

University of Bradford eThesis

This thesis is hosted in [Bradford Scholars](#) – The University of Bradford Open Access repository. Visit the repository for full metadata or to contact the repository team



© University of Bradford. This work is licenced for reuse under a [Creative Commons Licence](#).

Analytic representations of quantum systems with Theta functions

Pavlos EVANGELIDES

Submitted for the Degree
of Doctor of Philosophy

Faculty of Engineering and Informatics
Department of Computing
University of Bradford

2015

Abstract

Pavlos Evangelides

Analytic representations of quantum systems with Theta functions.

Keywords: Analytic functions, Theta functions, Bargmann functions, Finite quantum systems.

Quantum systems in a d -dimensional Hilbert space are considered, where the phase space is $\mathbb{Z}(d) \times \mathbb{Z}(d)$. An analytic representation in a cell \mathcal{S} in the complex plane using Theta functions, is defined. The analytic functions have exactly d zeros in a cell \mathcal{S} . The reproducing kernel plays a central role in this formalism. Wigner and Weyl functions are also studied.

Quantum systems with positions in a circle \mathbb{S} and momenta in \mathbb{Z} are also studied. An analytic representation in a strip \mathcal{A} in the complex plane is also defined. Coherent states on a circle are studied. The reproducing kernel is given. Wigner and Weyl functions are considered.

Acknowledgements

First of all, I would like to thank Professor Apostolos Vourdas, my supervisor, for his great support and guidance throughout this research. His inspiring suggestions and constant encouragement had a great effect on my study. It has been a really valuable opportunity and a special experience to work with him for the past four years, as his ideas pushed me into considering new perspectives on Mathematics.

Then, I would like to thank Dr. C. Lei for the great collaboration we had and for his overall support. His technical assistance in all aspects of my research has been very helpful for me. His experience in conducting PhD researches in the area of Quantum computing provided crucial guidance in organizing and composing this thesis.

Furthermore, I am especially grateful to Panayiotis Agathangelou for carrying out proofreading during the development and completion of this thesis and for some constructive discussions we had about my research topic in general.

Moreover, I would to thank my parents (Evangelos and Stella) and my sisters (Stamatia and Evie) for the financial support, as well as for the confidence boost they have been generously giving me while undertaking this

research. Further, I would like to thank Depso Ioannou for the financial support she provided.

Besides, I would like to thank Christiana Constantinou for the support she has been whole-heartedly providing me with.

Declaration

Some parts of the work presented in this thesis have been published in the following :

P.Evangelides, C. Lei and A. Vourdas, "*Analytic representations with Theta functions for systems on $\mathbb{Z}(d)$ and on \mathbb{S}* ", arXiv.1502.05309.

Table of Contents

Abstract	i
Acknowledgements	ii
Declaration	iv
Table of contents	v
List of Notations	ix
List of Figures	x
1 Introduction	1
1.1 Structure of the thesis	2
2 Mathematical tools	5
2.1 Definition	5
2.1.1 Properties	5
3 Quantum systems on \mathbb{R}	8
3.1 Introduction	8

TABLE OF CONTENTS

3.1.1	Dirac notation	10
3.1.2	Position and momentum operators	10
3.2	Fourier transform	12
3.3	Quantum Harmonic oscillator	14
3.3.1	Number states	16
3.4	Displacement and parity operators	17
3.4.1	Coherent states	18
3.5	Wigner and Weyl functions	19
3.6	Bargmann analytic representation	26
3.6.1	Operators of Bargmann function	27
3.7	Discussion and Conclusion	28
4	Quantum systems on $\mathbb{Z}(d)$	29
4.1	Introduction	29
4.2	Position and momentum states	30
4.3	Fourier Transform	31
4.4	Displacement and parity operators	32
4.5	Wigner and Weyl functions in finite systems	35
4.5.1	Marginal properties of Weyl and Wigner functions . . .	41
4.6	Discussion and Conclusion	41
5	Quantum systems on \mathbb{S}	42
5.1	Introduction	42
5.2	Position and momentum operators on a circle \mathbb{S}	43
5.3	Displacement and displaced parity operators on \mathbb{S}	45

TABLE OF CONTENTS

5.4	Wigner and Weyl functions for systems on a circle	48
5.5	Conclusion	49
6	Analytic representations for quantum systems on $\mathbb{Z}(d)$ with Theta functions	50
6.1	Introduction	50
6.2	Analytic Representation on a torus	51
6.3	Zeros of analytic function	58
6.3.1	Construction of the analytic representation from the zeros	59
6.4	Coherent states for finite systems	61
6.5	Analytic representation in terms of coherent states using $\mathcal{F}_f \psi\rangle$ as a fiducial vector	75
6.6	Entropic uncertainty relations	78
6.7	Wigner and Weyl functions for finite quantum systems	79
6.8	Discussion and Conclusion	86
7	Finite quantum systems:Periodic paths of zeros	87
7.1	Introduction	87
7.2	Periodic systems using time evolution operator	88
7.2.1	Hamiltonians with rational ratio of the eigenvalues . . .	89
7.3	Discussion	98
8	Analytic representation of quantum systems on \mathbb{S} using Theta functions	100
8.1	Introduction	100

TABLE OF CONTENTS

8.2	Analytic representation for quantum systems on a circle	101
8.3	Coherent states on a circle	105
8.4	Wigner and Weyl functions for systems on a circle	117
8.5	Discussion and Conclusion	119
9	Conclusion	121
9.1	Future work	122
	Bibliography	123
10	Appendix	131

List of Notations

Notation	Definition
\mathbb{R}	Real space.
\mathbb{Z}	Integer space.
$\mathbb{Z}(d)$	Integer modulo d .
\mathbb{S}	$[0, 2\pi] \times \mathbb{Z}$.
\mathcal{S}	$[ML, (M + 1)L) \times [NL, (N + 1)L]$.
\mathcal{A}	$[0, 2\pi] \times \mathbb{R}$.

List of figures

3.1	Real part of Weyl function of state $ 1 + i\rangle$ using Eq.(3.57)	22
3.2	Imaginary part of Weyl function of state $ 1 + i\rangle$ using Eq.(3.57)	23
3.3	Absolute value of Weyl function of state $ 1 + i\rangle$ using Eq.(3.57)	24
3.4	Wigner function of state $ 1 + i\rangle$ using Eq.(3.56)	25
4.1	Real part of Weyl function of a pure state in Eq.(3.27), for $d = 5$	37
4.2	Imaginary part of Weyl function of a pure state in Eq.(3.27), for $d = 5$	38
4.3	Absolute value of Weyl function of a pure state in Eq.(3.27), for $d = 5$	39
4.4	Wigner function of a pure state in Eq.(3.27), for $d = 5$	40
6.1	Using the coefficients of Eq.(6.15) the real part of function $\Phi(z)$ is demonstrated, for $d = 5$	55
6.2	Using the coefficients of Eq.(6.15) the imaginary part of func- tion $\Phi(z)$ is shown, for $d = 5$.	56
6.3	Using the coefficients of Eq.(6.15) the absolute value of func- tion $\Phi(z)$ is illustrated, for $d = 5$.	57

LIST OF FIGURES

6.4	The real part of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is demonstrated, for $d = 5$	65
6.5	The imaginary part of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is presented, for $d = 5$	66
6.6	The absolute value of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is illustrated, for $d = 5$	67
6.7	Real part of Weyl function of the coefficients of Eq.(6.84) . . .	82
6.8	Imaginary part of Weyl function of the coefficients of Eq.(6.84)	83
6.9	Absolute value of Weyl function using the coefficients of Eq.(6.84)	84
6.10	Wigner function using the coefficients of Eq.(6.84)	85
7.1	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.4) using the Hamiltonian of Eq.(7.3)	91
7.2	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.5) using the Hamiltonian of Eq.(7.3)	92
7.3	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.6) using the Hamiltonian of Eq.(7.3)	93
7.4	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.4) using the Hamiltonian of Eq.(7.7)	95
7.5	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.5) using the Hamiltonian of Eq.(7.7)	96
7.6	Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.6) using the Hamiltonian of Eq.(7.7)	97

Chapter 1

Introduction

In this thesis analytic representations for quantum systems with finite dimensional Hilbert space and also for quantum systems on a circle are studied.

Various analytic representations [1–4] have been studied in quantum mechanics. The most popular analytic representation is the Bargmann representation in the complex plane for the harmonic oscillator [5–7], which uses the resolution of identity of coherent states [8–15].

There is a lot of work on finite quantum systems [16–23] where the phase space is $\mathbb{Z}(d) \times \mathbb{Z}(d)$. An example of such systems, is a system with angular momentum j [24] (in this case $d = 2j - 1$). Theta functions [25–30] are used to study an analytic representation in these systems [31, 32]. Theta functions are very important since they are Gaussians when working on discretized circle. Gaussians are important in quantum systems because they can be normalized easily. Also, theta functions have several properties.

An important tool in the theory of analytic functions are the zeros [33–41], which are investigated in this thesis. In finite quantum systems, the zeros

define the state uniquely. As a result, when the zeros are known the state of the quantum system can be found. The zeros of analytic functions in a square cell \mathcal{S} is equal to d and the zeros obey a constraint [42, 43].

Our analytic representation is based on d^2 coherent states. In this case we assume that d is an odd number. The reproducing kernel [44] is also given.

The paths of the zeros [45] during time evolution are considered. In the case of Hamiltonians with rational ratio of the eigenvalues (so that there exists t with $\exp(it\mathcal{H}) = \mathbf{1}$) the system is periodic and the paths of the zeros follow a closed curve.

A brief introduction on quantum systems on a circle [46–53] is also given, where the phase space is $[0, 2\pi] \times \mathbb{Z}$. Basic concepts in this phase space are described. Using Theta functions, an analytic representation on a circle is defined. Coherent states on a circle [54–58] are studied. Finally, the Wigner and Weyl functions [59] on these systems are discussed.

Our results can be used in order to describe a mathematical formalism of qudits, based on the dimension of d . Qudits are the building blocks of quantum mechanics. Moreover, our results can be used in order to study the mesoscopic rings. Also, these results can be used in the theory of harmonic analysis as well as in the theory of analytic functions.

1.1 Structure of the thesis

This thesis consists of ten chapters. The first chapter gives an introduction to this thesis.

In chapter two the definition and some properties of theta functions are

considered.

In chapter three, some basic concepts of quantum mechanics in infinite Hilbert space are considered. Two of the most important functions Wigner and Weyl functions in phase space $\mathbb{R} \times \mathbb{R}$ are studied. Then, the Bargmann functions and Bargmann operators are investigated in the same context.

