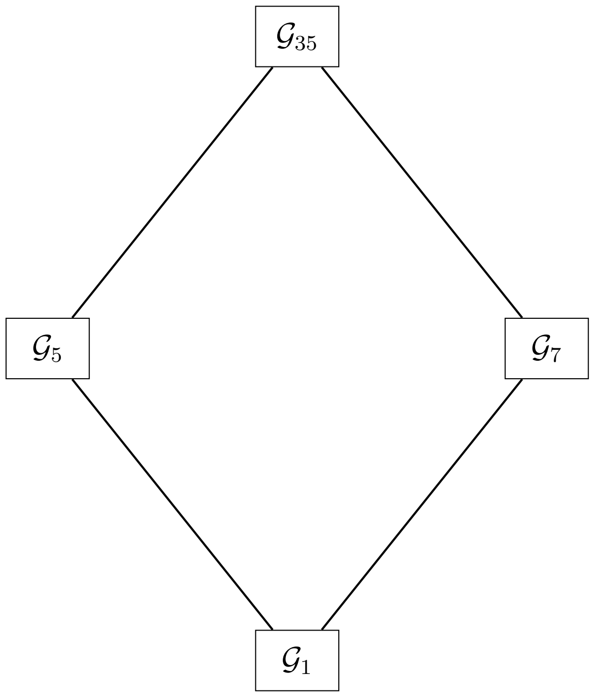


Table 5.17: A table of maximal lines in finite geometry $\mathcal{G}_{21}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{21}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6),$ $L(1,7), L(1,8), L(1,9), L(1,10), L(1,11), L(1,12)L(1,13), L(1,14),$ $L(1,15), L(1,16), L(1,17), L(1,18), L(1,19), L(1,20), L(3,1), L(3,2),$ $L(3,4), L(3,5), L(3,7), L(3,10), L(3,13), L(7,1), L(7,2), L(7,3)$ |
| $\mathcal{G}_7$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9), L(3,12), L(3,15), L(3,18)$ |
| $\mathcal{G}_3$ | $L(0,7), L(7,0), L(7,7), L(7,14)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.7: The Hasse diagram showing the geometry $\mathcal{G}_{21}$ and its subgeometries

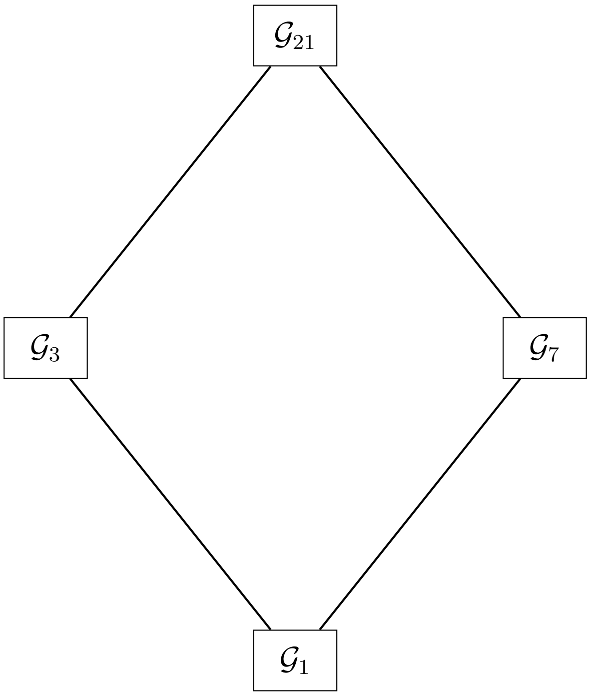


Table 5.18: A table of maximal lines in finite geometry $\mathcal{G}_{15}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{15}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7), L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(1,14), L(3,1), L(3,2), L(3,4), L(3,5), L(3,8), L(5,1), L(5,2), L(5,3)$ |
| $\mathcal{G}_5$ | $L(0,3), L(3,0), L(3,3), L(3,6), L(3,9), L(3,12)$ |
| $\mathcal{G}_3$ | $L(0,5), L(5,0), L(5,5), L(5,10)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.8: The Hasse diagram showing the geometry $\mathcal{G}_{15}$ and its subgeometries

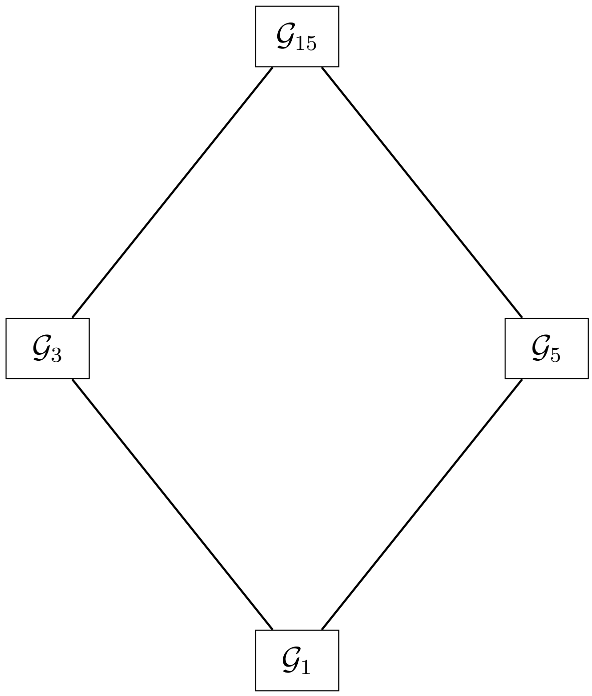


Table 5.19: A table of maximal lines in finite geometry $\mathcal{G}_{14}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{14}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6),$ $L(1,7), L(1,8), L(1,9), L(1,10), L(1,11), L(1,12), L(1,13), L(2,1),$ $L(2,3), L(2,5), L(2,7), L(2,9), L(2,11), L(2,13), L(7,1), L(7,2)$ |
| $\mathcal{G}_7$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8), L(2,10), L(2,12)$ |
| $\mathcal{G}_2$ | $L(0,7), L(7,0), L(7,7)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.9: The Hasse diagram showing the geometry $\mathcal{G}_{14}$ and its subgeometries

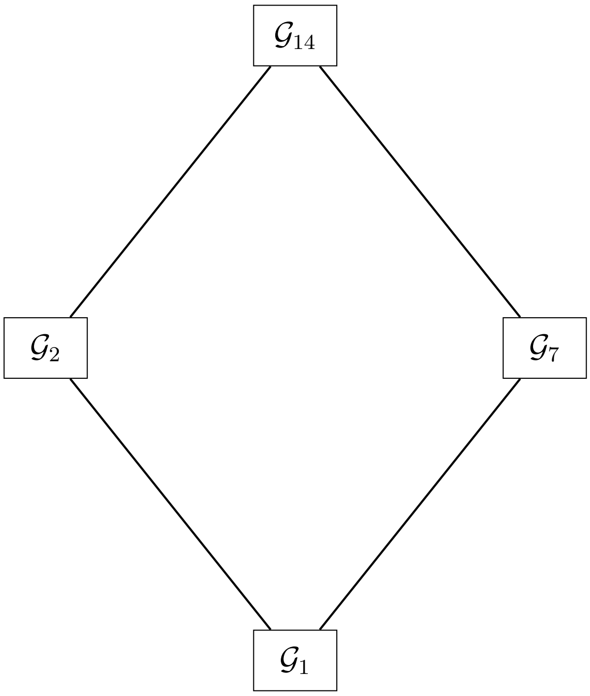


Table 5.20: A table of maximal lines in finite geometry $\mathcal{G}_{10}$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_{10}$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6), L(1,7), L(1,8), L(1,9), L(2,1), L(2,3), L(2,5), L(2,7), L(2,9), L(5,1), L(5,2).$ |
| $\mathcal{G}_5$ | $L(0,2), L(2,0), L(2,2), L(2,4), L(2,6), L(2,8)$ |
| $\mathcal{G}_2$ | $L(0,5), L(5,0), L(5,5)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.10: The Hasse diagram showing the geometry $\mathcal{G}_{10}$ and its subgeometries

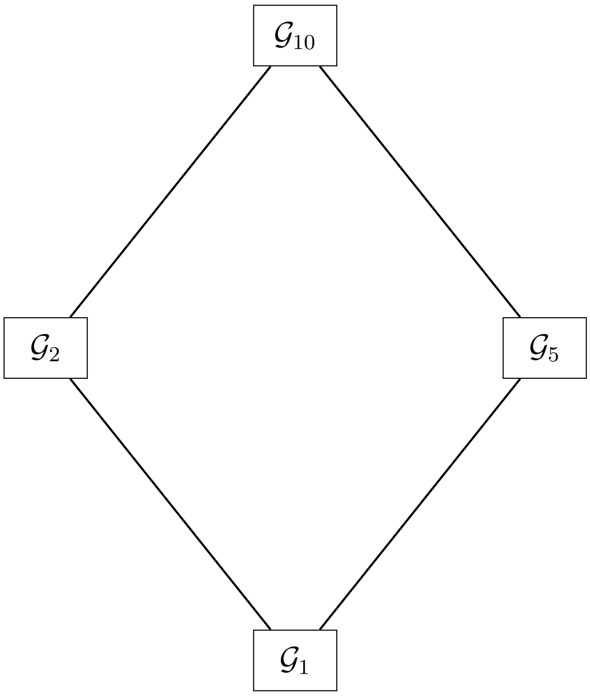


Table 5.21: A table of maximal lines in finite geometry $\mathcal{G}_6$ and its subgeometries.

| | |
|---|---|
| $\mathcal{G}_6$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(2,1), L(2,3), L(2,5), L(3,1), L(3,2).$ |
| $\mathcal{G}_3$ | $L(0,2), L(2,0), L(2,2), L(2,4)$ |
| $\mathcal{G}_2$ | $L(0,3), L(3,0), L(3,3)$ |
| $\mathcal{G}_1$ | $L(0,0)$ |

Figure 5.11: The Hasse diagram showing the geometry $\mathcal{G}_6$ and its subgeometries

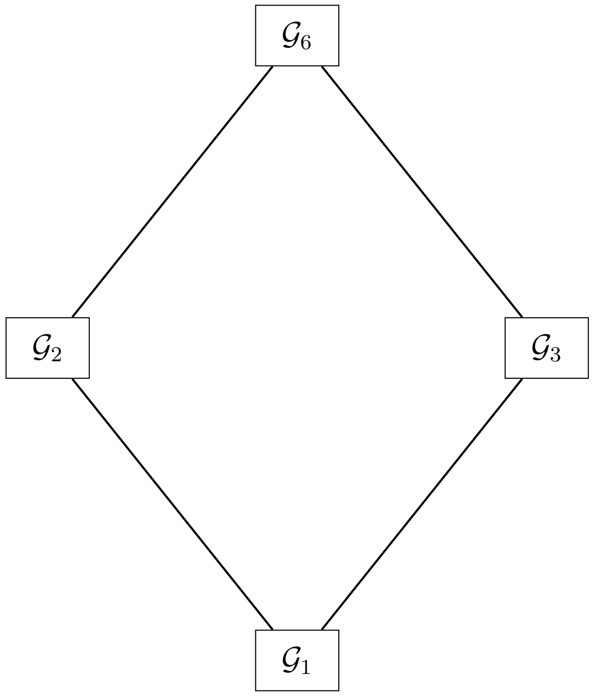


Table 5.22: A table of maximal lines in finite geometries

$\mathcal{G}_7, \mathcal{G}_5, \mathcal{G}_3,$ and $\mathcal{G}_1$ .

| | |
|---|---|
| $\mathcal{G}_7$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(1,6).$ |
| $\mathcal{G}_5$ | $L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4).$ |
| $\mathcal{G}_3$ | $L(0,1), L(1,0), L(1,1), L(1,2).$ |
| $\mathcal{G}_2$ | $L(0,1), L(1,0), L(1,1).$ |
| $\mathcal{G}_1$ | $L(0,0).$ |

Hence from the above examples, we confirm that there exists

$$|\mathbf{L}_d| = \sum_{\mathbf{w} \in \mathcal{D}(d)} \psi(\mathbf{w}) \tag{5.2}$$

maximal lines in the subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$ as follows;

(i) $\psi(105) = 192$ maximal lines in the geometry $\mathcal{G}_{105}$

(ii) $\psi(70) = 144$ maximal lines in the geometry $\mathcal{G}_{70}$

(iii) $\psi(42) = 96$ maximal lines in the geometry $\mathcal{G}_{42}$

(iv) $\psi(35) = 48$ maximal lines in the geometry $\mathcal{G}_{35}$

(v) $\psi(30) = 72$ maximal lines in the geometry $\mathcal{G}_{30}$

(vi) $\psi(21) = 32$ maximal lines in the geometry $\mathcal{G}_{21}$

(vi) $\psi(15) = 24$ maximal lines in the geometry $\mathcal{G}_{15}$

(vii) $\psi(14) = 24$ maximal lines in the geometry $\mathcal{G}_{14}$

(viii) $\psi(10) = 18$ maximal lines in the geometry $\mathcal{G}_{10}$

(ix) $\psi(7) = 8$ maximal lines in the geometry $\mathcal{G}_{7}$

(x) $\psi(6) = 12$ maximal lines in the geometry $\mathcal{G}_{6}$

(xi) $\psi(5) = 6$ maximal lines in the geometry $\mathcal{G}_{5}$

(xii) $\psi(3) = 4$ maximal lines in the geometry $\mathcal{G}_{3}$

(xiii) $\psi(2) = 3$ maximal lines in the geometry $\mathcal{G}_{2}$

This set that is $\{\mathcal{D}(d)\}$ is isomorphic to the set $\{\mathbf{G}_d\}$ of all subgometries of $\mathcal{G}_d$.