In chapter four, a brief introduction to finite quantum systems is given. Some important tools in the context of finite systems are examined. The Wigner and Weyl functions in phase space $\mathbb{Z}(d) \times \mathbb{Z}(d)$ are considered.

In chapter five some basic concepts on quantum systems on a circle \mathbb{S} are introduced. The Wigner and Weyl functions on \mathbb{S} are also considered.

Chapter six examines an analytic representation in finite quantum systems. An analytic representations in terms of coherent states is also defined. We study some properties of this analytic representation. Also, the zeros of analytic functions are considered. The novel part in this chapter is the properties of the analytic representation based on coherent states and the Wigner and Weyl functions.

In chapter seven, various examples of periodic systems in d -dimensional Hilbert space are considered and the paths of their zeros are studied. The novel part in this chapter is the behaviour of the paths of the zeros using different Hamiltonians.

In chapter eight, an analytic representation for systems on a circle \mathbb{S} is defined. Also, an analytic representation using coherent states is studied. Some important properties of this analytic representation is studied. The novel part in this chapter is the properties of the analytic representation based on coherent states as well as the Wigner and Weyl functions.

CHAPTER 1. INTRODUCTION

In chapter nine, a discussion of this thesis is given.

Chapter 2

Mathematical tools

2.1 Definition

The Jacobian theta function is given by

$$\Theta_3[u; \tau] = \sum_{n=-\infty}^{\infty} \exp(2inu + i\pi\tau n^2) \quad (2.1)$$

2.1.1 Properties

Theta function obeys the following quasi-periodicity properties

$$\Theta_3[u; \tau] = \Theta_3[u + \pi; \tau] \quad (2.2)$$

$$\Theta_3[u; \tau] = \Theta_3[u + \pi m + \pi n\tau; \tau] \exp(2inu + i\pi\tau n^2) \quad (2.3)$$

and

$$\Theta_3[u; \tau] = (-i\tau)^{-1/2} \exp\left(\frac{u^2}{i\pi\tau}\right) \Theta_3\left[\frac{u}{\tau}; -\frac{1}{\tau}\right]. \quad (2.4)$$

Another important property of theta function is the following

$$\Theta_3[u; \tau] = \Theta_3[-u; \tau]. \quad (2.5)$$

Also, since the τ in our analytic representation is imaginary then

$$[\Theta_3[u; \tau]]^* = \Theta_3[u^*; \tau]. \quad (2.6)$$

The last property is valid only when τ is imaginary number. In our case the following theta function is considered

$$\Theta_3\left[\frac{\pi m}{d} - z\frac{\pi}{L}; \frac{i}{d}\right] \quad (2.7)$$

Using Eqs.(2.1), (2.7) it can be concluded that

$$\begin{aligned} \Theta_3[u; \tau]^* &= \sum_{n=-\infty}^{\infty} \exp\left[-2in\left(\frac{\pi m}{d} - z^*\frac{\pi}{L}\right) - \frac{\pi}{d}n^2\right] \\ &= \sum_{n=-\infty}^{\infty} \exp\left(-2in\frac{\pi m}{d} + 2inz^*\frac{\pi}{L} - \frac{\pi}{d}n^2\right) \end{aligned} \quad (2.8)$$

Using Eq.(2.5) it can be proved that

$$\begin{aligned}
 \Theta_3[u; \tau]^* &= \sum_{n=-\infty}^{\infty} \exp\left(2in\frac{\pi m}{d} - 2inz^*\frac{\pi}{L} - \frac{\pi}{d}n^2\right) \\
 &= \Theta_3\left[\frac{\pi m}{d} - z^*\frac{\pi}{L}; \frac{i}{d}\right]
 \end{aligned} \tag{2.9}$$

Chapter 3

Quantum systems on \mathbb{R}

3.1 Introduction

Quantum mechanics [60–62] studies the behaviour of photons, electrons and other atomic objects. The state of a particle is represented by the wavefunction, which is the solution of the Schrödinger equation. The Hamiltonian operator \mathcal{H} gives the energy of a system which corresponds to the wavefunction. The time independent Schrödinger equation is given by

$$\mathcal{H}\Psi(x) = E\Psi(x) \tag{3.1}$$

In Eq.(3.1) the energy E of the system is the eigenvalues of Hamiltonian \mathcal{H} , while the eigenvectors represent the wavefunction $\Psi(x)$, where x is a position. The momentum and position operators are denoted by \mathcal{P} and \mathcal{X} , respectively.

In the x -representation the momentum and position operators are given by

$$\begin{aligned}\mathcal{P} &= -i\hbar \frac{\partial}{\partial x} \\ \mathcal{X} &= x.\end{aligned}\tag{3.2}$$

The commutator of two operators is defined by

$$[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}.\tag{3.3}$$

The position and momentum operators obey the canonical commutation relation. As a result, these operators do not commute

$$[\mathcal{X}, \mathcal{P}] = i\hbar.\tag{3.4}$$

The basic concepts of quantum mechanics, which will be used at later point are described in this chapter. These include the position and momentum operators as well as the Fourier transform, as reviewed in Sections 3.2 and 3.3, respectively. In Section 3.4 a brief introduction to the quantum harmonic oscillator is provided, for the one dimensional case. Furthermore expressions for special states are given, such as number states and coherent states. The displaced and parity operators are discussed. In Section 3.5 the Wigner and Weyl functions are defined. Finally, in Section 3.6 the Bargmann analytic representations are introduced.

3.1.1 Dirac notation

The Dirac notation represents the quantum states and their properties. The quantum state is represented by a ket vector as follows

$$f \equiv |f\rangle \quad (3.5)$$

and the complex conjugate by a bra vector as follows

$$f^* \equiv \langle f|. \quad (3.6)$$

3.1.2 Position and momentum operators

Throughout this section, the position \mathcal{X} and momentum \mathcal{P} operators are described in more detail. Position and momentum operators are defined in Eq.(3.4).

Position and momentum operators are Hermitian, hence their eigenvalues are real

$$\begin{aligned} \mathcal{X}|a\rangle_x &= a|a\rangle_x \\ \mathcal{P}|b\rangle_p &= b|b\rangle_p, \quad a, b \in \mathbb{R}. \end{aligned} \quad (3.7)$$

where $|a\rangle_x$ and $|b\rangle_p$ denotes position and momentum states, respectively. From Eq.(3.7) the position and momentum operators can only take real values. As a result, the phase space is $\mathbb{R} \times \mathbb{R}$.

The eigenstates $|a\rangle_x$ and $|b\rangle_p$ form improper orthogonal bases in Hilbert

space, therefore

$$\begin{aligned} {}_x\langle a|b\rangle_x &= \delta(a-b) \\ {}_p\langle \gamma|\delta\rangle_p &= \delta(\gamma-\delta). \end{aligned} \tag{3.8}$$

where $\delta(x)$ is the Dirac delta function.

The position and momentum eigenstates, have an important property called as completeness, which is given by

$$\begin{aligned} \int_{-\infty}^{\infty} dx |x\rangle_x {}_x\langle x| &= \mathbf{1} \\ \int_{-\infty}^{\infty} dp |p\rangle_p {}_p\langle p| &= \mathbf{1}. \end{aligned} \tag{3.9}$$

Using these resolutions of identity, an arbitrary state $|f\rangle$ can be written in terms of position and momentum states as follows

$$|f\rangle = \int_{-\infty}^{\infty} dx |x\rangle_x {}_x\langle x|f\rangle, \tag{3.10}$$

and assuming that ${}_x\langle x|f\rangle = f(x)$, then

$$|f\rangle = \int_{-\infty}^{\infty} dx |x\rangle_x f(x). \tag{3.11}$$

The corresponding expression for momentum states are

$$|g\rangle = \int_{-\infty}^{\infty} dp |p\rangle_p {}_p\langle p|g\rangle, \tag{3.12}$$

assuming that ${}_p\langle p|g\rangle = g(p)$, then

$$|g\rangle = \int_{-\infty}^{\infty} dp |p\rangle_p g(p). \quad (3.13)$$

Since, the particle can be found anywhere in the space, the probability to find the particle in the space is calculated by

$$\int_{-\infty}^{\infty} dx |f(x)|^2 = 1 \quad (3.14)$$

where $|f(x)|^2$ is the position probability density. The corresponding relation for the momentum is given by

$$\int_{-\infty}^{\infty} dp |g(p)|^2 = 1 \quad (3.15)$$

where $|g(p)|^2$ is the momentum probability density.

3.2 Fourier transform

The Fourier transform [63] is defined by

$$G(x) = \int_{-\infty}^{\infty} dp F(p) \exp(-2i\pi xp) \quad (3.16)$$

$G(x)$ is the Fourier transform of $F(p)$, where p represents momentum and x position, respectively. Also, for $G(x)$, the inverse Fourier transform is

$$F(p) = \int_{-\infty}^{\infty} dx G(x) \exp(2i\pi xp). \quad (3.17)$$

The Fourier transform can be written in terms of Fourier operator, which is defined as

$$\mathcal{F} = \int_{-\infty}^{\infty} d\eta |\eta\rangle_x \langle \eta|_p \quad (3.18)$$

Then,

$$\begin{aligned} \mathcal{F}|x\rangle_x &= |x\rangle_p, & \mathcal{F}|x\rangle_p &= |-x\rangle_x \\ \mathcal{F}^\dagger \mathcal{X} \mathcal{F} &= \mathcal{P}, & \mathcal{F}^\dagger \mathcal{P} \mathcal{F} &= -\mathcal{X} \end{aligned} \quad (3.19)$$

and

$$\mathcal{F}^4 = \mathbf{1}. \quad (3.20)$$

Based on Parseval's theorem, the integral of the absolute value of the square of a function $F(x)$ is equal to the integral of the absolute value of the square of the transform $\mathcal{F}(x)$,

$$\int_{-\infty}^{\infty} dx |F(x)|^2 = \int_{-\infty}^{\infty} dx |\mathcal{F}(x)|^2. \quad (3.21)$$

The Fourier transform of position and momentum states is given by

$$\begin{aligned} |x\rangle_x &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ixp) |p\rangle_p \\ |p\rangle_p &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp(-ixp) |x\rangle_x. \end{aligned} \quad (3.22)$$

The inner product of these states is given by

$${}_x\langle x|p\rangle_p = \frac{1}{2\pi} \exp(-ixp). \quad (3.23)$$

The wavefunction of position states can be expressed in terms of the wavefunction of momentum states using Fourier transform and vice versa. These equations are given by

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \exp(ixp) f(p) \\ g(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp(-ixp) g(x). \end{aligned} \quad (3.24)$$