## 5.2 Factorization of line in terms of prime factor lines

In this section, we use our new derived notation discussed in chapter four to factorize all the maximal lines in $\mathcal{G}_{210}$ and subsets of the set $\{\mathcal{D}(d)\}$ in terms of their primes factors. It is demonstrated as follows;
suppose $d = 210$; using eqs.(3.50) – (3.53); we obtain the following values for $\mathbf{p_1} = 2, \mathbf{p_2} = 3, \mathbf{p_3} = 5, \mathbf{p_4} = 7, \mathbf{t_1} = 1, \mathbf{t_2} = 4, \mathbf{t_3} = 3, \mathbf{t_4} = 4, \mathbf{r_1} = 105, \mathbf{r_2} = 70, \mathbf{r_3} = 42$, and $\mathbf{r_4} = 30$.
Each maximal lines $\mathcal{G}_{210}$ is factorized in term of their primes. This is achieved in our work by using the method of Good [34]. The detail of the results is expressed in the table below for $\mathcal{G}_{210}$.

Table 5.23: Maximal lines in finite geometry $\mathcal{G}_{210}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{210}$ | $L(0,1) = \Gamma(-1,-1,-1,-1)$, | $L(1,0) = \Gamma(0,0,0,0)$, | $L(1,1) = \Gamma(1,1,3,4)$, |
| | $L(1,2) = \Gamma(0,2,1,1)$, | $L(1,3) = \Gamma(1,0,4,5)$, | $L(1,4) = \Gamma(0,1,2,2)$, |
| | $L(1,5) = \Gamma(1,2,0,6)$, | $L(1,6) = \Gamma(0,0,3,3)$, | $L(1,7) = \Gamma(1,1,1,0)$, |
| | $L(1,8) = \Gamma(0,2,4,4)$, | $L(1,9) = \Gamma(1,0,2,1)$, | $L(1,10) = \Gamma(0,1,0,5)$, |
| | $L(1,11) = \Gamma(1,2,3,2)$, | $L(1,12) = \Gamma(0,0,1,6)$, | $L(1,13) = \Gamma(1,1,4,3)$, |
| | $L(1,14) = \Gamma(0,2,2,0)$, | $L(1,15) = \Gamma(1,0,0,4)$, | $L(1,16) = \Gamma(0,1,3,1)$, |
| | $L(1,17) = \Gamma(1,2,1,5)$, | $L(1,18) = \Gamma(0,0,4,2)$, | $L(1,19) = \Gamma(1,1,2,6)$, |
| | $L(1,20) = \Gamma(0,2,0,3)$, | $L(1,21) = \Gamma(1,0,3,0)$, | $L(1,22) = \Gamma(0,1,1,4)$, |
| | $L(1,23) = \Gamma(1,2,4,1)$, | $L(1,24) = \Gamma(0,0,2,5)$, | $L(1,25) = \Gamma(1,1,0,2)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{210}$ | $L(1,26)=\Gamma(0,2,3,6)$, | $L(1,27)=\Gamma(1,0,1,3)$, | $L(1,28)=\Gamma(0,1,4,0)$, |
| | $L(1,29)=\Gamma(1,2,2,4)$, | $L(1,30)=\Gamma(0,0,0,1)$, | $L(1,31)=\Gamma(1,1,3,1)$, |
| | $L(1,32)=\Gamma(0,2,1,2)$, | $L(1,33)=\Gamma(1,0,4,6)$, | $L(1,34)=\Gamma(0,1,2,3)$, |
| | $L(1,35)=\Gamma(1,2,0,0)$, | $L(1,36)=\Gamma(0,0,3,4)$, | $L(1,37)=\Gamma(1,1,1,1)$, |
| | $L(1,38)=\Gamma(0,2,4,5)$, | $L(1,39)=\Gamma(1,0,2,2)$, | $L(1,40)=\Gamma(0,1,0,6)$, |
| | $L(1,41)=\Gamma(1,2,3,3)$, | $L(1,42)=\Gamma(0,0,1,0)$, | $L(1,43)=\Gamma(1,1,4,4)$, |
| | $L(1,44)=\Gamma(0,2,2,1)$, | $L(1,45)=\Gamma(1,0,0,5)$, | $L(1,46)=\Gamma(0,1,3,2)$, |
| | $L(1,47)=\Gamma(1,2,1,6)$, | $L(1,48)=\Gamma(0,0,4,3)$, | $L(1,49)=\Gamma(1,1,2,0)$, |
| | $L(1,50)=\Gamma(0,2,0,4)$, | $L(1,51)=\Gamma(1,0,3,1)$, | $L(1,52)=\Gamma(0,1,1,5)$, |
| | $L(1,53)=\Gamma(1,2,4,2)$, | $L(1,54)=\Gamma(0,0,2,6)$, | $L(1,55)=\Gamma(1,1,0,3)$, |
| | $L(1,56)=\Gamma(0,2,3,0)$, | $L(1,57)=\Gamma(1,0,1,4)$, | $L(1,58)=\Gamma(0,1,4,1)$, |
| | $L(1,59)=\Gamma(1,2,2,5)$, | $L(1,60)=\Gamma(0,0,0,2)$, | $L(1,61)=\Gamma(1,1,3,6)$, |
| | $L(1,62)=\Gamma(0,2,1,3)$, | $L(1,63)=\Gamma(1,0,4,0)$, | $L(1,64)=\Gamma(0,1,2,4)$, |
| | $L(1,65)=\Gamma(1,2,0,1)$, | $L(1,66)=\Gamma(0,0,3,5)$, | $L(1,67)=\Gamma(1,1,1,2)$, |
| | $L(1,68)=\Gamma(0,2,4,6)$, | $L(1,69)=\Gamma(1,0,2,3)$, | $L(1,70)=\Gamma(0,1,0,0)$ |
| | $L(1,71)=\Gamma(1,2,3,4)$, | $L(1,72)=\Gamma(0,0,1,1)$, | $L(1,73)=\Gamma(1,1,4,5)$, |
| | $L(1,74)=\Gamma(0,2,2,2)$, | $L(1,75)=\Gamma(1,0,0,6)$, | $L(1,76)=\Gamma(0,1,3,3)$, |
| | $L(1,77)=\Gamma(1,2,1,0)$, | $L(1,78)=\Gamma(0,0,4,4)$, | $L(1,79)=\Gamma(1,1,2,1)$, |
| | $L(1,80)=\Gamma(0,2,0,5)$, | $L(1,81)=\Gamma(1,0,3,2)$, | $L(1,82)=\Gamma(0,1,1,6)$, |
| | $L(1,83)=\Gamma(1,2,4,3)$, | $L(1,84)=\Gamma(0,0,2,0)$, | $L(1,85)=\Gamma(1,1,0,4)$, |
| | $L(1,86)=\Gamma(0,2,3,1)$, | $L(1,87)=\Gamma(1,0,1,5)$, | $L(1,88)=\Gamma(0,1,4,2)$, |
| | $L(1,89)=\Gamma(1,2,2,6)$, | $L(1,90)=\Gamma(0,0,0,3)$, | $L(1,91)=\Gamma(1,1,3,0)$, |
| | $L(1,92)=\Gamma(0,2,1,4)$, | $L(1,93)=\Gamma(1,0,4,1)$, | $L(1,94)=\Gamma(0,1,2,5)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{210}$ | $L(1,95) = \Gamma(1,2,0,2)$, | $L(1,96) = \Gamma(0,0,3,6)$, | $L(1,97) = \Gamma(1,1,1,3)$, |
| | $L(1,98) = \Gamma(0,2,4,0)$, | $L(1,99) = \Gamma(1,0,2,4)$, | $L(1,100) = \Gamma(0,1,0,1)$, |
| | $L(1,101) = \Gamma(1,2,3,5)$, | $L(1,102) = \Gamma(0,0,1,2)$, | $L(1,103) = \Gamma(1,1,4,6)$, |
| | $L(1,104) = \Gamma(0,2,2,3)$, | $L(1,105) = \Gamma(1,0,0,0)$, | $L(1,106) = \Gamma(0,1,3,4)$, |
| | $L(1,107) = \Gamma(1,2,1,1)$, | $L(1,108) = \Gamma(0,0,4,5)$, | $L(1,109) = \Gamma(1,1,2,2)$, |
| | $L(1,110) = \Gamma(0,2,0,6)$, | $L(1,111) = \Gamma(1,0,3,3)$, | $L(1,112) = \Gamma(0,1,1,0)$, |
| | $L(1,113) = \Gamma(1,2,4,4)$, | $L(1,114) = \Gamma(0,0,2,1)$, | $L(1,115) = \Gamma(1,1,0,5)$, |
| | $L(1,116) = \Gamma(0,2,3,2)$, | $L(1,117) = \Gamma(1,0,1,6)$, | $L(1,118) = \Gamma(0,1,4,3)$, |
| | $L(1,119) = \Gamma(1,2,2,0)$, | $L(1,120) = \Gamma(0,0,0,4)$, | $L(1,121) = \Gamma(1,1,3,1)$, |
| | $L(1,122) = \Gamma(0,2,1,5)$, | $L(1,123) = \Gamma(1,0,4,2)$, | $L(1,124) = \Gamma(0,1,2,6)$, |
| | $L(1,125) = \Gamma(1,2,0,3)$, | $L(1,126) = \Gamma(0,0,3,0)$, | $L(1,127) = \Gamma(1,1,1,4)$, |
| | $L(1,128) = \Gamma(0,2,4,1)$, | $L(1,129) = \Gamma(1,0,2,5)$, | $L(1,130) = \Gamma(0,1,0,2)$, |
| | $L(1,131) = \Gamma(1,2,3,6)$, | $L(1,132) = \Gamma(0,0,1,3)$, | $L(1,133) = \Gamma(1,1,4,0)$, |
| | $L(1,134) = \Gamma(0,2,2,4)$, | $L(1,135) = \Gamma(1,0,0,1)$, | $L(1,136) = \Gamma(0,1,3,5)$, |
| | $L(1,137) = \Gamma(1,2,1,2)$, | $L(1,138) = \Gamma(0,0,4,6)$, | $L(1,139) = \Gamma(1,1,2,3)$, |
| | $L(1,140) = \Gamma(0,2,0,0)$, | $L(1,141) = \Gamma(1,0,3,4)$, | $L(1,142) = \Gamma(0,1,1,1)$ |
| | $L(1,143) = \Gamma(1,2,4,5)$, | $L(1,144) = \Gamma(0,0,2,2)$, | $L(1,145) = \Gamma(1,1,0,6)$, |
| | $L(1,146) = \Gamma(0,2,3,3)$, | $L(1,147) = \Gamma(1,0,1,0)$, | $L(1,148) = \Gamma(0,1,4,4)$, |
| | $L(1,149) = \Gamma(1,2,2,1)$, | $L(1,150) = \Gamma(0,0,0,5)$, | $L(1,151) = \Gamma(1,1,3,2)$, |
| | $L(1,152) = \Gamma(0,2,1,6)$, | $L(1,153) = \Gamma(1,0,4,3)$, | $L(1,154) = \Gamma(0,1,2,0)$, |
| | $L(1,155) = \Gamma(1,2,0,4)$, | $L(1,156) = \Gamma(0,0,3,1)$, | $L(1,157) = \Gamma(1,1,1,5)$, |
| | $L(1,158) = \Gamma(0,2,4,2)$, | $L(1,159) = \Gamma(1,0,2,6)$, | $L(1,160) = \Gamma(0,1,0,3)$, |
| | $L(1,161) = \Gamma(1,2,3,0)$, | $L(1,162) = \Gamma(0,0,1,4)$, | $L(1,163) = \Gamma(1,1,4,1)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{210}$ | $L(1,164)=\Gamma(0,2,2,5)$, | $L(1,165)=\Gamma(1,0,0,2)$, | $L(1,166)=\Gamma(0,1,3,6)$, |
| | $L(1,167)=\Gamma(1,2,1,3)$, | $L(1,168)=\Gamma(0,0,4,0)$, | $L(1,169)=\Gamma(1,1,2,4)$, |
| | $L(1,170)=\Gamma(0,2,0,1)$, | $L(1,171)=\Gamma(1,0,3,5)$, | $L(1,172)=\Gamma(0,1,1,2)$, |
| | $L(1,173)=\Gamma(1,2,4,6)$, | $L(1,174)=\Gamma(0,0,2,3)$, | $L(1,175)=\Gamma(1,1,0,0)$, |
| | $L(1,176)=\Gamma(0,2,3,4)$, | $L(1,177)=\Gamma(1,0,1,1)$, | $L(1,178)=\Gamma(0,1,4,5)$, |
| | $L(1,179)=\Gamma(1,2,2,2)$, | $L(1,180)=\Gamma(0,0,0,6)$, | $L(1,181)=\Gamma(1,1,3,3)$, |
| | $L(1,182)=\Gamma(0,2,1,0)$, | $L(1,183)=\Gamma(1,0,4,4)$, | $L(1,184)=\Gamma(0,1,2,1)$, |
| | $L(1,185)=\Gamma(1,2,0,5)$, | $L(1,186)=\Gamma(0,0,3,2)$, | $L(1,187)=\Gamma(1,1,1,6)$, |
| | $L(1,188)=\Gamma(0,2,4,3)$, | $L(1,189)=\Gamma(1,0,2,0)$, | $L(1,190)=\Gamma(0,1,0,4)$, |
| | $L(1,191)=\Gamma(1,2,3,1)$, | $L(1,192)=\Gamma(0,0,1,5)$, | $L(1,193)=\Gamma(1,1,4,2)$, |
| | $L(1,194)=\Gamma(0,2,2,6)$, | $L(1,195)=\Gamma(1,0,0,3)$, | $L(1,196)=\Gamma(0,1,3,0)$, |
| | $L(1,197)=\Gamma(1,2,1,4)$, | $L(1,198)=\Gamma(0,0,4,1)$, | $L(1,199)=\Gamma(1,1,2,5)$, |
| | $L(1,200)=\Gamma(0,2,0,2)$, | $L(1,201)=\Gamma(1,0,3,6)$, | $L(1,202)=\Gamma(0,1,1,3)$, |
| | $L(1,203)=\Gamma(1,2,4,0)$, | $L(1,204)=\Gamma(0,0,2,4)$, | $L(1,205)=\Gamma(1,1,0,1)$, |
| | $L(1,206)=\Gamma(0,2,3,5)$, | $L(1,207)=\Gamma(1,0,1,2)$, | $L(1,208)=\Gamma(0,1,4,6)$, |
| | $L(1,209)=\Gamma(1,2,2,3)$, | $L(2,1)=\Gamma(-1,2,4,2)$, | $L(2,3)=\Gamma(-1,0,2,6)$, |
| | $L(2,5)=\Gamma(-1,1,0,3)$, | $L(2,7)=\Gamma(-1,2,3,0)$, | $L(2,9)=\Gamma(-1,0,1,4)$, |
| | $L(2,11)=\Gamma(-1,1,4,1)$, | $L(2,13)=\Gamma(-1,2,2,5)$, | $L(2,15)=\Gamma(-1,0,0,2)$, |
| | $L(2,17)=\Gamma(-1,1,3,6)$, | $L(2,19)=\Gamma(-1,2,1,3)$, | $L(2,21)=\Gamma(-1,0,4,0)$, |
| | $L(2,23)=\Gamma(-1,1,2,4)$, | $L(2,25)=\Gamma(-1,2,0,1)$, | $L(2,27)=\Gamma(-1,0,3,5)$, |
| | $L(2,29)=\Gamma(-1,1,1,2)$, | $L(2,31)=\Gamma(-1,2,4,6)$, | $L(2,33)=\Gamma(-1,0,2,3)$, |
| | $L(2,35)=\Gamma(-1,1,0,0)$, | $L(2,37)=\Gamma(-1,2,3,4)$, | $L(2,39)=\Gamma(-1,0,1,1)$, |
| | $L(2,41)=\Gamma(-1,1,4,5)$, | $L(2,47)=\Gamma(-1,1,3,3)$, | $L(2,49)=\Gamma(-1,2,1,0)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{210}$ | $L(2,51)=\Gamma(-1,0,4,4)$, | $L(2,53)=\Gamma(-1,1,2,1)$, | $L(2,55)=\Gamma(-1,2,0,5)$, |
| | $L(2,57)=\Gamma(-1,0,3,2)$, | $L(2,59)=\Gamma(-1,1,1,6)$, | $L(2,61)=\Gamma(-1,2,4,3)$, |
| | $L(2,63)=\Gamma(-1,0,2,0)$, | $L(2,65)=\Gamma(-1,1,0,4)$, | $L(2,67)=\Gamma(-1,2,3,1)$, |
| | $L(2,69)=\Gamma(-1,0,1,5)$, | $L(2,71)=\Gamma(-1,1,4,2)$, | $L(2,73)=\Gamma(-1,2,2,6)$, |
| | $L(2,75)=\Gamma(-1,0,0,3)$, | $L(2,77)=\Gamma(-1,1,3,0)$, | $L(2,79)=\Gamma(-1,2,1,4)$, |
| | $L(2,81)=\Gamma(-1,0,4,1)$, | $L(2,83)=\Gamma(-1,1,2,5)$, | $L(2,85)=\Gamma(-1,2,0,2)$, |
| | $L(2,87)=\Gamma(-1,0,3,6)$, | $L(2,89)=\Gamma(-1,1,1,3)$, | $L(2,91)=\Gamma(-1,2,4,0)$, |
| | $L(2,93)=\Gamma(-1,0,2,4)$, | $L(2,95)=\Gamma(-1,1,0,1)$, | $L(2,97)=\Gamma(-1,2,3,5)$, |
| | $L(2,99)=\Gamma(-1,0,1,2)$, | $L(2,101)=\Gamma(-1,1,4,6)$, | $L(2,103)=\Gamma(-1,2,2,3)$, |
| | $L(2,105)=\Gamma(-1,0,0,0)$, | $L(2,107)=\Gamma(-1,1,3,4)$, | $L(2,109)=\Gamma(-1,2,1,1)$ |
| | $L(2,111)=\Gamma(-1,0,4,5)$, | $L(2,113)=\Gamma(-1,1,2,2)$, | $L(2,115)=\Gamma(-1,2,0,6)$, |
| | $L(2,117)=\Gamma(-1,0,3,3)$, | $L(2,119)=\Gamma(-1,1,1,0)$, | $L(2,121)=\Gamma(-1,2,4,4)$, |
| | $L(2,123)=\Gamma(-1,0,2,1)$, | $L(2,125)=\Gamma(-1,1,0,5)$, | $L(2,127)=\Gamma(-1,2,3,2)$, |
| | $L(2,129)=\Gamma(-1,0,1,6)$, | $L(2,131)=\Gamma(-1,1,4,3)$, | $L(2,133)=\Gamma(-1,2,2,0)$, |
| | $L(2,135)=\Gamma(-1,0,0,4)$, | $L(2,137)=\Gamma(-1,1,3,1)$, | $L(2,139)=\Gamma(-1,2,1,5)$, |
| | $L(2,141)=\Gamma(-1,0,4,2)$, | $L(2,143)=\Gamma(-1,1,2,6)$, | $L(2,145)=\Gamma(-1,2,0,3)$, |
| | $L(2,147)=\Gamma(-1,0,3,0)$, | $L(2,149)=\Gamma(-1,1,1,4)$, | $L(2,151)=\Gamma(-1,2,4,1)$, |
| | $L(2,153)=\Gamma(-1,0,2,5)$, | $L(2,155)=\Gamma(-1,1,0,2)$, | $L(2,157)=\Gamma(-1,2,3,6)$, |
| | $L(2,159)=\Gamma(-1,0,1,3)$, | $L(2,161)=\Gamma(-1,1,4,0)$, | $L(2,163)=\Gamma(-1,2,2,4)$, |
| | $L(2,165)=\Gamma(-1,0,0,1)$, | $L(2,167)=\Gamma(-1,1,3,5)$, | $L(2,169)=\Gamma(-1,2,1,2)$, |
| | $L(2,171)=\Gamma(-1,0,4,6)$, | $L(2,173)=\Gamma(-1,1,2,3)$, | $L(2,175)=\Gamma(-1,2,0,0)$, |
| | $L(2,177)=\Gamma(-1,0,3,4)$, | $L(2,179)=\Gamma(-1,1,1,1)$, | $L(2,181)=\Gamma(-1,2,4,5)$, |
| | $L(2,183)=\Gamma(-1,0,2,2)$, | $L(2,185)=\Gamma(-1,1,0,6)$, | $L(2,187)=\Gamma(-1,2,3,3)$, |

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| $\mathcal{G}_{210}$ | $L(2,189) = \Gamma(-1,0,1,0)$, | $L(2,191) = \Gamma(-1,1,4,4)$, | $L(2,193) = \Gamma(-1,2,2,1)$, |
| | $L(2,195) = \Gamma(-1,0,0,5)$, | $L(2,197) = \Gamma(-1,1,3,2)$, | $L(2,199) = \Gamma(-1,2,1,6)$, |
| | $L(2,201) = \Gamma(-1,0,4,3)$, | $L(2,203) = \Gamma(-1,1,2,0)$, | $L(2,205) = \Gamma(-1,2,0,4)$, |
| | $L2,207) = \Gamma(-1,0,3,1)$, | $L(2,209) = \Gamma(-1,1,1,5)$, | $L(3,1) = \Gamma(1,-1,1,6)$, |
| | $L(3,2) = \Gamma(0,-1,2,5)$, | $L(3,4) = \Gamma(0,-1,4,3)$, | $L(3,5) = \Gamma(1,-1,0,2)$, |
| | $L(3,7) = \Gamma(1,-1,2,0)$, | $L(3,8) = \Gamma(0,-1,3,6)$, | $L(3,10) = \Gamma(0,-1,0,4)$, |
| | $L(3,11) = \Gamma(1,-1,1,3)$, | $L(3,13) = \Gamma(1,-1,3,1)$, | $L(3,14) = \Gamma(0,-1,4,0)$, |
| | $L(3,16) = \Gamma(0,-1,2,3)$, | $L(3,17) = \Gamma(1,-1,2,4)$, | $L(3,19) = \Gamma(1,-1,4,2)$, |
| | $L(3,20) = \Gamma(0,-1,0,1)$, | $L(3,22) = \Gamma(0,-1,2,6)$, | $L(3,23) = \Gamma(1,-1,3,5)$, |
| | $L(3,25) = \Gamma(1,-1,0,3)$, | $L(3,26) = \Gamma(0,-1,1,2)$, | $L(3,28) = \Gamma(0,-1,3,0)$, |
| | $L(3,29) = \Gamma(1,-1,4,6)$, | $L(3,31) = \Gamma(1,-1,1,4)$, | $L(3,32) = \Gamma(0,-1,2,3)$ |
| | $L(3,34) = \Gamma(0,-1,4,1)$, | $L(3,35) = \Gamma(1,-1,0,0)$, | $L(3,37) = \Gamma(1,-1,2,5)$, |
| | $L(3,38) = \Gamma(0,-1,3,4)$, | $L(3,40) = \Gamma(0,-1,0,2)$, | $L(3,41) = \Gamma(1,-1,1,1)$, |
| | $L(3,43) = \Gamma(1,-1,3,6)$, | $L(3,44) = \Gamma(0,-1,4,5)$, | $L(3,46) = \Gamma(0,-1,1,3)$, |
| | $L(3,47) = \Gamma(1,-1,2,2)$, | $L(3,49) = \Gamma(1,-1,4,0)$, | $L(3,50) = \Gamma(0,-1,0,6)$, |
| | $L(3,52) = \Gamma(0,-1,2,4)$, | $L(3,53) = \Gamma(1,-1,3,3)$, | $L(3,55) = \Gamma(1,-1,0,1)$, |
| | $L(3,56) = \Gamma(0,-1,1,0)$, | $L(3,58) = \Gamma(0,-1,3,5)$, | $L(3,59) = \Gamma(1,-1,4,4)$, |
| | $L(3,61) = \Gamma(1,-1,1,2)$, | $L(3,62) = \Gamma(0,-1,2,1)$, | $L(3,64) = \Gamma(0,-1,4,6)$, |
| | $L(3,65) = \Gamma(1,-1,0,5)$, | $L(3,67) = \Gamma(1,-1,2,3)$, | $L(3,68) = \Gamma(0,-1,3,2)$, |
| | $L(3,70) = \Gamma(0,-1,0,0)$, | $L(3,73) = \Gamma(1,-1,3,4)$, | $L(3,76) = \Gamma(0,-1,1,1)$, |
| | $L(3,79) = \Gamma(1,-1,4,5)$, | $L(3,82) = \Gamma(0,-1,2,2)$, | $L(3,85) = \Gamma(1,-1,0,6)$, |
| | $L(3,88) = \Gamma(0,-1,3,3)$, | $L(3,91) = \Gamma(1,-1,1,0)$, | $L(3,94) = \Gamma(0,-1,4,4)$, |
| | $L(3,97) = \Gamma(1,-1,2,1)$, | $L(3,100) = \Gamma(0,-1,0,5)$, | $L(3,103) = \Gamma(1,-1,3,2)$, |