3.3 Quantum Harmonic oscillator

The wavefunction of the system is evaluated by solving the Schrödinger equation, using the potential of the harmonic oscillator. Assuming an one dimensional harmonic oscillator, the potential energy of the system in position x is given by

$$U(x) = \frac{1}{2}x^2. \quad (3.25)$$

The Hamiltonian operator is given by

$$\mathcal{H} = \frac{1}{2}\mathcal{P}^2 + \frac{1}{2}\mathcal{X}^2 \quad (3.26)$$

where $\hbar = m = \omega = 1$. Applying the potential energy of harmonic oscillator Eq.(3.26) in time independent Schrodinger equation (Eq.(3.1)), leads to

$$\left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 \right] \Psi(x) = E \Psi(x) \quad (3.27)$$

The solution of Eq.(3.27) is

$$\Psi_n(x) = \left(\frac{2^n \sqrt{\pi}}{n!} \right) H_n(x) \exp \left(-\frac{1}{2}x^2 \right) \quad (3.28)$$

where $H_n(x)$ is any Hermite polynomial and is given by

$$H_n(x) = (-1)^n \exp(x^2) \frac{\partial^n}{\partial x^n} \exp(-x^2) = \left(2x - \frac{\partial}{\partial x} \right)^n. \quad (3.29)$$

At this point, using Eq.(3.26) the creation and destruction operators are introduced and are given by

$$\begin{aligned} a^\dagger &= \sqrt{\frac{1}{2}} (\mathcal{X} - i\mathcal{P}) \\ a &= \sqrt{\frac{1}{2}} (\mathcal{X} + i\mathcal{P}). \end{aligned} \quad (3.30)$$

There is a canonical commutation relation between the creation and destruction operators which is

$$[a, a^\dagger] = 1. \quad (3.31)$$

The state $|0\rangle$ is defined using the destruction operator in Eq.(3.30)

$$a|0\rangle = 0; \quad \langle 0|0\rangle = 1. \quad (3.32)$$

The $|0\rangle$ is the vacuum state.

The number operator is defined, using the operators given in Eq.(3.30), as follows

$$\mathcal{N} = a^\dagger a. \quad (3.33)$$

Hence, the Hamiltonian operator is written, in terms of number operator as follows

$$\mathcal{H} = \frac{1}{2} (aa^\dagger - a^\dagger a) = \mathcal{N} + \frac{1}{2}\mathbf{1}. \quad (3.34)$$

3.3.1 Number states

The number operator is symbolized by \mathcal{N} and is given in Eq.(3.33). The number states are defined by

$$\begin{aligned} |n\rangle &= (a^\dagger)^n n!^{-1/2} |0\rangle \\ \mathcal{N}|n\rangle &= n|n\rangle. \end{aligned} \quad (3.35)$$

When the creation and destruction operators act on number states, the result is evaluated by

$$a|n\rangle = \sqrt{n}|n-1\rangle; \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \quad (3.36)$$

The scalar product of two number states $|n\rangle, |m\rangle$ is equal to δ_{nm} , which is also equal to the Kronecker delta function,

$$\langle n|m\rangle = \delta_{nm}. \quad (3.37)$$

The number eigenstates form a complete set, therefore

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \quad (3.38)$$

3.4 Displacement and parity operators

The displacement operator of the harmonic oscillator in the $x - p$ phase-space is symbolized by $\mathcal{D}(z)$ and is defined as

$$\mathcal{D}(z) = \exp(za^\dagger - z^*a); \quad z \in \mathbb{C}. \quad (3.39)$$

where $z = z_R + iz_I$. A coherent state can be defined when the displacement operator acts on the vacuum state, as follows

$$|z\rangle = \mathcal{D}(z)|0\rangle. \quad (3.40)$$

The definition of the parity operator around the origin is the following

$$\mathcal{P}_0 = \int_{-\infty}^{\infty} dp | -p\rangle_p {}_p\langle p| = \int_{-\infty}^{\infty} dx | -x\rangle_x {}_x\langle x|. \quad (3.41)$$

When the parity operator acts on position or momentum states then

$$\mathcal{P}_0|x\rangle_x = | -x\rangle_x; \quad \mathcal{P}_0|p\rangle_p = | -p\rangle_p. \quad (3.42)$$

The displaced parity operator is symbolized by $\mathcal{P}(z)$ and is described as

$$\mathcal{P}(z) = \mathcal{D}(z)\mathcal{P}_0[\mathcal{D}(z)]^\dagger. \quad (3.43)$$

3.4.1 Coherent states

The coherent states are functions of the z variable, which runs throughout the entire complex plane. These coherent states are also called Glauber states, which are given by the expressions

$$\begin{aligned}
 |z\rangle &= \pi^{-1/2} \exp\left(\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{a^n}{(n!)^{-1/2}} |n\rangle \\
 &= \pi^{-1/2} \exp\left(\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{(aa^\dagger)^n}{n!} |0\rangle \\
 &= \pi^{-1/2} \exp\left(\frac{1}{2}|z|^2\right) \exp(aa^\dagger) |0\rangle.
 \end{aligned} \tag{3.44}$$

Alternatively, the coherent state is defined as

$$a|z\rangle = z|z\rangle. \tag{3.45}$$

The resolution of unity for coherent states is given by

$$\frac{1}{2\pi} \int_{\mathbb{C}} d^2z |z\rangle \langle z| = \mathbf{1} \tag{3.46}$$

where $d^2z = dz_{real}dz_{imag}$ and \mathbb{C} is the complex plane.

An arbitrary state $|q\rangle$ is given in terms of coherent state as

$$|q\rangle = \frac{1}{2\pi} \int_{\mathbb{C}} d^2z |z\rangle \langle z|q\rangle. \tag{3.47}$$

The scalar product of two coherent states can be found using the definition of coherent state

$$\langle z|z_1\rangle = \pi^{-1} \exp\left(z^* z_1 - \frac{1}{2}|z|^2 - \frac{1}{2}|z_1|^2\right) \quad (3.48)$$

The scalar product of position and coherent states is equal to

$${}_x\langle x|z\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + \sqrt{2}zx - zz_R\right). \quad (3.49)$$

The scalar product of coherent and momentum states can be found using Fourier transform and is given by

$${}_p\langle p|z\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2}p^2 + i\sqrt{2}zp - zz_R\right). \quad (3.50)$$

3.5 Wigner and Weyl functions

Wigner functions were found by E.Wigner in 1932. They are quasiprobability distribution functions in the phase space. The Wigner function for the density operator ρ is given by

$$W(\rho; x, p) = \text{Tr}[\rho \mathcal{P}(z)]. \quad (3.51)$$

Using this definition the following significant properties regarding the quantum states can be derived

$$\begin{aligned} W(\rho; x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dX \exp(ipX) \langle x - \frac{1}{2}X | \rho | x + \frac{X}{2} \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dP \exp(iPx) \langle p - \frac{1}{2}P | \rho | p + \frac{P}{2} \rangle \end{aligned} \quad (3.52)$$

X and P are the position and the momentum variables, respectively. The Weyl function is given in terms of displacement operator

$$\widetilde{W}(\rho; X, P) = \text{Tr}[\rho \mathcal{D}(z)]. \quad (3.53)$$

Moreover, Weyl functions can be defined by a density operator which is described as

$$\begin{aligned} \widetilde{W}(\rho; X, P) &= \int_{-\infty}^{\infty} dx \exp(iPx) \langle x - \frac{1}{2}X | \rho | x + \frac{X}{2} \rangle \\ &= \int_{-\infty}^{\infty} dp \exp(-ipX) \langle p - \frac{1}{2}P | \rho | p + \frac{P}{2} \rangle. \end{aligned} \quad (3.54)$$

The Weyl function is derived using the two dimensional Fourier transform within the Wigner functions, as follows

$$W(x, p) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX dP \widetilde{W}(X, P) \exp[-i(Xp - Px)]. \quad (3.55)$$

The Wigner function for a coherent state $|z\rangle$ is given in terms of

$$W(z; x, p) = \frac{1}{\pi} \exp \left[- \left(x - \sqrt{2}z_R \right) - \left(p - \sqrt{2}z_I \right) \right]. \quad (3.56)$$

Eq.(3.56) is proved using Eq.(3.52).

The Weyl function for a coherent state $|z\rangle$ is given in terms of

$$\widetilde{W}(z; X, P) = \exp \left[-\frac{1}{4} (X^2 + P^2) + i\sqrt{2} (Xz_I - Pz_R) \right]. \quad (3.57)$$

Eq.(3.57) is proved using Eq.(3.54).

Fig.(2.1) shows the real part of Weyl function of the state $|1+i\rangle$. In Fig.(2.2), the imaginary part of Weyl functions of the state $|1+i\rangle$ is presented. In Fig.(2.3) the absolute value of Weyl function of the same state is shown. In Fig.(2.4) the Wigner function of this state, is plotted.

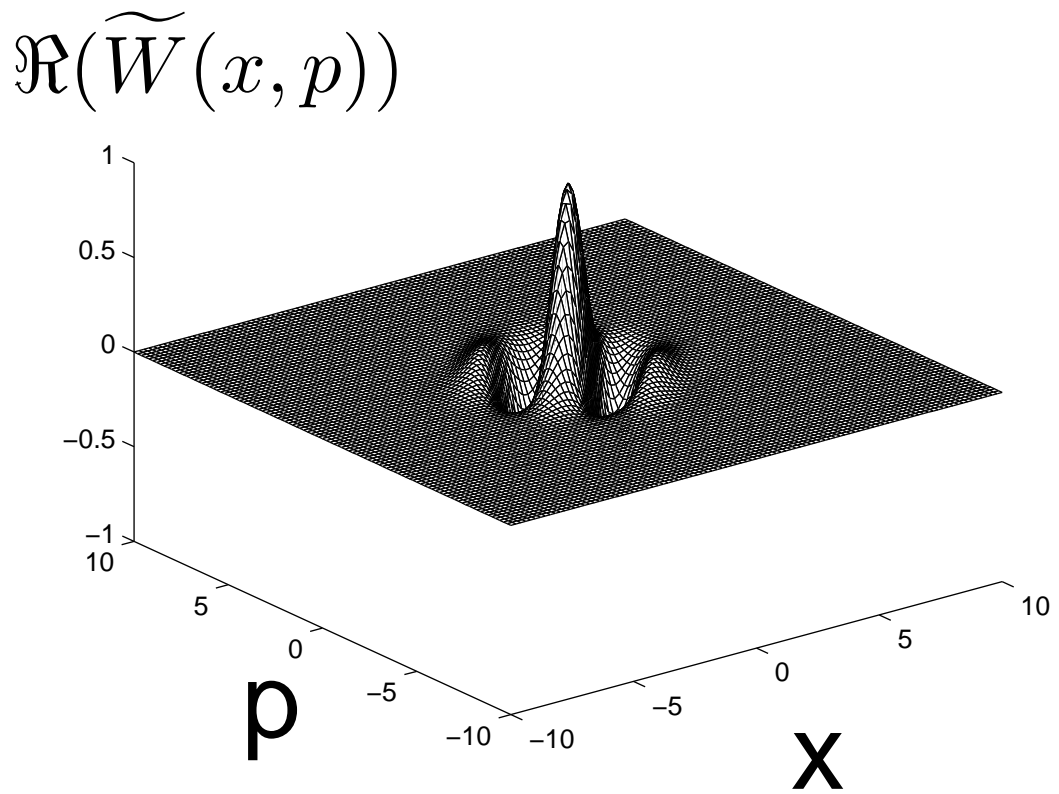


Figure 3.1: Real part of Weyl function of state $|1 + i\rangle$ using Eq.(3.57)