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| $\mathcal{G}_{210}$ | $L(3,106) = \Gamma(0,-1,1,6)$, | $L(3,109) = \Gamma(1,-1,4,3)$, | $L(3,112) = \Gamma(0,-1,2,0)$, |
| | $L(3,115) = \Gamma(1,-1,0,4)$, | $L(3,118) = \Gamma(0,-1,3,1)$, | $L(3,121) = \Gamma(1,-1,1,5)$, |
| | $L(3,124) = \Gamma(0,-1,4,2)$, | $L(3,127) = \Gamma(1,-1,2,6)$, | $L(3,130) = \Gamma(0,-1,0,3)$, |
| | $L(3,133) = \Gamma(1,-1,3,0)$, | $L(3,136) = \Gamma(0,-1,1,4)$, | $L(3,139) = \Gamma(1,-1,4,1)$, |
| | $L(5,1) = \Gamma(1,2,-1,5)$, | $L(5,2) = \Gamma(0,1,-1,3)$, | $L(5,3) = \Gamma(1,0,-1,1)$, |
| | $L(5,4) = \Gamma(0,2,-1,6)$, | $L(5,6) = \Gamma(0,0,-1,2)$, | $L(5,7) = \Gamma(1,2,-1,0)$, |
| | $L(5,8) = \Gamma(0,1,-1,5)$, | $L(5,9) = \Gamma(1,0,-1,3)$, | $L(5,11) = \Gamma(1,1,-1,6)$, |
| | $L(5,12) = \Gamma(0,0,-1,4)$, | $L(5,13) = \Gamma(1,2,-1,2)$, | $L(5,14) = \Gamma(0,1,-1,0)$, |
| | $L(5,16) = \Gamma(0,2,-1,3)$, | $L(5,17) = \Gamma(1,1,-1,1)$, | $L(5,18) = \Gamma(0,0,-1,6)$, |
| | $L(5,19) = \Gamma(1,2,-1,4)$, | $L(5,21) = \Gamma(1,0,-1,0)$, | $L(5,22) = \Gamma(0,2,-1,5)$, |
| | $L(5,23) = \Gamma(1,1,-1,3)$, | $L(5,24) = \Gamma(0,0,-1,1)$, | $L(5,26) = \Gamma(0,1,-1,4)$, |
| | $L(5,27) = \Gamma(1,0,-1,2)$, | $L(5,28) = \Gamma(0,2,-1,0)$, | $L(5,29) = \Gamma(1,1,-1,5)$, |
| | $L(5,31) = \Gamma(1,2,-1,1)$, | $L(5,32) = \Gamma(0,1,-1,6)$, | $L(5,33) = \Gamma(1,0,-1,4)$, |
| | $L(5,34) = \Gamma(0,2,-1,2)$, | $L(5,36) = \Gamma(0,0,-1,5)$, | $L(5,37) = \Gamma(1,2,-1,3)$, |
| | $L(5,38) = \Gamma(0,1,-1,1)$, | $L(5,39) = \Gamma(1,0,-1,6)$, | $L(5,41) = \Gamma(1,1,-1,2)$, |
| | $L(5,42) = \Gamma(0,0,-1,0)$, | $L(5,47) = \Gamma(1,1,-1,4)$, | $L(5,52) = \Gamma(0,2,-1,1)$, |
| | $L(5,57) = \Gamma(1,0,-1,5)$, | $L(5,62) = \Gamma(0,1,-1,2)$, | $L(5,67) = \Gamma(1,2,-1,6)$, |
| | $L(5,72) = \Gamma(0,0,-1,3)$, | $L(5,77) = \Gamma(1,1,-1,0)$, | $L(5,82) = \Gamma(0,3,-1,4)$, |
| | $L(6,1) = \Gamma(-1,-1,3,3)$, | $L(6,5) = \Gamma(-1,-1,0,1)$, | $L(6,7) = \Gamma(-1,-1,1,0)$, |
| | $L(6,11) = \Gamma(-1,-1,3,5)$, | $L(6,13) = \Gamma(-1,-1,4,4)$, | $L(6,17) = \Gamma(-1,-1,1,2)$, |
| | $L(6,19) = \Gamma(-1,-1,2,1)$, | $L(6,23) = \Gamma(-1,-1,4,6)$, | $L(6,25) = \Gamma(-1,-1,0,5)$, |
| | $L(6,29) = \Gamma(-1,-1,2,3)$, | $L(6,31) = \Gamma - 1,-1,3,2)$, | $L(6,35) = \Gamma(-1,-1,0,0)$, |
| | $L(6,37) = \Gamma(-1,-1,1,6)$, | $L(6,41) = \Gamma(-1,-1,3,4)$, | $L(6,43) = \Gamma(-1,-1,4,3)$, |

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| $\mathcal{G}_{210}$ | $L(6,47)=\Gamma(-1,-1,1,1)$, | $L(6,49)=\Gamma(-1,-1,2,0)$, | $L(6,53)=\Gamma(-1,-1,4,5)$, |
| | $L(6,55)=\Gamma(-1,-1,0,4)$, | $L(6,59)=\Gamma(-1,-1,2,2)$, | $L(6,61)=\Gamma(-1,-1,3,1)$, |
| | $L(6,65)=\Gamma(-1,-1,0,6)$, | $L(6,67)=\Gamma(-1,-1,1,5)$, | $L(6,73)=\Gamma(-1,-1,4,2)$, |
| | $L(6,79)=\Gamma(-1,-1,2,6)$, | $L(6,85)=\Gamma(-1,-1,0,3)$, | $L(6,91)=\Gamma(-1,-1,3,0)$, |
| | $L(6,97)=\Gamma(-1,-1,1,4)$, | $L(6,103)=\Gamma(-1,-1,4,1)$, | $L(6,109)=\Gamma(-1,-1,2,5)$ |
| | $L(6,115)=\Gamma(-1,-1,0,2)$, | $L(6,121)=\Gamma(-1,-1,3,6)$, | $L(6,127)=\Gamma(-1,-1,1,3)$, |
| | $L(6,133)=\Gamma(-1,-1,4,0)$, | $L(6,139)=\Gamma(-1,-1,2,4)$, | $L(7,1)=\Gamma(1,1,4,-1)$, |
| | $L(7,2)=\Gamma(0,2,3,-1)$, | $L(7,3)=\Gamma(1,0,2,-1)$, | $L(7,4)=\Gamma(0,1,1,-1)$, |
| | $L(7,5)=\Gamma(1,2,0,-1)$, | $L(7,6)=\Gamma(0,0,4,-1)$, | $L(7,8)=\Gamma(0,2,2,-1)$, |
| | $L(7,9)=\Gamma(1,0,1,-1)$, | $L(7,10)=\Gamma(0,1,0,-1)$, | $L(7,11)=\Gamma(1,2,4,-1)$, |
| | $L(7,12)=\Gamma(0,0,3,-1)$, | $L(7,13)=\Gamma(1,1,2,-1)$, | $L(7,15)=\Gamma(1,0,0,-1)$, |
| | $L(7,16)=\Gamma(0,1,4,-1)$, | $L(7,17)=\Gamma(1,2,3,-1)$, | $L(7,18)=\Gamma(0,0,2,-1)$, |
| | $L(7,19)=\Gamma(1,1,1,-1)$, | $L(7,20)=\Gamma(0,2,0,-1)$, | $L(7,22)=\Gamma(0,1,3,-1)$, |
| | $L(7,23)=\Gamma(1,2,2,-1)$, | $L(7,24)=\Gamma(0,0,1,-1)$, | $L(7,25)=\Gamma(1,1,0,-1)$, |
| | $L(7,26)=\Gamma(0,2,4,-1)$, | $L(7,27)=\Gamma(1,0,3,-1)$, | $L(7,29)=\Gamma(1,2,1,-1)$, |
| | $L(7,30)=\Gamma(0,0,0,-1)$, | $L(7,37)=\Gamma(1,1,3,-1)$, | $L(7,44)=\Gamma(0,2,1,-1)$, |
| | $L(7,51)=\Gamma(1,0,4,-1)$, | $L(7,58)=\Gamma(0,1,2,-1)$, | $L(10,1)=\Gamma(-1,1,-1,6)$, |
| | $L(10,3)=\Gamma(-1,0,-1,4)$, | $L(10,7)=\Gamma(-1,1,-1,0)$, | $L(10,9)=\Gamma(-1,0,-1,5)$, |
| | $L(10,11)=\Gamma(-1,2,-1,3)$, | $L(10,13)=\Gamma(-1,1,-1,1)$, | $L(10,17)=\Gamma(-1,2,-1,4)$, |
| | $L(10,19)=\Gamma(-1,1,-1,2)$, | $L(10,21)=\Gamma(-1,0,-1,0)$, | $L(10,23)=\Gamma(-1,2,-1,5)$, |
| | $L(10,27)=\Gamma(-1,0,-1,1)$, | $L(10,29)=\Gamma(-1,2,-1,6)$, | $L(10,31)=\Gamma(-1,1,-1,4)$, |
| | $L(10,33)=\Gamma(-1,0,-1,2)$, | $L(10,37)=\Gamma(-1,1,-1,5)$, | $L(10,39)=\Gamma(-1,0,-1,3)$, |
| | $L(10,41)=\Gamma(-1,2,-1,1)$, | $L(10,47)=\Gamma(-1,2,-1,2)$, | $L(10,57)=\Gamma(-1,0,-1,6)$, |

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| $\mathcal{G}_{210}$ | $L(10,67)=\Gamma(-1,1,-1,3)$, | $L(10,77)=\Gamma(-1,2,-1,0)$, |
| | $L(14,1)=\Gamma(-1,2,2,-1)$, | $L(14,3)=\Gamma(-1,0,1,-1)$, |
| | $L(14,5)=\Gamma(-1,1,0,-1)$, | $L(14,9)=\Gamma(-1,0,3,-1)$, |
| | $L(14,11)=\Gamma(-1,1,2,-1)$, | $L(14,13)=\Gamma(-1,2,1,-1)$, |
| | $L(14,15)=\Gamma(-1,0,0,-1)$, | $L(14,17)=\Gamma(-1,1,4,-1)$, |
| | $L(14,19)=\Gamma(-1,2,3,-1)$, | $L(14,23)=\Gamma(-1,1,1,-1)$, |
| | $L(14,25)=\Gamma(-1,2,0,-1)$, | $L(14,27)=\Gamma(-1,0,4,-1)$, |
| | $L(14,29)=\Gamma(-1,1,3,-1)$, | $L(14,37)=\Gamma(-1,2,4,-1)$, |
| | $L(14,51)=\Gamma(-1,0,2,-1)$, | $L(15,1)=\Gamma(1,-1,-1,4)$, |
| | $L(15,2)=\Gamma(0,-1,-1,1)$, | $L(15,4)=\Gamma(0,-1,-1,2)$, |
| | $L(15,7)=\Gamma(1,-1,-1,0)$, | $L(15,8)=\Gamma(0,-1,-1,4)$, |
| | $L(15,11)=\Gamma(1,-1,-1,2)$, | $L(15,13)=\Gamma(1,-1,-1,3)$, |
| | $L(15,14)=\Gamma(0,-1,-1,0)$, | $L(15,17)=\Gamma(1,-1,-1,5)$, |
| | $L(15,19)=\Gamma(1,-1,-1,6)$, | $L(15,23)=\Gamma(1,-1,-1,1)$, |
| | $L(15,26)=\Gamma(0,-1,-1,6)$, | $L(15,34)=\Gamma(0,-1,-1,3)$, |
| | $L(15,38)=\Gamma(0,-1,-1,5)$, | $L(21,1)=\Gamma(1,-1,3,-1)$, |
| | $L(21,2)=\Gamma(0,-1,1,-1)$, | $L(21,4)=\Gamma(0,-1,2,-1)$, |
| | $L(21,5)=\Gamma(1,-1,0,-1)$, | $L(21,8)=\Gamma(0,-1,4,-1)$, |
| | $L(21,10)=\Gamma(0,-1,0,-1)$, | $L(21,13)=\Gamma(1,-1,4,-1)$, |
| | $L(21,16)=\Gamma(0,-1,3,-1)$, | $L(21,17)=\Gamma(1,-1,1,-1)$, |
| | $L(21,19)=\Gamma(1,-1,2,-1)$, | $L(30,1)=\Gamma(-1,-1,-1,2)$, |
| | $L(30,7)=\Gamma(-1,-1,-1,0)$, | $L(30,11)=\Gamma(-1,-1,-1,1)$, |
| | $L(30,13)=\Gamma(-1,-1,-1,5)$, | $L(30,17)=\Gamma(-1,-1,-1,6)$, |

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|---|---|---|
| $\mathcal{G}_{210}$ | $L(30,19)=\Gamma(-1,-1,-1,3)$, | $L(30,23)=\Gamma(-1,-1,-1,4)$, |
| | $L(35,1)=\Gamma(1,2,-1,-1)$, | $L(35,2)=\Gamma(0,1,-1,-1)$, |
| | $L(35,3)=\Gamma(1,0,-1,-1)$, | $L(35,4)=\Gamma(0,2,-1,-1)$, |
| | $L(35,6)=\Gamma(0,0,-1,-1)$, | $L(35,11)=\Gamma(1,1,-1,-1)$, |
| | $L(42,1)=\Gamma(-1,-1,4,-1)$, | $L(42,5)=\Gamma(-1,-1,0,-1)$, |
| | $L(42,13)=\Gamma(-1,-1,2,-1)$, | $L(42,17)=\Gamma(-1,-1,3,-1)$, |
| | $L(42,19)=\Gamma(-1,-1,1,-1)$, | $L(70,1)=\Gamma(-1,1,-1,-1)$, |
| | $L(70,3)=\Gamma(-1,0,1,-1)$, | $L(70,11)=\Gamma(-1,2,-1,-1)$, |
| | $L(2,43)=\Gamma(-1,2,2,2)$, | $L(105,1)=\Gamma(1,-1,-1,-1)$, |
| | $L(105,2)=\Gamma(0,-1,-1,-1)$, | $L(2,45)=\Gamma(-1,0,0,6)$. |

Suppose $d=105, \mathbf{p_1}=3, \mathbf{p_2}=5, \mathbf{p_3}=7, \mathbf{t_1}=2, \mathbf{t_2}=1, \mathbf{t_3}=1, \mathbf{r_1}=35, \mathbf{r_2}=21$, and $\mathbf{r_3}=15$, we obtain the following results.