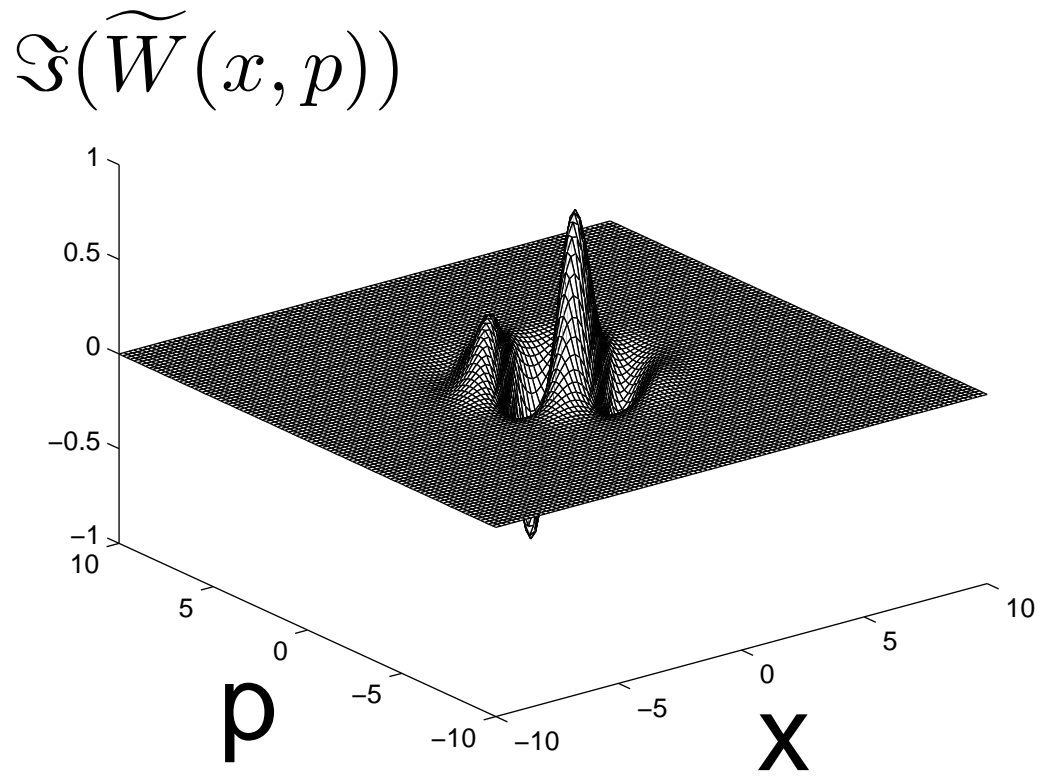


Figure 3.2: Imaginary part of Weyl function of state $|1 + i\rangle$ using Eq.(3.57)

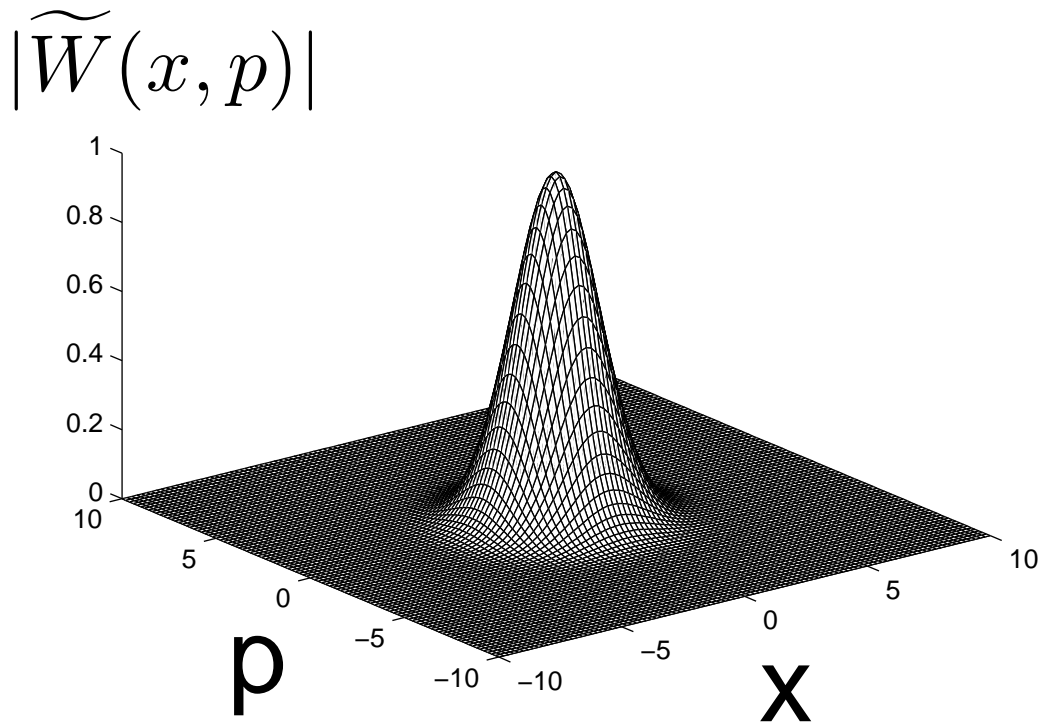


Figure 3.3: Absolute value of Weyl function of state $|1 + i\rangle$ using Eq.(3.57)

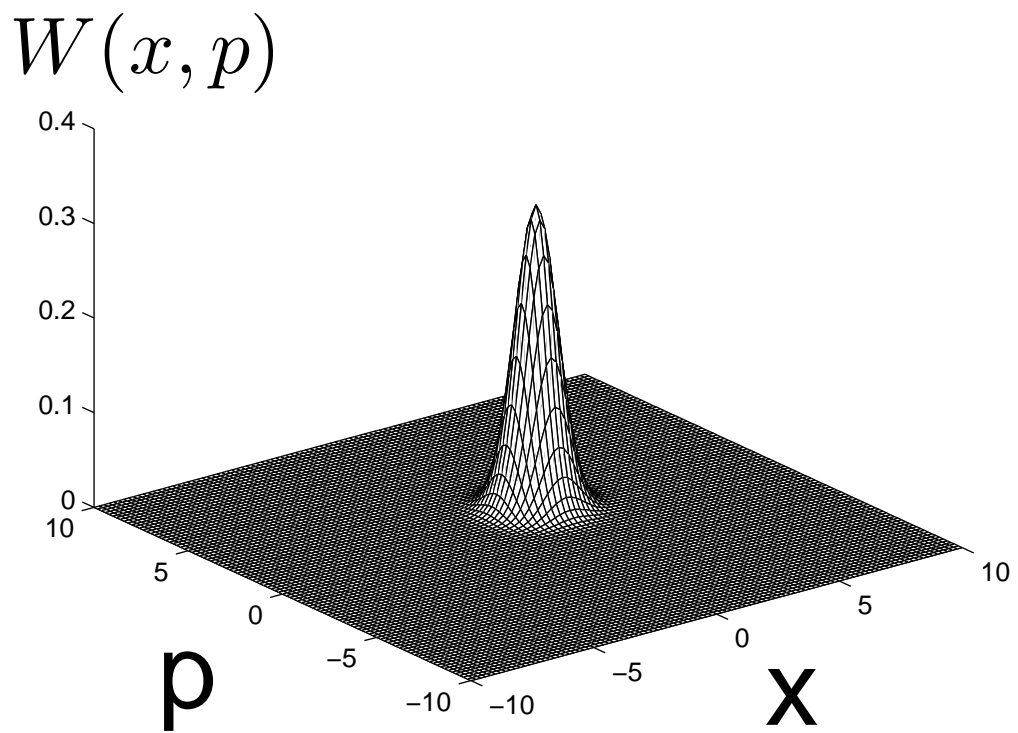


Figure 3.4: Wigner function of state $|1 + i\rangle$ using Eq.(3.56)

3.6 Bargmann analytic representation

In this section, the Bargmann analytic function in a complex plane is investigated, which is defined by a coherent state. The space of these functions is defined as the space of the entire functions with no singularities. Furthermore, the kernel operator of a Bargmann function is discussed.

Let $|r\rangle$ be an arbitrary state

$$|r\rangle = \sum_{m=0}^{\infty} r_m |m\rangle \quad (3.58)$$

and the normalization condition is shown below

$$\sum_{m=0}^{\infty} |r_m|^2 = 1. \quad (3.59)$$

The following notation is used

$$\langle r| = \sum_{m=0}^{\infty} r_m^* \langle m|; \quad |r^*\rangle = \sum_{m=0}^{\infty} r_m^* |m\rangle. \quad (3.60)$$

The state $|r\rangle$, can be expressed in terms of the Bargmann representation as

$$r(z) = \exp\left(\frac{1}{2}|z|^2\right) \langle z^*|r\rangle = \sum_{m=0}^{\infty} \frac{r_m z^m}{m!^{1/2}} \quad (3.61)$$

Using Eq.(3.46) an arbitrary state $|r\rangle$ can be expressed as follows

$$|r\rangle = \frac{1}{2\pi} \int_{\mathbb{C}} d^2z \exp\left(-\frac{1}{2}z^2\right) r(z) |z^*\rangle \quad (3.62)$$

The scalar product of two states is given by

$$\langle r_1 | r_2 \rangle = \frac{1}{\pi} \int_{\mathbb{C}} d^2 z r_1^*(z) r_2(z) \exp(-|z|^2) \quad (3.63)$$

which can be proved by using the resolution of identity of coherent states. The creation and destruction operators in terms of Bargmann analytic representation are represented by

$$\begin{aligned} a &\rightarrow \partial_z \\ a^\dagger &\rightarrow z. \end{aligned} \quad (3.64)$$

The Bargmann analytic function can be expressed in terms of the wavefunction of position states as

$$\int_{\mathbb{C}} dz_I r(z) \exp\left(-\frac{1}{2}z_I^2\right) = 2^{1/2}\pi^{3/4} \exp\left(\frac{1}{4}z_R^2\right) r_x\left(-\frac{1}{\sqrt{2}}z_R\right) \quad (3.65)$$

while in the case of the momentum wavefunction is equal to

$$\int_{\mathbb{C}} dz_R r(z) \exp\left(-\frac{1}{2}z_R^2\right) = 2^{1/2}\pi^{3/4} \exp\left(\frac{1}{4}z_I^2\right) r_p\left(-\frac{1}{\sqrt{2}}z_I\right) \quad (3.66)$$

3.6.1 Operators of Bargmann function

Operators in the Bargmann formalism are represented as

$$L(z, w^*; \mathcal{L}) = \exp\left(\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2\right) \langle z^* | \mathcal{L} | w^* \rangle = \sum_{m,n=0}^{\infty} \mathcal{L}_{mn} z^m (w^*)^n \quad (3.67)$$

where \mathcal{L} is an arbitrary operator with the matrix elements $\mathcal{L}_{mn} = \langle n|\mathcal{L}|m\rangle$.