Table 5.33: Maximal lines in finite geometry $\mathcal{G}_{105}$ in terms of its prime factor lines.

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|---|---|---|---|
| $\mathcal{G}_{105}$ | $L(0,1)=\Gamma(-1,-1,-1)$, | $L(1,0)=\Gamma(0,0,0)$, | $L(1,1)=\Gamma(2,1,1)$, |
| | $L(1,2)=\Gamma(1,2,2)$, | $L(1,3)=\Gamma(0,3,3)$, | $L(1,4)=\Gamma(2,4,4)$, |
| | $L(1,5)=\Gamma(1,0,5)$, | $L(1,6)=\Gamma(0,1,6)$, | $L(1,7)=\Gamma(2,2,0)$, |
| | $L(1,8)=\Gamma(1,3,1)$, | $L(1,9)=\Gamma(0,4,2)$, | $L(1,10)=\Gamma(2,0,3)$, |
| | $L(1,11)=\Gamma(1,1,4)$, | $L(1,12)=\Gamma(0,2,5)$, | $L(1,13)=\Gamma(2,3,6)$, |
| | $L(1,14)=\Gamma(1,4,0)$, | $L(1,15)=\Gamma(0,0,1)$, | $L(1,16)=\Gamma(2,1,2)$, |
| | $L(1,17)=\Gamma(1,2,3)$, | $L(1,18)=\Gamma(0,3,4)$, | $L(1,19)=\Gamma(2,4,5)$, |

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|---|---|---|---|
| $\mathcal{G}_{105}$ | $L(1,20)=\Gamma(1,0,6)$, | $L(1,21)=\Gamma(0,1,0)$, | $L(1,22)=\Gamma(2,2,1)$, |
| | $L(1,23)=\Gamma(1,3,2)$, | $L(1,24)=\Gamma(0,4,3)$, | $L(1,25)=\Gamma(2,0,4)$, |
| | $L(1,26)=\Gamma(1,1,5)$, | $L(1,27)=\Gamma(0,2,6)$, | $L(1,28)=\Gamma(2,3,0)$, |
| | $L(1,29)=\Gamma(1,4,1)$, | $L(1,30)=\Gamma(0,0,2)$, | $L(1,31)=\Gamma(2,1,3)$, |
| | $L(1,32)=\Gamma(1,2,4)$, | $L(1,33)=\Gamma(0,3,5)$, | $L(1,34)=\Gamma(2,4,6)$, |
| | $L(1,35)=\Gamma(1,0,0)$, | $L(1,36)=\Gamma(0,1,1)$, | $L(1,37)=\Gamma(2,2,2)$, |
| | $L(1,38)=\Gamma(1,3,3)$, | $L(1,39)=\Gamma(0,4,4)$, | $L(1,40)=\Gamma(2,0,5)$, |
| | $L(1,41)=\Gamma(1,1,6)$, | $L(1,42)=\Gamma(0,2,0)$, | $L(1,43)=\Gamma(2,3,1)$, |
| | $L(1,44)=\Gamma(1,4,2)$, | $L(1,45)=\Gamma(0,0,3)$, | $L(1,46)=\Gamma(2,1,4)$, |
| | $L(1,47)=\Gamma(1,2,5)$, | $L(1,48)=\Gamma(0,3,6)$, | $L(1,49)=\Gamma(2,4,0)$, |
| | $L(1,50)=\Gamma(1,0,1)$, | $L(1,51)=\Gamma(0,1,2)$, | $L(1,52)=\Gamma(2,2,3)$, |
| | $L(1,53)=\Gamma(1,3,4)$, | $L(1,54)=\Gamma(0,4,5)$, | $L(1,55)=\Gamma(2,0,6)$, |
| | $L(1,56)=\Gamma(1,1,0)$, | $L(1,57)=\Gamma(0,2,1)$, | $L(1,58)=\Gamma(2,3,2)$, |
| | $L(1,59)=\Gamma(1,4,3)$, | $L(1,60)=\Gamma(0,0,4)$, | $L(1,61)=\Gamma(2,1,5)$, |
| | $L(1,62)=\Gamma(1,2,6)$, | $L(1,63)=\Gamma(0,3,0)$, | $L(1,64)=\Gamma(2,4,1)$, |
| | $L(1,65)=\Gamma(1,0,2)$, | $L(1,66)=\Gamma(0,1,3)$, | $L(1,67)=\Gamma(2,2,4)$, |
| | $L(1,68)=\Gamma(1,3,5)$, | $L(1,69)=\Gamma(0,4,6)$, | $L(1,70)=\Gamma(2,0,0)$ |
| | $L(1,71)=\Gamma(1,1,1)$, | $L(1,72)=\Gamma(0,2,2)$, | $L(1,73)=\Gamma(2,3,3)$, |
| | $L(1,74)=\Gamma(1,4,4)$, | $L(1,75)=\Gamma(0,0,5)$, | $L(1,76)=\Gamma(2,1,6)$, |
| | $L(1,77)=\Gamma(1,2,0)$, | $L(1,78)=\Gamma(0,3,1)$, | $L(1,79)=\Gamma(2,4,2)$, |
| | $L(1,80)=\Gamma(1,0,3)$, | $L(1,81)=\Gamma(0,1,4)$, | $L(1,82)=\Gamma(2,2,5)$, |
| | $L(1,83)=\Gamma(1,3,6)$, | $L(1,84)=\Gamma(0,4,0)$, | $L(1,85)=\Gamma(2,0,1)$, |
| | $L(1,86)=\Gamma(1,1,2)$, | $L(1,87)=\Gamma(0,2,3)$, | $L(1,88)=\Gamma(2,3,4)$, |

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| $\mathcal{G}_{105}$ | $L(1,89) = \Gamma(1,4,5)$, | $L(1,90) = \Gamma(0,0,6)$, | $L(1,91) = \Gamma(2,1,0)$, |
| | $L(1,92) = \Gamma(1,2,1)$, | $L(1,93) = \Gamma(0,3,2)$, | $L(1,94) = \Gamma(2,4,3)$, |
| | $L(1,95) = \Gamma(1,0,4)$, | $L(1,96) = \Gamma(0,1,5)$, | $L(1,97) = \Gamma(2,2,6)$, |
| | $L(1,98) = \Gamma(1,3,0)$, | $L(1,99) = \Gamma(0,4,1)$, | $L(1,100) = \Gamma(2,0,2)$, |
| | $L(1,101) = \Gamma(1,1,3)$, | $L(1,102) = \Gamma(0,2,4)$, | $L(1,103) = \Gamma(2,3,5)$, |
| | $L(1,104) = \Gamma(1,4,6)$, | $L(3,1) = \Gamma(-1,2,5)$, | $L(3,2) = \Gamma(-1,4,3)$, |
| | $L(3,4) = \Gamma(-1,3,6)$, | $L(3,5) = \Gamma(-1,0,4)$, | $L(3,7) = \Gamma(-1,4,0)$, |
| | $L(3,8) = \Gamma(-1,1,5)$, | $L(3,10) = \Gamma(-1,0,1)$, | $L(3,11) = \Gamma(-1,2,6)$, |
| | $L(3,13) = \Gamma(-1,1,2)$, | $L(3,14) = \Gamma(-1,3,0)$, | $L(3,16) = \Gamma(-1,2,3)$, |
| | $L(3,17) = \Gamma(-1,4,1)$, | $L(3,19) = \Gamma(-1,3,4)$, | $L(3,20) = \Gamma(-1,0,2)$, |
| | $L(3,22) = \Gamma(-1,4,5)$, | $L(3,23) = \Gamma(-1,1,3)$, | $L(3,25) = \Gamma(-1,0,6)$, |
| | $L(3,26) = \Gamma(-1,2,4)$, | $L(3,28) = \Gamma(-1,1,0)$, | $L(3,29) = \Gamma(-1,3,5)$, |
| | $L(3,31) = \Gamma(-1,2,1)$, | $L(3,32) = \Gamma(-1,4,6)$, | $L(3,34) = \Gamma(-1,3,2)$, |
| | $L(3,35) = \Gamma(-1,0,0)$, | $L(3,38) = \Gamma(-1,1,1)$, | $L(3,41) = \Gamma(-1,2,2)$, |
| | $L(3,44) = \Gamma(-1,3,3)$, | $L(3,47) = \Gamma(-1,4,4)$, | $L(3,50) = \Gamma(-1,0,5)$, |
| | $L(3,53) = \Gamma(-1,1,6)$, | $L(3,56) = \Gamma(-1,2,0)$, | $L(3,59) = \Gamma(-1,3,1)$, |
| | $L(3,62) = \Gamma(-1,4,2)$, | $L(3,65) = \Gamma(-1,0,3)$, | $L(3,68) = \Gamma(-1,1,4)$, |
| | $L(5,1) = \Gamma(1,-1,3)$, | $L(5,2) = \Gamma(2,-1,6)$, | $L(5,3) = \Gamma(0,-1,2)$ |
| | $L(5,4) = \Gamma(1,-1,5)$, | $L(5,6) = \Gamma(0,-1,4)$, | $L(5,7) = \Gamma(1,-1,0)$, |
| | $L(5,8) = \Gamma(2,-1,3)$, | $L(5,9) = \Gamma(0,-1,6)$, | $L(5,11) = \Gamma(2,-1,5)$, |
| | $L(5,12) = \Gamma(0,-1,1)$, | $L(5,13) = \Gamma(1,-1,4)$, | $L(5,14) = \Gamma(2,-1,0)$, |
| | $L(5,16) = \Gamma(1,-1,6)$, | $L(5,17) = \Gamma(2,-1,2)$, | $L(5,18) = \Gamma(0,-1,5)$, |
| | $L(5,19) = \Gamma(1,-1,1)$, | $L(5,21) = \Gamma(0,-1,0)$, | $L(5,26) = \Gamma(2,-1,1)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{105}$ | $L(5,31) = \Gamma(1,-1,2)$, | $L(5,36) = \Gamma(0,-1,3)$, | $L(5,41) = \Gamma(2,-1,4)$, |
| | $L(7,1) = \Gamma(2,3,-1)$, | $L(7,2) = \Gamma(1,1,-1)$, | $L(7,3) = \Gamma(0,4,-1)$, |
| | $L(7,4) = \Gamma(2,2,-1)$, | $L(7,5) = \Gamma(1,0,-1)$, | $L(7,6) = \Gamma(0,3,-1)$, |
| | $L(7,8) = \Gamma(1,4,-1)$, | $L(7,9) = \Gamma(0,2,-1)$, | $L(7,10) = \Gamma(2,0,-1)$, |
| | $L(7,11) = \Gamma(1,3,-1)$, | $L(7,12) = \Gamma(0,1,-1)$, | $L(7,13) = \Gamma(2,4,-1)$, |
| | $L(7,15) = \Gamma(0,0,-1)$, | $L(7,22) = \Gamma(2,1,-1)$, | $L(7,29) = \Gamma(1,2,-1)$, |
| | $L(15,1) = \Gamma(-1,-1,1)$, | $L(15,2) = \Gamma(-1,-1,2)$, | $L(15,4) = \Gamma(-1,-1,4)$, |
| | $L(15,7) = \Gamma(-1,-1,0)$, | $L(15,13) = \Gamma(-1,-1,6)$, | $L(15,17) = \Gamma(-1,-1,3)$, |
| | $L(15,19) = \Gamma(-1,-1,5)$, | $L(21,1) = \Gamma(-1,1,-1)$, | $L(21,2) = \Gamma(-1,2,-1)$, |
| | $L(21,4) = \Gamma(-1,4,-1)$, | $L(21,5) = \Gamma(-1,0,-1)$, | $L(21,8) = \Gamma(-1,3,-1)$, |
| | $L(35,1) = \Gamma(1,-1,-1)$, | $L(35,2) = \Gamma(2,-1,-1)$, | $L(35,3) = \Gamma(0,-1,-1)$. |

Suppose $d = 70; \mathbf{p_1} = 2, \mathbf{p_2} = 5, \mathbf{p_3} = 7, \mathbf{r_1} = 35, \mathbf{r_2} = 14, \mathbf{r_3} = 10, \mathbf{t_1} = 1, \mathbf{t_2} = 4$, and $\mathbf{t_3} = 5$, we obtain the following results.

Table 5.37: Maximal lines in finite geometry $\mathcal{G}_{70}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{70}$ | $L(0,1) = \Gamma(-1,-1,-1)$, | $L(1,0) = \Gamma(0,0,0)$, | $L(1,1) = \Gamma(1,4,5)$, |
| | $L(1,2) = \Gamma(0,3,3)$, | $L(1,3) = \Gamma(1,2,1)$, | $L(1,4) = \Gamma(0,1,6)$, |
| | $L(1,8) = \Gamma(0,2,5)$, | $L(1,9) = \Gamma(1,1,3)$, | $L(1,10) = \Gamma(0,0,1)$, |
| | $L(1,11) = \Gamma(1,4,6)$, | $L(1,12) = \Gamma(0,3,4)$, | $L(1,13) = \Gamma(1,2,2)$, |
| | $L(1,14) = \Gamma(0,1,0)$, | $L(1,15) = \Gamma(1,0,5)$, | $L(1,16) = \Gamma(0,4,3)$, |
| | $L(1,17) = \Gamma(1,3,1)$, | $L(1,18) = \Gamma(0,2,6$, | $L(1,19) = \Gamma(1,1,4)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{70}$ | $L(1,20) = \Gamma(0,0,2)$, | $L(1,21) = \Gamma(1,4,0)$, | $L(1,22) = \Gamma(0,3,5)$, |
| | $L(1,23) = \Gamma(1,2,3)$, | $L(1,24) = \Gamma(0,1,1)$, | $L(1,25) = \Gamma(1,0,6)$, |
| | $L(1,26) = \Gamma(0,4,4)$, | $L(1,27) = \Gamma(1,3,2)$, | $L(1,28) = \Gamma(0,2,0)$, |
| | $L(1,29) = \Gamma(1,1,5)$, | $L(1,30) = \Gamma(0,0,3)$, | $L(1,31) = \Gamma(1,4,1)$, |
| | $L(1,32) = \Gamma(0,3,6)$, | $L(1,33) = \Gamma(1,2,4)$, | $L(1,34) = \Gamma(0,1,2)$, |
| | $L(1,35) = \Gamma(1,0,0)$, | $L(1,36) = \Gamma(0,4,5)$, | $L(1,37) = \Gamma(1,3,3)$, |
| | $L(1,38) = \Gamma(0,2,1)$, | $L(1,39) = \Gamma(1,1,6)$, | $L(1,40) = \Gamma(0,0,4)$, |
| | $L(1,41) = \Gamma(1,4,2)$, | $L(1,42) = \Gamma(0,3,0)$, | $L(1,43) = \Gamma(1,2,5)$, |
| | $L(1,44) = \Gamma(0,1,3)$, | $L(1,45) = \Gamma(1,0,1)$, | $L(1,46) = \Gamma(0,4,6)$, |
| | $L(1,47) = \Gamma(1,3,4)$, | $L(1,48) = \Gamma(0,2,2)$, | $L(1,49) = \Gamma(1,1,0)$, |
| | $L(1,50) = \Gamma(0,0,5)$, | $L(1,51) = \Gamma(1,4,3)$, | $L(1,52) = \Gamma(0,3,1)$, |
| | $L(1,53) = \Gamma(1,2,6)$, | $L(1,54) = \Gamma(0,1,4)$, | $L(1,55) = \Gamma(1,0,2)$, |
| | $L(1,56) = \Gamma(0,4,0)$, | $L(1,57) = \Gamma(1,3,5)$, | $L(1,58) = \Gamma(0,2,3)$, |
| | $L(1,59) = \Gamma(1,1,1)$, | $L(1,60) = \Gamma(0,0,6)$, | $L(1,61) = \Gamma(1,4,4)$, |
| | $L(1,62) = \Gamma(0,3,2)$, | $L(1,63) = \Gamma(1,2,0)$, | $L(1,64) = \Gamma(0,1,5)$, |
| | $L(1,65) = \Gamma(1,0,3)$, | $L(1,66) = \Gamma(0,4,1)$, | $L(1,67) = \Gamma(1,3,6)$, |
| | $L(1,68) = \Gamma(0,2,4)$, | $L(1,69) = \Gamma(1,1,2)$, | $L(2,1) = \Gamma(-1,2,6)$ |
| | $L(2,3) = \Gamma(-1,1,4)$, | $L(2,5) = \Gamma(-1,0,1)$, | $L(2,7) = \Gamma(-1,4,0)$, |
| | $L(2,9) = \Gamma(-1,3,5)$, | $L(2,11) = \Gamma(-1,2,3)$, | $L(2,13) = \Gamma(-1,1,1)$, |
| | $L(2,15) = \Gamma(-1,0,6)$, | $L(2,17) = \Gamma(-1,4,4)$, | $L(2,19) = \Gamma(-1,3,3)$, |
| | $L(2,21) = \Gamma(-1,2,0)$, | $L(2,23) = \Gamma(-1,1,5)$, | $L(2,25) = \Gamma(-1,0,3)$, |
| | $L(2,27) = \Gamma(-1,4,1)$, | $L(2,29) = \Gamma(-1,3,6)$, | $L(2,31) = \Gamma(-1,2,4)$, |
| | $L(2,33) = \Gamma(-1,1,2)$, | $L(2,35) = \Gamma(-1,0,0)$, | $L(2,37) = \Gamma(-1,4,5)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{70}$ | $L(2,39) = \Gamma(-1,3,3)$, | $L(2,41) = \Gamma(-1,2,1)$, | $L(2,43) = \Gamma(-1,1,6)$, |
| | $L(2,45) = \Gamma(-1,0,4)$, | $L(2,47) = \Gamma(-1,4,2)$, | $L(2,49) = \Gamma(-1,3,0)$, |
| | $L(2,51) = \Gamma(-1,2,5)$, | $L(2,53) = \Gamma(-1,1,3)$, | $L(2,55) = \Gamma(-1,0,1)$, |
| | $L(2,57) = \Gamma(-1,4,6)$, | $L(2,59) = \Gamma(-1,3,4)$, | $L(2,61) = \Gamma(-1,2,2)$, |
| | $L(2,63) = \Gamma(-1,1,0)$, | $L(2,65) = \Gamma(-1,0,5)$, | $L(2,67) = \Gamma(-1,4,3)$, |
| | $L(2,69) = \Gamma(-1,3,1)$, | $L(5,1) = \Gamma(1,-1,1)$, | $L(5,2) = \Gamma(0,-1,2)$, |
| | $L(5,3) = \Gamma(1,-1,3)$, | $L(5,4) = \Gamma(0,-1,3)$, | $L(5,6) = \Gamma(0,-1,6)$, |
| | $L(5,7) = \Gamma(1,-1,0)$, | $L(5,8) = \Gamma(0,-1,1)$, | $L(5,9) = \Gamma(1,-1,2)$, |
| | $L(5,11) = \Gamma(1,-1,4)$, | $L(5,12) = \Gamma(0,-1,5)$, | $L(5,13) = \Gamma(1,-1,6)$, |
| | $L(5,14) = \Gamma(0,-1,0)$, | $L(5,19) = \Gamma(1,-1,5)$, | $L(5,24) = \Gamma(0,-1,3)$, |
| | $L(7,1) = \Gamma(1,2,-1)$, | $L(7,2) = \Gamma(0,4,-1)$, | $L(7,3) = \Gamma(1,1,-1)$, |
| | $L(7,4) = \Gamma(0,3,-1)$, | $L(7,5) = \Gamma(1,0,-1)$, | $L(7,6) = \Gamma(0,2,-1)$, |
| | $L(7,8) = \Gamma(0,1,-1)$, | $L(7,9) = \Gamma(1,3,-1)$, | $L(7,10) = \Gamma(0,0,-1)$, |
| | $L(7,17) = \Gamma(1,4,-1)$, | $L(10,1) = \Gamma(-1,-1,4)$, | $L(10,3) = \Gamma(-1,-1,5)$, |
| | $L(10,7) = \Gamma(-1,-1,0)$, | $L(10,9) = \Gamma(-1-1,1)$, | $L(10,11) = \Gamma(-1,-1,2)$, |
| | $L(10,13) = \Gamma(-1,-1,3)$, | $L(10,19) = \Gamma(-1,-1,6)$, | $L(14,1) = \Gamma(-1,1,-1)$, |
| | $L(14,3) = \Gamma(-1,3,-1)$, | $L(14,5) = \Gamma(-1,0,-1)$, | $L(14,9) = \Gamma(-1,4,-1)$, |
| | $L(14,17) = \Gamma(-1,2,-1)$, | $L(35,1) = \Gamma(1,-1,-1)$, | $L(35,2) = \Gamma(0,-1,-1)$ |

Suppose $d = 15, \mathbf{p_1} = 3, \mathbf{p_2} = 5, \mathbf{t_1} = 2, \mathbf{t_2} = 2, \mathbf{r_1} = 5$, and $\mathbf{r_2} = 3$, we obtain the following results.

Table 5.40: Maximal lines in finite geometry $\mathcal{G}_{15}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{15}$ | $L(0,1) = \Gamma(-1,-1)$, | $L(1,0) = \Gamma(0,0)$, | $L(1,1) = \Gamma(2,2)$, |
| | $L(1,2) = \Gamma(1,4)$, | $L(1,3) = \Gamma(0,1)$, | $L(1,4) = \Gamma(2,3)$, |
| | $L(1,5) = \Gamma(1,0)$, | $L(1,6) = \Gamma(0,2)$, | $L(1,7) = \Gamma(2,4)$, |
| | $L(1,8) = \Gamma(1,1)$, | $L(1,9) = \Gamma(0,3)$, | $L(1,10) = \Gamma(2,0)$, |
| | $L(1,11) = \Gamma(1,2)$, | $L(1,12) = \Gamma(0,4)$, | $L(1,13) = \Gamma(2,1)$, |
| | $L(1,14) = \Gamma(1,3)$, | $L(3,1) = \Gamma(-1,4)$, | $L(3,2) = \Gamma(-1,3)$, |
| | $L(3,4) = \Gamma(-1,1)$, | $L(3,5) = \Gamma(-1,0)$, | $L(3,8) = \Gamma(-1,2)$, |
| | $L(5,1) = \Gamma(1,-1)$, | $L(5,2) = \Gamma(2,-1)$, | $L(5,3) = \Gamma(0,-1)$. |

Suppose $d = 35, \mathbf{p_1} = 5, \mathbf{p_2} = 7, \mathbf{t_1} = 3, \mathbf{t_2} = 3, \mathbf{r_1} = 7$, and $\mathbf{r_2} = 5$, we obtain the following results.

Table 5.41: Maximal lines in finite geometry $\mathcal{G}_{35}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{35}$ | $L(0,1) = \Gamma(-1,-1)$, | $L(1,0) = \Gamma(0,0)$, | $L(1,1) = \Gamma(3,3)$, |
| | $L(1,2) = \Gamma(1,6)$, | $L(1,3) = \Gamma(4,2)$, | $L(1,4) = \Gamma(2,5)$, |
| | $L(1,5) = \Gamma(0,1)$, | $L(1,6) = \Gamma(3,4)$, | $L(1,7) = \Gamma(1,0)$, |
| | $L(1,8) = \Gamma(4,3)$, | $L(1,9) = \Gamma(2,6)$, | $L(1,10) = \Gamma(0,2)$, |
| | $L(1,11) = \Gamma(3,5)$, | $L(1,12) = \Gamma(1,1)$, | $L(1,13) = \Gamma(4,4)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{35}$ | $L(1,14) = \Gamma(2,0)$, | $L(1,15) = \Gamma(0,3)$, | $L(1,16) = \Gamma(3,6)$, |
| | $L(1,17) = \Gamma(1,2)$, | $L(1,18) = \Gamma(4,5)$, | $L(1,19) = \Gamma(2,1)$, |
| | $L(1,20) = \Gamma(0,4)$, | $L(1,21) = \Gamma(3,0)$, | $L(1,22) = \Gamma(1,3)$, |
| | $L(1,23) = \Gamma(4,6)$, | $L(1,24) = \Gamma(2,2)$, | $L(1,25) = \Gamma(0,5)$, |
| | $L(1,26) = \Gamma(3,1)$, | $L(1,27) = \Gamma(1,4)$, | $L(1,28) = \Gamma(4,0)$, |
| | $L(1,29) = \Gamma(2,3)$, | $L(1,30) = \Gamma(0,6)$, | $L(1,31) = \Gamma(3,2)$, |
| | $L(1,32) = \Gamma(1,5)$, | $L(1,33) = \Gamma(4,1)$, | $L(1,34) = \Gamma(2,4)$, |
| | $L(5,1) = \Gamma(-1,2)$, | $L(5,2) = \Gamma(-1,4)$, | $L(5,3) = \Gamma(-1,6)$, |
| | $L(5,4) = \Gamma(-1,1)$, | $L(5,6) = \Gamma(-1,5)$, | $L(5,7) = \Gamma(-1,0)$, |
| | $L(5,12) = \Gamma(-1,3)$, | $L(7,1) = \Gamma(4,-1)$, | $L(7,2) = \Gamma(3,-1)$, |
| | $L(7,3) = \Gamma(2,-1)$, | $L(7,4) = \Gamma(1,-1)$, | $L(7,5) = \Gamma(0,-1)$, |