The kernel L is an analytic function with respect to z, w^* .

An arbitrary state $\mathcal{L}|r\rangle$ can be expressed in terms of kernel as follows

$$\mathcal{L}|r\rangle = \frac{1}{2\pi} \int_{\mathbb{C}} d^2w \exp(-|w|^2) L(z, w^*; \mathcal{L}) r(w). \quad (3.68)$$

When the arbitrary operator \mathcal{L} is equal to $\mathbf{1}$, then the result of this kernel is

$$L(z, w^*, \mathbf{1}) = \exp(zw^*), \quad (3.69)$$

and when $L(z, w^*, \mathbf{1})$ acts on a state, the result is equal to the same state

$$r(z) = \frac{1}{2\pi} \int_{\mathbb{C}} d^2w \exp(-|w|^2 + wz^*) r(w). \quad (3.70)$$

3.7 Discussion and Conclusion

In this chapter, a brief introduction of the phase space methods on Hilbert space for a particle on the real line \mathbb{R} is given. The phase space in this formalism is $\mathbb{R} \times \mathbb{R}$. Therefore the position and momenta belongs to \mathbb{R} . Furthermore, the Fourier transform on the infinite-dimensional Hilbert space is defined. Some special states such as number and coherent states, as well as displaced and parity displaced operators are studied. Hence, two of the most important functions in this phase space, which are the Wigner and Weyl functions, are defined. Finally, Bargmann analytic representations and the Hilbert space of Bargmann functions are considered in the complex plane, as defined by Glauber coherent states.

Chapter 4

Quantum systems on $\mathbb{Z}(d)$

4.1 Introduction

In this chapter, finite systems in d -dimensional Hilbert space are introduced. It is known that in d dimensional Hilbert space a different formalism is used when d is an odd or even integer number. Here, it is assumed that d is an odd integer, which means that the inverse modulo 2, (2^{-1}) , exists. The existence of 2^{-1} is used in several formulae. Some basic definitions are given in finite Hilbert space which will be used at a later point. Sections 4.2 and 4.3 introduce the Fourier transform and the position and momentum operators in a toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$, where $\mathbb{Z}(d)$ is the set of integers modulo d . Hence, the phase space in finite quantum systems is clearly the toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$. In Section 4.4, the displacement and parity operators in finite quantum systems are examined. Furthermore, in Section 4.5 the Wigner and Weyl functions in d - dimensional Hilbert space are investigated.

4.2 Position and momentum states

The position states are denoted by $|m\rangle_x$ where the values of $m \in \mathbb{Z}(d)$, forming an orthonormal basis in $L^2(\mathbb{Z}(d))$. It obeys the following relations

$${}_x\langle m|n\rangle_x = \delta(n, m); \quad \sum_{n=0}^{d-1} |n\rangle_x {}_x\langle n| = \mathbf{1}. \quad (4.1)$$

The momentum states are stated by $|m\rangle_p$ and they form an orthonormal basis in $L^2(\mathbb{Z}(d))$. The corresponding relations are

$${}_p\langle m|n\rangle_p = \delta(n, m); \quad \sum_{n=0}^{d-1} |n\rangle_p {}_p\langle n| = \mathbf{1} \quad (4.2)$$

where $\delta(n, m)$ is the Kronecker delta. These identities are proved using the following identity

$$\frac{1}{d} \sum_{m=0}^{d-1} \omega[m(k-l)] = \delta(k, l) \quad (4.3)$$

where $\omega(b) = \exp(\frac{2i\pi b}{d})$; $b \in \mathbb{Z}(d)$.

Position and momentum operators in finite system are described as

$$\begin{aligned} \mathcal{X}_f &= \sum_{m=0}^{d-1} m |m\rangle_x {}_x\langle m| \\ \mathcal{P}_f &= \sum_{m=0}^{d-1} m |m\rangle_p {}_p\langle m|. \end{aligned} \quad (4.4)$$

The index ' f ' denotes finite Hilbert space.

4.3 Fourier Transform

The definition of finite Fourier transform is the following

$$\mathcal{F}_f = \frac{1}{\sqrt{d}} \sum_{m,n=0}^{d-1} \omega(nm) |m\rangle_x \langle n| \quad (4.5)$$

and the finite Fourier operator obeys the following properties

$$\begin{aligned} \mathcal{F}_f \mathcal{F}_f^\dagger &= \mathcal{F}_f^\dagger \mathcal{F}_f = \mathbf{1} \\ \mathcal{F}_f^4 &= \mathbf{1}. \end{aligned} \quad (4.6)$$

The relation between position and momentum states is calculated by applying the finite Fourier transform onto the position states, resulting in the momentum states

$$|n\rangle_p = \mathcal{F}_f |n\rangle_x = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega(nm) |m\rangle_x. \quad (4.7)$$

Subsequently, the position/momentum operators in terms of momentum/position operators, respectively, are derived using the finite Fourier transform, as follows

$$\begin{aligned} -\mathcal{X}_f &= \mathcal{F}_f \mathcal{P}_f \mathcal{F}_f^\dagger \\ \mathcal{P}_f &= \mathcal{F}_f \mathcal{X}_f \mathcal{F}_f^\dagger. \end{aligned} \quad (4.8)$$

4.4 Displacement and parity operators

The position and momentum eigenvalues belongs to $\mathbb{Z}(d)$. As a result, the phase space is the toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$. In this phase space the displacement operators are described by

$$\mathcal{X}_f = \exp \left[i \frac{2\pi}{d} \mathcal{U}_f \right]; \quad \mathcal{P}_f = \exp \left[-i \frac{2\pi}{d} \mathcal{V}_f \right]. \quad (4.9)$$

Operators \mathcal{X}_f and \mathcal{P}_f are parts of displacement operator between the X and P axes in $\mathbb{Z}(d) \times \mathbb{Z}(d)$ phase space. There are two useful relations among these operators

$$\mathcal{X}_f^d = \mathcal{P}_f^d = \mathbf{1}; \quad \mathcal{X}_f^b \mathcal{P}_f^a = \mathcal{P}_f^a \mathcal{X}_f^b \omega(-ab). \quad (4.10)$$

where $a, b \in \mathbb{Z}(d)$.

When these operators act on position and momentum states, the result is equal to

$$\begin{aligned} \mathcal{X}_f^b |m\rangle_x &= |m+b\rangle_x; & \mathcal{P}_f^a |m\rangle_p &= |m+a\rangle_p \\ \mathcal{P}_f^a |m\rangle_x &= \omega(am) |m\rangle_x; & \mathcal{X}_f^b |m\rangle_p &= \omega(-mb) |m\rangle_p. \end{aligned} \quad (4.11)$$

The definition of displacement operator in finite systems is

$$\mathcal{D}_f(a, b) = \mathcal{P}_f^a \mathcal{X}_f^b \omega(-2^{-1}ab) = \mathcal{X}_f^b \mathcal{P}_f^a \omega(2^{-1}ab) \quad (4.12)$$

where 2^{-1} is 2 inverse. The complex conjugate of displacement operator is $[\mathcal{D}_f(a, b)]^\dagger = \mathcal{D}_f(-a, -b)$.

When the displacement operator acts on position or momentum states, the result is

$$\begin{aligned}\mathcal{D}_f(a, b)|m\rangle_x &= \omega(2^{-1}ab + am)|m + b\rangle_x \\ \mathcal{D}_f(a, b)|m\rangle_p &= \omega(-2^{-1}ab - am)|m + a\rangle_p.\end{aligned}\quad (4.13)$$

The definitions of operators \mathcal{X}_f and \mathcal{P}_f can be used to prove that

$$\mathcal{D}_f(a, b)\mathcal{D}_f(c, d) = \mathcal{D}_f(a + c, b + d)\omega[2^{-1}(ad - bc)]. \quad (4.14)$$

The displaced Fourier operator is defined as

$$\mathcal{F}_f(a, b) = \mathcal{D}_f(a, b)\mathcal{F}_f\mathcal{D}_f^\dagger(a, b) = \omega[2^{-1}(a^2 + b^2)]\mathcal{F}_f\mathcal{D}_f(-a - b, a - b). \quad (4.15)$$

Eq.(4.15) is proved using the following relation

$$\mathcal{F}_f\mathcal{D}_f(a, b)\mathcal{F}_f^\dagger = \mathcal{D}_f(b, -a). \quad (4.16)$$

The parity operator around the origin is defined as

$$\mathcal{P}_f(0, 0) = \mathcal{F}_f^2; \quad [\mathcal{P}_f(0, 0)]^2 = \mathbf{1}. \quad (4.17)$$

When the parity operator acts on position or momentum states then

$$\mathcal{P}_f(0,0)|m\rangle_x = |-m\rangle_x; \quad \mathcal{P}_f(0,0)|m\rangle_p = |-m\rangle_p. \quad (4.18)$$

The displaced parity operator is given in terms of the displacement operator as follows

$$\begin{aligned} \mathcal{P}_f(a,b) &= \mathcal{D}_f(a,b)\mathcal{P}_f(0,0)\mathcal{D}_f(-a,-b) = \mathcal{D}_f(2a,2b)\mathcal{P}_f(0,0) \\ &= \mathcal{P}_f(0,0)\mathcal{D}_f(-2a,-2b). \end{aligned} \quad (4.19)$$

It is easily seen that

$$[\mathcal{P}_f(a,b)]^2 = \mathbf{1}. \quad (4.20)$$

The marginal properties of the displacement operator in finite systems are

$$\begin{aligned} \frac{1}{d} \sum_{a=0}^{d-1} \mathcal{D}_f(a,b) &= |2^{-1}b\rangle_x \langle -2^{-1}b| \\ \frac{1}{d} \sum_{b=0}^{d-1} \mathcal{D}_f(a,b) &= |2^{-1}a\rangle_p \langle -2^{-1}a| \\ \frac{1}{d} \sum_{a,b=0}^{d-1} \mathcal{D}_f(a,b) &= \mathcal{P}_f(0,0). \end{aligned} \quad (4.21)$$

The first two properties are proved by multiplying both sides by position and momentum states, respectively. The third one is proved from the first with the extra summation $\sum_{b=0}^{d-1}$.

Similarly, the marginal properties of the displacement parity operator in the

d -dimensional Hilbert space are

$$\begin{aligned}
 \frac{1}{d} \sum_{a=0}^{d-1} \mathcal{P}_f(a, b) &= |b\rangle_x \langle -b| \\
 \frac{1}{d} \sum_{b=0}^{d-1} \mathcal{P}_f(a, b) &= |a\rangle_p \langle -a| \\
 \frac{1}{d} \sum_{a,b=0}^{d-1} \mathcal{P}_f(a, b) &= \mathbf{1}.
 \end{aligned} \tag{4.22}$$

The marginal properties of the displaced parity operator are proved in a similar way to that of the marginal properties of the displacement operator.

The relation between the displaced parity operator in terms of the displacement operator and vice versa, given by

$$\begin{aligned}
 \mathcal{P}_f(\gamma, \delta) &= \frac{1}{d} \sum_{a,b=0}^{d-1} \mathcal{D}_f(a, b) \omega(b\gamma - a\delta) \\
 \mathcal{D}_f(a, b) &= \frac{1}{d} \sum_{a,b=0}^{d-1} \mathcal{P}_f(\gamma, \delta) \omega(-b\gamma + a\delta).
 \end{aligned} \tag{4.23}$$

4.5 Wigner and Weyl functions in finite systems

The Wigner function is defined as

$$W(\rho; a, b) = \text{Tr}[\rho \mathcal{P}_f(a, b)] \tag{4.24}$$

where ρ is a density operator. When the operator ρ is Hermitian the Wigner function can only take real values. In comparison, for a non-Hermitian op-

erator the Wigner function can take complex values, as well.

Substituting the displaced parity operator into the definition of the Wigner function it concluded

$$W(\rho; a, b) = \sum_{m=0}^{d-1} \omega(-2am + 2ab) {}_x\langle m|\rho|2b - m\rangle_x. \quad (4.25)$$

The Weyl function is defined as

$$\begin{aligned} \widetilde{W}(\rho; a, b) &= \text{Tr}[\rho \mathcal{D}_f(a, b)] \\ &= \sum_{m=0}^{d-1} \omega(am + 2^{-1}ab) {}_x\langle m|\rho|b + m\rangle_x. \end{aligned} \quad (4.26)$$

As an example we consider the state in five dimensional Hilbert space

$$\begin{aligned} |q\rangle &= \frac{1}{3.09} [1|0\rangle_x + (2 - i)|1\rangle_x + i|2\rangle_x] \\ &\quad + \frac{1}{3.09} [(1 + 0.5i)|3\rangle_x + (1.2 + 0.7i)|4\rangle_x]. \end{aligned} \quad (4.27)$$

In Fig.(3.1) the real part of the Weyl function of the state $|q\rangle$ is demonstrated. In Fig.(3.2), the imaginary part of the Weyl functions of this state is presented. In Fig.(3.3) the absolute value of the Weyl function of the same state is shown. In Fig.(3.4) the Wigner function of the pure state $|q\rangle$ when $d = 5$ can be seen. Due to the fact that ρ is Hermitian, the Wigner function can only take real values.

From the Figs.(4.1-4.4) it shows the fact that $a, b \in \mathbb{Z}$. From the Fig.(4.4) it can be observed that the Wigner function can get also negative values.

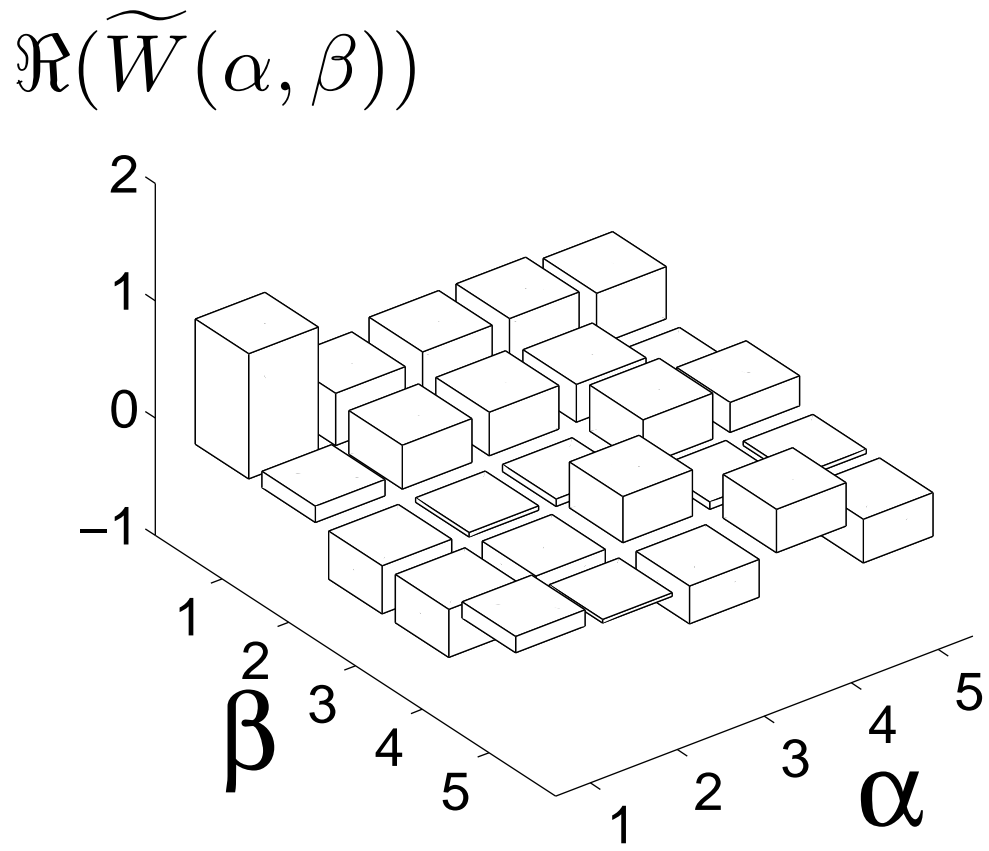


Figure 4.1: Real part of Weyl function of a pure state in Eq.(3.27), for $d = 5$

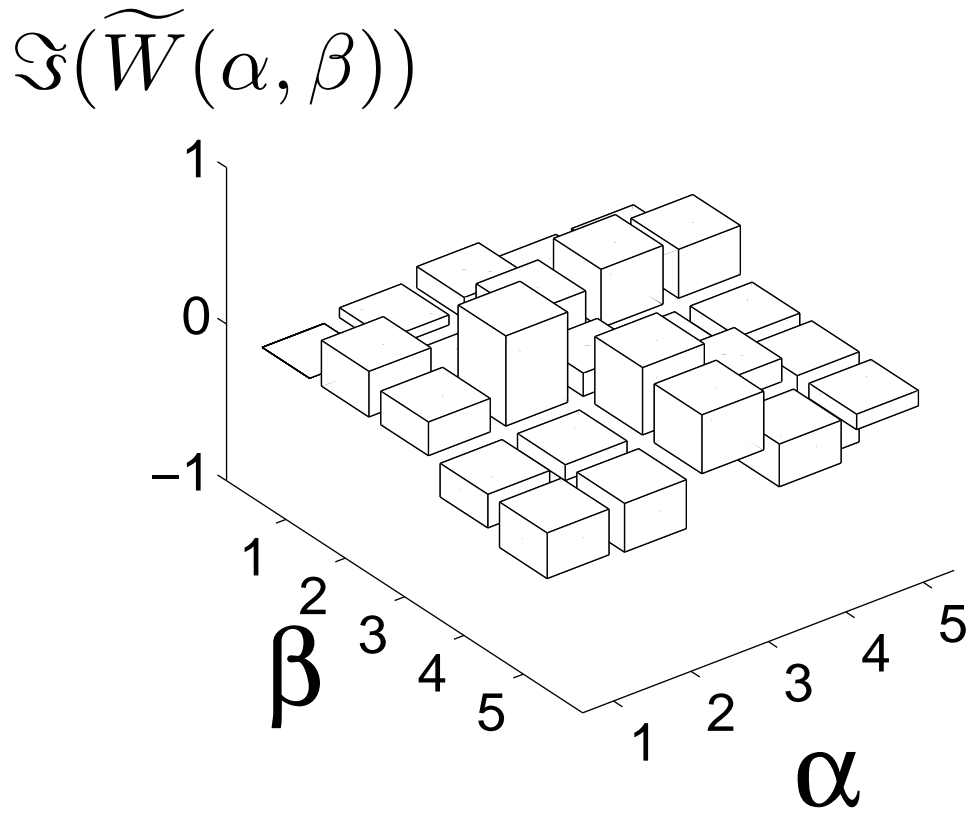


Figure 4.2: Imaginary part of Weyl function of a pure state in Eq.(3.27), for $d = 5$