Suppose $d = 14; \mathbf{p_1} = 2, \mathbf{p_2} = 7, \mathbf{t_1} = 1, \mathbf{t_2} = 4, \mathbf{r_1} = 7$, and $\mathbf{r_2} = 2$ we obtain the following results

Table 5.43: Maximal lines in finite geometry $\mathcal{G}_{14}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{14}$ | $L(0,1) = \Gamma(-1,-1)$, | $L(1,0) = \Gamma(0,0)$, | $L(1,1) = \Gamma(1,4)$, |
| | $L(1,2) = \Gamma(0,1)$, | $L(1,3) = \Gamma(1,5)$, | $L(1,4) = \Gamma(0,2)$, |
| | $L(1,5) = \Gamma(1,6)$, | $L(1,6) = \Gamma(0,3)$, | $L(1,7) = \Gamma(1,0)$, |
| | $L(1,8) = \Gamma(0,4)$, | $L(1,9) = \Gamma(1,1)$, | $L(1,10) = \Gamma(0,5)$, |
| | $L(1,11) = \Gamma(1,2)$, | $L(1,12) = \Gamma(0,6)$, | $L(1,13) = \Gamma(1,4)$, |
| | $L(2,1) = \Gamma(-1,2)$, | $L(2,3) = \Gamma(-1,6)$, | $L(2,5) = \Gamma(-1,3)$, |

| | | | |
|---|---|---|---|
| $\mathcal{G}_{14}$ | $L(2,7)=\Gamma(-1,0)$, | $L(2,9)=\Gamma(-1,4)$, | $L(2,11)=\Gamma(-1,1)$, |
| | $L(2,13)=\Gamma(-1,5)$, | $L(7,1)=\Gamma(1,-1)$, | $L(7,2)=\Gamma(0,-1)$. |

Suppose $d=21, \mathbf{p_1}=3, \mathbf{p_2}=7, \mathbf{t_1}=1, \mathbf{t_2}=5, \mathbf{r_1}=7$, and $\mathbf{r_2}=3$, we obtain the following results

Table 5.45: Maximal lines in finite geometry $\mathcal{G}_{21}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{21}$ | $L(0,1)=\Gamma(-1,-1)$, | $L(1,0)=\Gamma(0,0)$, | $L(1,1)=\Gamma(1,5)$, |
| | $L(1,2)=\Gamma(2,3)$, | $L(1,3)=\Gamma(0,1)$, | $L(1,4)=\Gamma(1,6)$, |
| | $L(1,5)=\Gamma(2,4)$, | $L(1,6)=\Gamma(0,2)$, | $L(1,7)=\Gamma(1,0)$, |
| | $L(1,8)=\Gamma(2,5)$, | $L(1,9)=\Gamma(0,3)$, | $L(1,10)=\Gamma(1,1)$, |
| | $L(1,11)=\Gamma(2,6)$, | $L(1,12)=\Gamma(0,4)$, | $L(1,13)=\Gamma(1,2)$. |
| | $L(1,14)=\Gamma(2,0)$, | $L(1,15)=\Gamma(0,5)$, | $L(1,16)=\Gamma(1,3)$, |
| | $L(1,17)=\Gamma(2,1)$, | $L(1,18)=\Gamma(0,6)$, | $L(1,19)=\Gamma(1,4)$, |
| | $L(1,20)=\Gamma(2,2)$, | $L(3,1)=\Gamma(-1,4)$, | $L(3,2)=\Gamma(-1,1)$, |
| | $L(3,4)=\Gamma(-1,2)$, | $L(3,5)=\Gamma(-1,6)$, | $L(3,7)=\Gamma(-1,0)$, |
| | $L(3,10)=\Gamma(-1,5)$, | $L(3,13)=\Gamma(-1,3)$, | $L(7,1)=\Gamma(1,-1)$, |
| | $L(7,2)=\Gamma(2,-1)$, | $L(7,3)=\Gamma(0,-1)$. | |

Suppose $d = 6, \mathbf{p_1} = 2, \mathbf{p_2} = 3, \mathbf{t_1} = 1, \mathbf{t_2} = 2, \mathbf{r_1} = 3$, and $\mathbf{r_2} = 2$

Table 5.46: Maximal lines in finite geometries $\mathcal{G}_6$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_6$ | $L(0,1) = \Gamma(-1,-1)$, | $L(1,0) = \Gamma(0,0)$, | $L(1,1) = \Gamma(1,2)$, |
| | $L(1,2) = \Gamma(0,1)$, | $L(1,3) = \Gamma(1,0)$, | $L(1,4) = \Gamma(0,2)$, |
| | $L(1,5) = \Gamma(1,1)$, | $L(2,1) = \Gamma(-1,1)$, | $L(2,3) = \Gamma(-1,0)$, |
| | $L(2,5) = \Gamma(-1,2)$, | $L(3,1) = \Gamma(1,-1)$, | $L(3,2) = \Gamma(0,-1)$. |

For $d = 10, \mathbf{p_1} = 2, \mathbf{p_2} = 5, \mathbf{t_1} = 1, \mathbf{t_2} = 3, \mathbf{r_1} = 5$, and $\mathbf{r_2} = 2$, we obtain the following results

Table 5.47: Maximal lines in finite geometry $\mathcal{G}_{10}$ in terms of its prime factor lines.

| | | | |
|---|---|---|---|
| $\mathcal{G}_{10}$ | $L(0,1) = \Gamma(-1,-1)$, | $L(1,0) = \Gamma(0,0)$, | $L(1,1) = \Gamma(1,3)$, |
| | $L(1,2) = \Gamma(0,1)$, | $L(1,3) = \Gamma(1,4)$, | $L(1,4) = \Gamma(0,2)$, |
| | $L(1,5) = \Gamma(1,0)$, | $L(1,6) = \Gamma(0,3)$, | $L(1,7) = \Gamma(1,1)$, |
| | $L(1,8) = \Gamma(0,4)$, | $L(1,9) = \Gamma(1,2)$, | $L(2,1) = \Gamma(-1,4)$, |
| | $L(2,3) = \Gamma(-1,2)$, | $L(2,5) = \Gamma(-1,0)$, | $L(2,7) = \Gamma(-1,3)$, |
| | $L(2,9) = \Gamma(-1,1)$, | $L(5,1) = \Gamma(1,-1)$, | $L(5,2) = \Gamma(0,-1)$. |

Table 5.48: Maximal lines in finite geometries $\mathcal{G}_7, \mathcal{G}_5, \mathcal{G}_3, \mathcal{G}_2, \mathcal{G}_1$.

| | | | |
|---|---|---|---|
| $\mathcal{G}_7$ | $L(0,1) = \Gamma(-1)$,<br>$L(1,2) = \Gamma(2)$<br>$L(1,5) = \Gamma(5)$, | $L(1,0) = \Gamma(0)$,<br>$L(1,3) = \Gamma(3)$,<br>$L(1,6) = \Gamma(6)$. | $L(1,1) = \Gamma(1)$,<br>$L(1,4) = \Gamma(4)$ |
| $\mathcal{G}_5$ | $L(0,1) = \Gamma(-1)$,<br>$L(1,2) = \Gamma(2)$, | $L(1,0) = \Gamma(0)$,<br>$L(1,3) = \Gamma(3)$, | $L(1,1) = \Gamma(1)$,<br>$L(1,4) = \Gamma(4)$. |
| $\mathcal{G}_3$ | $L(0,1) = \Gamma(-1)$,<br>$L(1,2) = \Gamma(2)$ | $L(1,0) = \Gamma(0)$, | $L(1,1) = \Gamma(1)$, |
| $\mathcal{G}_2$ | $L(0,1) = \Gamma(-1)$, | $L(1,0) = \Gamma(0)$, | $L(1,1) = \Gamma(1)$ |
| $\mathcal{G}_1$ | $L(0,0) = \Gamma(0)$ | | |

## 5.3 Weak mutually unbiased bases ($\mathcal{WMUB}$)

In this section, we factorize bases in a finite dimensional Hilbert space $\mathcal{H}_d$ as products of spaces $\bigotimes_{\mathbf{j=1}}^{\mathbf{k}} \mathcal{H}_{\mathbf{p_j}}$ where $\mathbf{p}$ is a prime number. It is called the weak mutually unbiased bases ($\mathcal{WMUB}$) and was formulated by [23]. Using our new derived notation discussed in eq.(4.33), we demonstrate the existence of duality between lines in finite geometry $\mathcal{G}_d$ and $\mathcal{WMUB}$ in finite Hilbert space $\mathcal{H}_d$. This is achieved in our work by using the scheme of [34] in fast Fourier transform explained earlier in chapter four using an example. More of such examples are shown below. However, due to insufficient space to show-case for finitely dimensional Hilbert space $\mathcal{H}_{210} = \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_5 \otimes \mathcal{H}_7$ in this write up. We show this concept only for subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$ for $d = 210$.

Suppose $q = 70 = 2 \times 5 \times 7$, the 144 maximal number of weak mutually unbiased are summarised thus;

Table 5.49: Weak mutually unbiased bases for $\mathcal{H}_{70} = \mathcal{H}_2 \otimes \mathcal{H}_5 \otimes \mathcal{H}_7$ .

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(-1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,5);m\rangle$ | $\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,3,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,4,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert -1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(0,0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(0,3,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,3,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,3,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,0);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,0);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\lvert\mathcal{B}(1,1,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\lvert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,0);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\lvert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |

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| $\vert\mathcal{B}(-1,0,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,5);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,5);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,5);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,3,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,3,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,3,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |

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| $\|\mathcal{B}(-1,3,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |

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| $\|\mathcal{B}(1,-1,5);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,6);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6;\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,1,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,3,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,4,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |

Table 5.56: Weak mutually unbiased bases for $\mathcal{H}_{105} = \mathcal{H}_3 \otimes \mathcal{H}_5 \otimes \mathcal{H}_7$ .

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(-1,-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,-1,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(-1,-1,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,3,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,4,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert -1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert -1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert -1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert -1,6);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(0,2,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,3,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,4,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(1,0,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,5);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,6);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,5);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,6);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,5);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,6);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,3,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,3,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,3,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,3,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |

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| $\lvert\mathcal{B}(1,3,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,3,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,3}(0,1\lvert-1,3);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,5);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,4,6);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\lvert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,4}(0,1\lvert-1,4);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,0);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,1);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,2);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,3);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,4);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,5);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,0,6);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\lvert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,6}(0,1\lvert-1,6);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,0);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\lvert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,1);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\lvert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,2);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\lvert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,3);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\lvert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,4);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\lvert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(2,1,5);m\rangle$ | $\lvert\mathcal{X}_{1,2}(0,1\lvert-1,2);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\lvert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,5}(0,1\lvert-1,5);\bar{m}_3\rangle$ |

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| $\|\mathcal{B}(2,1,6);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,0);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,2);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,3);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,4);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,5);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,2,6);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,0);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,2);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,3);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,4);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,5);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,3,6);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,0);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,2);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,3);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,4);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,5);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(2,4,6);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |

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| $\|\mathcal{B}(-1,0,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |

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| $\|\mathcal{B}(-1,3,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,3,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,4,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,5}(0,1\|-1,5);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,6}(0,1\|-1,6);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |

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| $\vert\mathcal{B}(1,-1,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,-1,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6;\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,0);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,2);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,3);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,4);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,5);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,5}(0,1\vert-1,5);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,-1,6);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,6}(0,1\vert-1,6);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,3,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,4,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,3,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,4,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,3,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(2,4,-1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |

Table 5.65: Weak mutually unbiased bases for $\mathcal{H}_{30} = \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_5$.

| | | | |
|---|---|---|---|
| $\vert\mathcal{B}(-1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,-1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,0,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(-1,2,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(1,-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(1,0\vert 0,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,-1}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,0,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,0}(0,1\vert -1,0);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,1}(0,1\vert -1,1);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,2}(0,1\vert -1,2);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,3}(0,1\vert -1,3);\bar{m}_3\rangle$ |
| $\vert\mathcal{B}(0,1,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ | $\vert\mathcal{X}_{3,4}(0,1\vert -1,4);\bar{m}_3\rangle$ |

| | | | |
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| $\lvert\mathcal{B}(0,2,0);m\rangle$ | $\lvert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(0,2,1);m\rangle$ | $\lvert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(0,2,2);m\rangle$ | $\lvert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(0,2,3);m\rangle$ | $\lvert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(0,2,4);m\rangle$ | $\lvert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,0,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,0,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,0,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,0,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,0,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,1,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,0);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,1);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,2);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,3);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,3}(0,1\vert-1,3);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(1,2,4);m\rangle$ | $\lvert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,4}(0,1\vert-1,4);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,0);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,0}(0,1\vert-1,0);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,1);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,1}(0,1\vert-1,1);\bar{m}_3\rangle$ |
| $\lvert\mathcal{B}(-1,0,2);m\rangle$ | $\lvert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\lvert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ | $\lvert\mathcal{X}_{3,2}(0,1\vert-1,2);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(-1,0,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,0,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(-1,2,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,-1,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,0}(0,1\|-1,0);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,1}(0,1\|-1,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,2}(0,1\|-1,2);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,3}(0,1\|-1,3);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,-1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,4}(0,1\|-1,4);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |

| | | | |
|---|---|---|---|
| $\|\mathcal{B}(0,1,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(0,2,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,0,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |
| $\|\mathcal{B}(1,2,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ | $\|\mathcal{X}_{3,-1}(1,0\|0,1);\bar{m}_3\rangle$ |