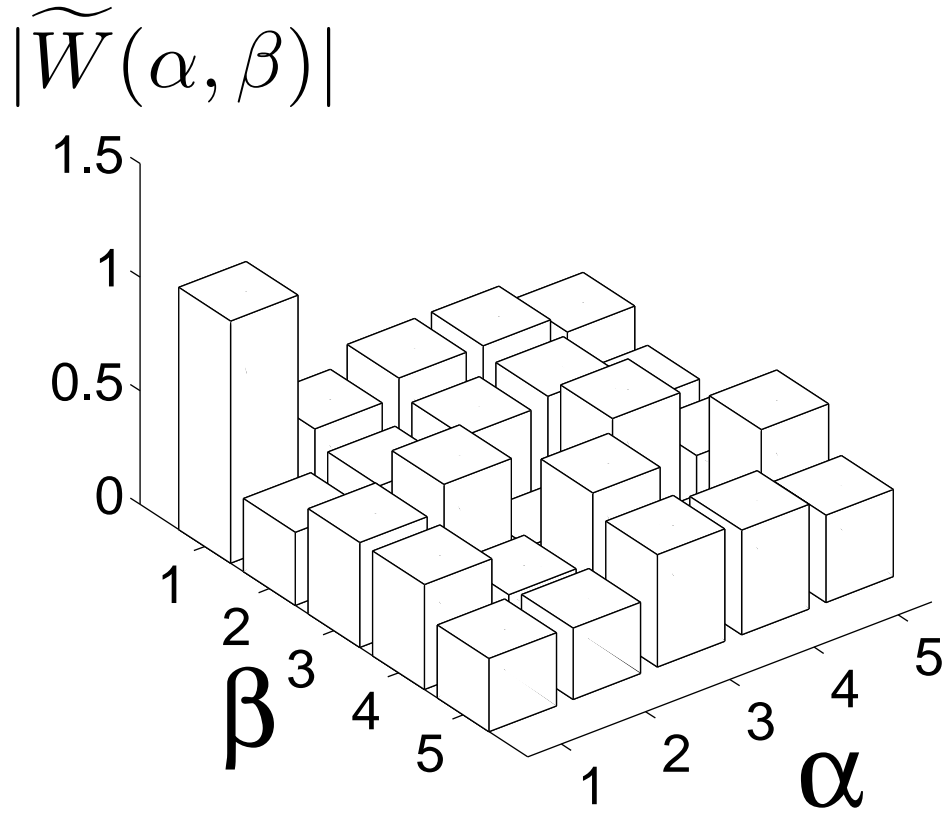


Figure 4.3: Absolute value of Weyl function of a pure state in Eq.(3.27), for $d = 5$

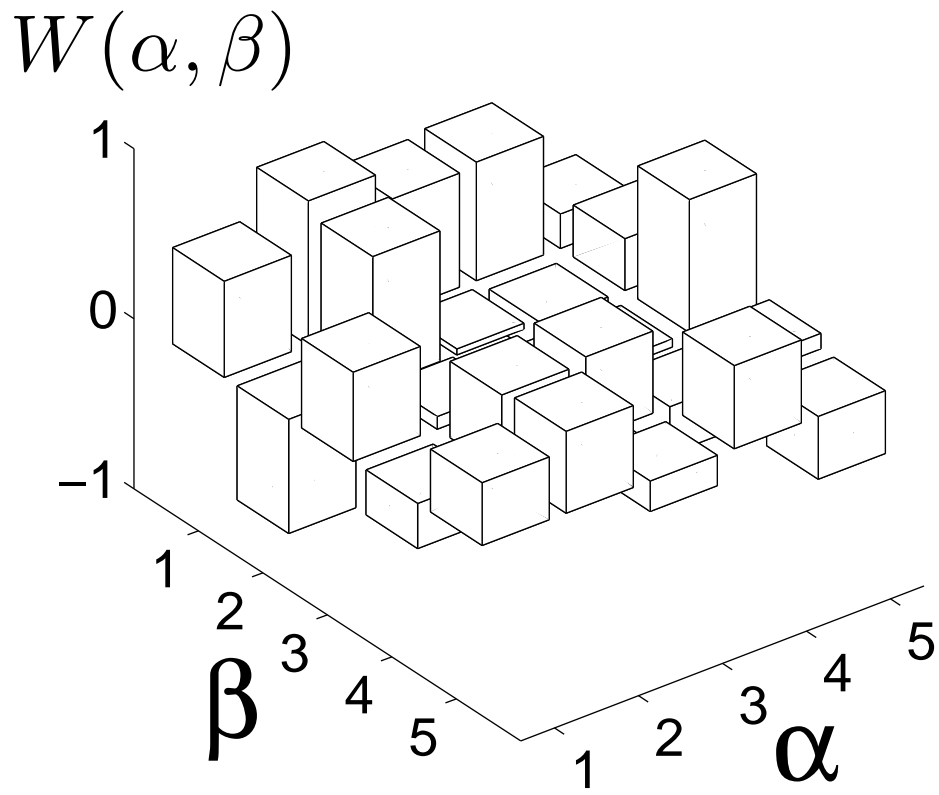


Figure 4.4: Wigner function of a pure state in Eq.(3.27), for $d = 5$

4.5.1 Marginal properties of Weyl and Wigner functions

The marginal properties of Wigner function are

$$\begin{aligned}\frac{1}{d} \sum_{b=0}^{d-1} W(\rho; a, b) &= {}_p\langle a | \rho | a \rangle_p \\ \frac{1}{d} \sum_{a=0}^{d-1} W(\rho; a, b) &= {}_x\langle b | \rho | b \rangle_x\end{aligned}\tag{4.28}$$

The marginal properties of the Weyl function are

$$\begin{aligned}\frac{1}{d} \sum_{b=0}^{d-1} \widetilde{W}(\rho; a, b) &= {}_p\langle -2^{-1}a | \rho | -2^{-1}a \rangle_p \\ \frac{1}{d} \sum_{a=0}^{d-1} \widetilde{W}(\rho; a, b) &= {}_x\langle -2^{-1}b | \rho | -2^{-1}b \rangle_x\end{aligned}\tag{4.29}$$

4.6 Discussion and Conclusion

In this chapter, the quantum systems in $\mathbb{Z}(d)$ are considered. The phase space is the toroidal lattice $\mathbb{Z}(d) \times \mathbb{Z}(d)$, therefore the eigenvalues of the position and momentum operators are integers modulo d . Moreover, the Fourier transform and the displacement operator in the finite Hilbert space are defined. The Wigner and Weyl functions in this phase space, are defined.

Chapter 5

Quantum systems on \mathbb{S}

5.1 Introduction

In this chapter, some basic concepts of quantum systems on a circle are presented. The position and momentum eigenvalues belongs to $[0, 2\pi] \times \mathbb{Z}$, respectively. Hence, the phase space is $[0, 2\pi] \times \mathbb{Z}$. In Section 5.1 the Fourier transform and the position and momentum operators are introduced. In Section 5.2 the displacement operators are considered. Finally, in the last Section, 5.3, Wigner and Weyl functions on \mathbb{S} are defined.

5.2 Position and momentum operators on a circle \mathbb{S}

The wavefunction $v(x)$ on a circle \mathbb{S} is periodic and is given by

$$v(x + 2\pi) = v(x); \quad x \in [0, 2\pi] \quad (5.1)$$

with normalization condition

$$\frac{1}{2\pi} \int_0^{2\pi} dx |v(x)|^2 = 1 \quad (5.2)$$

assuming that the radius of a circle \mathbb{S} is equal to 1.

The position and momentum operators on a circle are denoted by \mathcal{X}_c and \mathcal{P}_c , respectively, and they are defined as

$$\begin{aligned} \mathcal{X}_c &= \frac{1}{2\pi} \int_0^{2\pi} dx \, x |x\rangle \langle x| \\ \mathcal{P}_c &= \sum_{N=-\infty}^{\infty} N |N\rangle_p {}_p \langle N|. \end{aligned} \quad (5.3)$$

The index ' c ' denotes systems on a circle \mathbb{S} .

Using the Fourier transform, the wavefunction $v(x)$ can be written as

$$v(x) = \sum_{N=-\infty}^{\infty} v_N \exp(iNx) \quad (5.4)$$

where $N \in \mathbb{Z}$, and the inverse Fourier transform is given by

$$v_N = \frac{1}{2\pi} \int_0^{2\pi} dx \exp(-iNx) v(x). \quad (5.5)$$

Let $v(x)$ be the position representation of a state $|v\rangle$, then Eq.(5.4) can be written as

$$\langle x|v\rangle = \sum_{N=-\infty}^{\infty} \langle N|v\rangle \exp(iNx) \quad (5.6)$$

and v_N be a momentum representation of the same state, then Eq.(5.5) can be written as

$${}_p\langle N|v\rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle x|v\rangle \exp(-iNx) dx. \quad (5.7)$$

Let $|x\rangle$ and $|N\rangle_p$ be position and momentum states on \mathbb{S} , respectively, using Eqs.(5.6), (5.7) it can be concluded that

$$\begin{aligned} |x\rangle &= \sum_{N=-\infty}^{\infty} \exp(-iNx) |N\rangle_p \\ |N\rangle_p &= \frac{1}{2\pi} \int_0^{2\pi} dx \exp(iNx) |x\rangle. \end{aligned} \quad (5.8)$$

The scalar product of position states is equal to

$$\langle x|\psi\rangle = 2\pi\delta_c(x - \psi) \quad (5.9)$$

where $\delta_c(x - \psi)$ is Dirac comb delta function and it is defined as follows

$$\delta_c(x - \psi) = \frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \exp[iN(x - \psi)]. \quad (5.10)$$

The scalar product of the momentum states is given by

$${}_p\langle K|N\rangle_p = \delta_{KN}. \quad (5.11)$$

The resolution of identity for position and momentum states is given by

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} dx |x\rangle\langle x| &= \mathbf{1} \\ \sum_{N=-\infty}^{\infty} |N\rangle_p {}_p\langle N| &= \mathbf{1}. \end{aligned} \quad (5.12)$$

5.3 Displacement and displaced parity operators on \mathbb{S}

The displacement operator on \mathbb{S} is denoted by $\mathcal{D}_c(\alpha, N)$ and is given by

$$\mathcal{D}_c(\alpha, N) = \exp\left(-\frac{i\alpha N}{2}\right) \exp(iN\mathcal{X}_c) \exp(-i\alpha\mathcal{P}_c). \quad (5.13)$$

where $\alpha \in [0, 2\pi]$ and $N \in \mathbb{Z}$.