Table 5.69: Weak mutually unbiased bases for $\mathcal{H}_6 = \mathcal{H}_2 \otimes \mathcal{H}_3$.

| | | |
|---|---|---|
| $\|\mathcal{B}(-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,0);m\rangle$ | $\|\mathcal{X}_{1,0}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |

Table 5.70: Weak mutually unbiased bases for $\mathcal{H}_{21} = \mathcal{H}_3 \otimes \mathcal{H}_7$.

| | | |
|---|---|---|
| $\|\mathcal{B}(-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,5}(0,1\|-1,5);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,6}(0,1\|-1,6);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,-1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,5}(0,1\|-1,5);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,6}(0,1\|-1,6);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\vert\mathcal{B}(1,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,0);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,2);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,3);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,4);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,5);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,6);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |

Table 5.72: Weak mutually unbiased bases for $\mathcal{H}_{15} = \mathcal{H}_3 \otimes \mathcal{H}_5$.

| | | |
|---|---|---|
| $\vert\mathcal{B}(-1,-1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,0);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,1);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,2);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert-1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\|\mathcal{B}(2,-1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,0);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,2);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,3);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,4);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |

Table 5.74: Weak mutually unbiased bases for $\mathcal{H}_{10} = \mathcal{H}_2 \otimes \mathcal{H}_5$.

| | | |
|---|---|---|
| $\|\mathcal{B}(-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\|\mathcal{B}(-1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,2);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,3);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,4);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |

Table 5.76: Weak mutually unbiased bases for $\mathcal{H}_{14} = \mathcal{H}_2 \otimes \mathcal{H}_7$.

| | | |
|---|---|---|
| $\|\mathcal{B}(-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,2);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\vert\mathcal{B}(-1,3);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,4);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert -1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,5);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert -1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(-1,6);m\rangle$ | $\vert\mathcal{X}_{1,-1}(1,0\vert 0,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert -1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,-1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,-1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,-1}(1,0\vert 0,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,0);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,1);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,2);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,3);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,4);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert -1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,5);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert -1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(0,6);m\rangle$ | $\vert\mathcal{X}_{1,0}(0,1\vert -1,0);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert -1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,0);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert -1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert -1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert -1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert -1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert -1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert -1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert -1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert -1,6);\bar{m}_2\rangle$ |

Table 5.78: Weak mutually unbiased bases for $\mathcal{H}_{35} = \mathcal{H}_5 \otimes \mathcal{H}_7$.

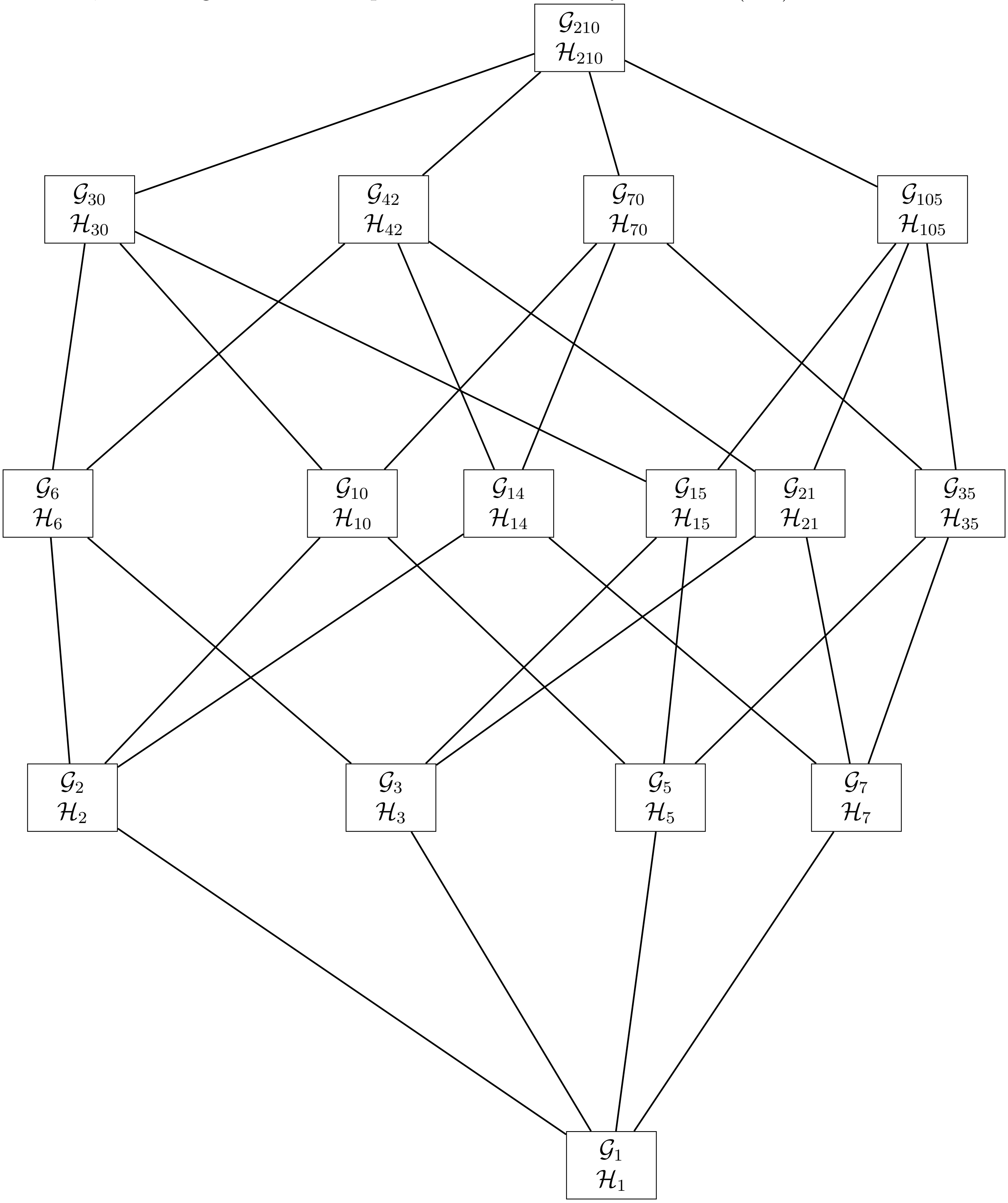
| | | |
|---|---|---|
| $\|\mathcal{B}(-1,-1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,0);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,1);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,2;m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,3);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,4);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,5);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,5}(0,1\|-1,5);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(-1,6);m\rangle$ | $\|\mathcal{X}_{1,-1}(1,0\|0,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,6}(0,1\|-1,6);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,-1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,-1);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(2,-1);m\rangle$ | $\|\mathcal{X}_{1,2}(0,1\|-1,2);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(3,-1);m\rangle$ | $\|\mathcal{X}_{1,3}(0,1\|-1,3);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(4,-1);m\rangle$ | $\|\mathcal{X}_{1,4}(0,1\|-1,4);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,-1}(1,0\|0,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,0);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,1);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,1}(0,1\|-1,1);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,2);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,2}(0,1\|-1,2);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,3);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,3}(0,1\|-1,3);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,4);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,4}(0,1\|-1,4);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,5);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,5}(0,1\|-1,5);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(0,6);m\rangle$ | $\|\mathcal{X}_{1,0}(0,1\|-1,0);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,6}(0,1\|-1,6);\bar{m}_2\rangle$ |
| $\|\mathcal{B}(1,0);m\rangle$ | $\|\mathcal{X}_{1,1}(0,1\|-1,1);\bar{m}_1\rangle$ | $\|\mathcal{X}_{2,0}(0,1\|-1,0);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\vert\mathcal{B}(1,1);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,2);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,3);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,4);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,5);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(1,6);m\rangle$ | $\vert\mathcal{X}_{1,1}(0,1\vert-1,1);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,0);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,1);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,2);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,3);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,4);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,5);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(2,6);m\rangle$ | $\vert\mathcal{X}_{1,2}(0,1\vert-1,2);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,0);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,1);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,2);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,3);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,4);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,5);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(3,6);m\rangle$ | $\vert\mathcal{X}_{1,3}(0,1\vert-1,3);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,0);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,0}(0,1\vert-1,0);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,1);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,1}(0,1\vert-1,1);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,2);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,2}(0,1\vert-1,2);\bar{m}_2\rangle$ |

| | | |
|---|---|---|
| $\vert\mathcal{B}(4,3);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,3}(0,1\vert-1,3);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,4);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,4}(0,1\vert-1,4);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,5);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,5}(0,1\vert-1,5);\bar{m}_2\rangle$ |
| $\vert\mathcal{B}(4,6);m\rangle$ | $\vert\mathcal{X}_{1,4}(0,1\vert-1,4);\bar{m}_1\rangle$ | $\vert\mathcal{X}_{2,6}(0,1\vert-1,6);\bar{m}_2\rangle$ |

Hence, from the above examples, it is obvious that there exists a duality between finite geometry $\mathcal{G}_d$ and finite dimensional Hilbert space $\mathcal{H}_d$.

Figure 5.12: The Hasse diagram showing the geometry $\mathcal{G}_{210}$ and its subgeometries, and along with Hilbert spaces $\mathcal{H}_{210}$ of the subsystems of $\Lambda(210)$



## 5.4 Conclusion

In this chapter, we presented more examples to show consistency in our formalism. From all the examples shown, it is obvious that for $q|d$, there exists a partial ordered relation between subgeometries $\mathcal{G}_q$ and geometry $\mathcal{G}_d$ with subgeometry as partial order. Also we confirm that the subsets of the set $\{D(d)\}$ of divisors of $d$ is isomorphic to the set of subgeometries of $\mathcal{G}_d$. Likewise, there exists a partial ordered relation between a subspace $\mathcal{H}_q$ and a $d-$ dimensional Hilbert space $\mathcal{H}_d$ in a finite quantum system $\Lambda(d)$ with subspace as partial order.

# Chapter 6

# Conclusion

In this work, we have studied a partially ordered set, the set $\{\mathbf{G}_d\}$ of subgeometries of $\mathcal{G}_d$ in relation to the finite quantum system $\Lambda(d)$. We defined our geometry $\mathcal{G}_d$ as

$$\mathcal{G}_d = \mathcal{Z}_d \times \mathcal{Z}_d, \tag{6.1}$$

where $\mathcal{Z}_d$ is a ring of integer modulo $d$. Our geometry is not a near-linear geometry. In a non-near-linear finite geometry, there are non-trivial subgeometries. The concept of non-near-linear geometry is related to non- prime integers which can be factorized as products of its non-trivial factors. Each maximal line in $\mathcal{G}_d$ is factorized in terms of prime factor lines in $\mathcal{Z}_{\mathbf{p_j}} \times \mathcal{Z}_{\mathbf{p_j}}$. Likewise for $q|d, \mathcal{Z}_q$ is a subgroup of $\mathcal{Z}_d$, $\Lambda(q)$ is a subsystem of $\Lambda(d)$, and as a result, there exists a partial ordered relation between:

(i) a finite geometry $\mathcal{G}_d$ and its subgeometry $\mathcal{G}_q$ with subgeometry as partial order.

(ii) a finite quantum system $\Lambda(d)$ and its susbsystems $\Lambda(q)$ with subsystem as partial order.

Also, we have shown that there exists a bijection between:

(i) the set $\{\mathbf{G}_d\}$ of subgeometries of a finite geometry and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

(ii) the set $\{\mathbf{h}_d\}$ of subspace of a finite Hilbert space and subsets of the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

(iii) the set $\{\Upsilon(d)\}$ of all subsystems of $d-$ dimensional quantum system $\Lambda(d)$ and the finite quantum systems obtained from the set $\{\mathcal{D}(d)\}$ of divisors of $d$.

The duality between the lines in $\mathcal{G}_d$ and $\mathcal{WMUB}$ in $\mathcal{H}_d$ is as follows (for $q|d$):

(i) each maximal lines in $\mathcal{G}_d$ has $d$ points, this corresponds to $d$ orthogonal vectors in each of $\mathcal{WMUB}$ in $\mathcal{H}_d$.

(ii) there are $\psi(d)$ maximal lines in $\mathcal{G}_d$, likewise there exists $\psi(d)$ $\mathcal{WMUB}$ in $\mathcal{H}_d$.

(iii) the subgeometry $\mathcal{G}_q$ of $\mathcal{G}_d$ corresponds to the subsystem $\Lambda(q)$ of $\Lambda(d)$.

(iv) there are $\sigma_0(d)$ subgeometries $\mathcal{G}_q$ of $\mathcal{G}_d$ and likewise there are $\sigma_0(d)$ subsystems $\Lambda(q)$ of $\Lambda(d)$.

This expresses the duality between lines in finite geometry and weak mutually unbiased bases in finite quantum systems.

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