When the displacement operator acts on a position eigenstate the result is

$$\mathcal{D}_c(\alpha, N)|x\rangle = \exp\left[iN\left(x + \frac{\alpha}{2}\right)\right] |x + \alpha\rangle \quad (5.14)$$

while when acting on momentum states the result is

$$\mathcal{D}_c(\alpha, N)|K\rangle_p = \exp\left(-\frac{iN\alpha}{2}\right)\exp(-i\alpha K)|N+K\rangle_p. \quad (5.15)$$

The displacement operator obeys the following relations

$$\begin{aligned} [\mathcal{D}_c(\alpha, N)]^* &= \mathcal{D}_c(-\alpha, -N) \\ \mathcal{D}_c(\alpha, N)\mathcal{D}_c(\alpha_1, K) &= \mathcal{D}_c(\alpha + \alpha_1, K + N)\exp\left[\frac{i}{2}(N\alpha_1 - K\alpha)\right] \\ \mathcal{D}(\alpha + 2\pi, N) &= (-1)^N\mathcal{D}_c(\alpha, N). \end{aligned} \quad (5.16)$$

The parity operator around the origin is denoted by $\mathcal{P}_c(0, 0)$ and it is defined as follows

$$\begin{aligned} \mathcal{P}_c(0, 0) &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, |-\alpha\rangle_x {}_x\langle\alpha| \\ \mathcal{P}_c(0, 0) &= \sum_{N=-\infty}^{\infty} | -N\rangle_p {}_p\langle N|. \end{aligned} \quad (5.17)$$

The parity operator obeys the following relations

$$\begin{aligned} \mathcal{P}_c(0, 0) &= \mathcal{P}_c(0, 0)^\dagger \\ \mathcal{P}_c(0, 0)^2 &= \mathbf{1}. \end{aligned} \quad (5.18)$$

The parity operator can be written in terms of displacement operator as follows

$$\mathcal{P}_c(0, 0) = \frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \, \mathcal{D}_c(\alpha, 2N). \quad (5.19)$$

In order to prove Eq.(5.19), Eq.(5.20) is used

$$\frac{1}{2\pi} \int_0^{2\pi} d\alpha \mathcal{D}_c(\alpha, N) = | - 2N \rangle_p \langle 2N|. \quad (5.20)$$

Using Eq.(5.17) and inserting an extra \sum_N into Eq.(5.20), Eq.(5.19) is derived.

The displaced parity operator is denoted by $\mathcal{P}_c(\alpha, N)$ and it is given by

$$\begin{aligned} \mathcal{P}_c(\alpha, N) &= \mathcal{D}_c(\alpha, N) \mathcal{P}_c(0, 0) \\ &= \mathcal{P}_c(0, 0) \mathcal{D}_c(-\alpha, -N). \end{aligned} \quad (5.21)$$

The displaced parity operator obey the following relation

$$\mathcal{P}_c(\alpha + 2\pi, N) = (-1)^N \mathcal{P}_c(\alpha, N). \quad (5.22)$$

The displaced parity operator is related through the Fourier transform with the displacement operator as follows

$$\mathcal{P}_c(\alpha, N) = \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta \mathcal{D}_c(\beta, N + 2M) \exp \left(\frac{i}{2} N \beta - i M \alpha - \frac{i}{2} \alpha N \right) \quad (5.23)$$

Eq.(5.23) is proved by multiplying the left and right hand sides of Eq.(5.19) by $\mathcal{D}_c(\alpha, N)$ and using Eq.(5.21).

5.4 Wigner and Weyl functions for systems on a circle

The Wigner function [49] on a circle is defined as

$$W(\rho; \alpha, N) = \text{Tr}[\rho \mathcal{P}_c(\alpha, N)]; \quad \alpha \in [0, 2\pi], \quad N \in \mathbb{Z}; \quad (5.24)$$

where ρ is a density operator. It can be proved that the Wigner function can get only real values. If ρ is not Hermitian then the Wigner function can get complex values, as well. Using the definition of the parity displaced operator, the Wigner function can be written as

$$\begin{aligned} W(\rho; x, N) &= \frac{1}{2\pi} \int_0^{2\pi} d\alpha \langle x - \alpha | \rho | x + \alpha \rangle \exp(2iN\alpha) \\ &= \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \langle N + K | \rho | K - N \rangle \exp(2ixN) \end{aligned} \quad (5.25)$$

where $x \in [0, 2\pi]$, $N \in \mathbb{Z}$.

The Weyl function is defined as

$$\widetilde{W}(\rho; \alpha, N) = \text{Tr}[\rho \mathcal{D}_c(\alpha, N)] \quad (5.26)$$

The Weyl function is also given by

$$\begin{aligned} \widetilde{W}(\rho; \alpha, N) &= \frac{1}{2\pi} \int_0^{2\pi} dx \langle x - \frac{1}{2}\alpha | \rho | x + \frac{1}{2}\alpha \rangle \exp(iNx) \\ &= \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} {}_p\langle N + K | \rho | N - K \rangle_p \exp \left[-i\alpha \left(N - \frac{1}{2}K \right) \right]. \end{aligned} \quad (5.27)$$

The Wigner and Weyl functions are related through the 2-dimensional Fourier transform as follows

$$W(\alpha, N) = \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \int_0^{2\pi} d\beta \widetilde{W}(\beta, N + 2K) \exp(-iK\alpha + \frac{i}{2}\beta N - \frac{i}{2}\alpha N) \quad (5.28)$$

5.5 Conclusion

Overall, the phase space methods on a circle \mathbb{S} , are described. Also, the basic formalism in quantum systems on \mathbb{S} is introduced in this chapter. Moreover, position and momentum states, as well as the Fourier transform on a circle \mathbb{S} are defined. Lastly, displacement operators are examined, while the Wigner and Weyl functions are also defined. The definitions which are given in this chapter are used extensively in the chapter seven of this thesis.

Chapter 6

Analytic representations for quantum systems on $\mathbb{Z}(d)$ with Theta functions

6.1 Introduction

An analytic representation of finite quantum systems in a cell \mathcal{S} is given. The number of zeros of the analytic function in a cell \mathcal{S} is exactly d , where the zeros obey the constraint defined in Eq.(6.19). Furthermore, an analytic representation in terms of d^2 coherent states for odd values of d . Wigner and Weyl functions are also defined. In Section 6.2 the analytic representation in cell \mathcal{S} is presented. In Section 6.3, the zeros of the analytic representation are considered. In Section 6.4, the analytic representation based on d^2 coherent states, together with their properties, are studied. Meanwhile, in Section 6.5, an analytic representation in terms of a fiducial state $\mathcal{F}|\psi\rangle$ is defined. In

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Section 6.6, the entropic uncertainty relations for these coherent states are given and in Section 6.7 Wigner and Weyl functions are given.

6.2 Analytic Representation on a torus

Let $|\phi\rangle$ be a normalized pure state

$$|\phi\rangle = \sum_{m=0}^{d-1} \phi_m |m\rangle_x; \quad \sum_{m=0}^{d-1} |\phi_m|^2 = 1. \quad (6.1)$$

The following notation shall be used

$$|\phi^*\rangle = \sum_{m=0}^{d-1} \phi_m^* |m\rangle_x; \quad \langle\phi| = \sum_{m=0}^{d-1} \phi_m^* {}_x\langle m|; \quad \langle\phi^*| = \sum_{m=0}^{d-1} \phi_m {}_x\langle m|. \quad (6.2)$$

The Fourier transform coefficients can be written as

$$\begin{aligned} \tilde{\phi}_m &= {}_p\langle m|\phi\rangle = {}_x\langle m|\mathcal{F}_f^\dagger|\phi\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{m_1, n=0}^{d-1} \omega(-m_1 n) {}_x\langle m|m_1\rangle_x {}_x\langle n|\phi\rangle \\ &= \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \omega(-mn) {}_x\langle n|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \omega(-mn) \phi_n. \end{aligned} \quad (6.3)$$

The analytic representation of state $|\phi\rangle$ is defined as follows,

$$\Phi(z) = \pi^{-1/4} \sum_{m=0}^{d-1} \phi_m \Theta_3 \left[\frac{2m\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right]; \quad L = \sqrt{2\pi d}, \quad (6.4)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

$\Theta_3[u; \tau]$ is defined as

$$\Theta_3[u; \tau] = \sum_{N=-\infty}^{\infty} \exp(2iNu + i\pi\tau N^2) \quad (6.5)$$

and it has the following property

$$\Theta_3[u; \tau] = (-i\tau)^{-1/2} \exp\left(\frac{u^2}{\pi i\tau}\right) \Theta_3\left[\frac{u}{\tau}; -\frac{1}{\tau}\right]. \quad (6.6)$$

The quasiperiodicity condition of $\Phi(z)$ along the real axis is proved by the use of Eq.(2.2)

$$\Phi(z + L) = \Phi(z). \quad (6.7)$$

The quasiperiodicity condition of $\Phi(z)$ along the imaginary axis is proved using Eq.(2.3)

$$\Phi(z + iL) = \Phi(z) \exp\left(\frac{L^2}{2} - iLz\right). \quad (6.8)$$

The analytic function $\Phi(z)$ is defined on a square cell $\mathcal{S} = [ML, (M+1)L) \times [NL, (N+1)L)$ where M, N are integers labelling the square cell $\mathcal{S} \in \mathbb{C}$.

The scalar product of these two states in terms of their analytic representations is equal to

$$\langle \phi_1^* | \phi_2 \rangle = \frac{2\pi}{L^3} \int_{\mathcal{S}} d^2z \exp(-z_I^2) \Phi_1(z) \Phi_2(z^*) \quad (6.9)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

and the coefficients ϕ_m are

$$\begin{aligned}\phi_m &= \frac{2\pi^{3/4}}{L^3} \int_S d^2z \exp(-z_I^2) \Theta_3 \left[\frac{2m\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right] \Phi(z^*) \\ \tilde{\phi}_m &= \frac{2\pi^{3/4}}{L^3} \sum_{n=0}^{d-1} \omega(-mn) \int_S d^2z \exp(-z_I^2) \Theta_3 \left[\frac{2n\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right] \Phi(z^*)\end{aligned}\tag{6.10}$$

where $\tilde{\phi}_m$ are the Fourier transform coefficients in terms of $\Phi(z)$.

The orthogonality relation proves the validity of Eq.(6.9) and (6.10), as given by

$$\begin{aligned}& \frac{2\pi^{1/2}}{L^3} \int_S d^2z \exp(-z_I^2) \Theta_3 \left[\frac{2n\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right] \\ & \times \Theta_3 \left[\frac{2m\pi^2}{L^2} - z^* \frac{\pi}{L}; \frac{i2\pi}{L^2} \right] = \delta(m, n).\end{aligned}\tag{6.11}$$

Furthermore, the analytic representation of position eigenstates is given by

$$|m\rangle_x \rightarrow \pi^{-1/4} \Theta_3 \left[\frac{2m\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right].\tag{6.12}$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

The momentum representation is proved according the following steps

$$\begin{aligned}
|k\rangle_p &= \frac{\sqrt{2\pi}}{L} \sum_{m=0}^{d-1} \omega(km) \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{2i\pi}{L^2} \right] \\
&= \frac{L}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp \left(-2in z \frac{\pi}{L} - \frac{2\pi}{L^2} n^2 \right) \frac{2\pi}{L^2} \sum_{m=0}^{d-1} \omega(km + nm) \\
&= \frac{L}{\sqrt{2\pi}} \sum_{n=-k+dN}^{\infty} \exp \left(-2in z \frac{\pi}{L} - \frac{2\pi}{L^2} n^2 \right) \\
&= \frac{L}{\sqrt{2\pi}} \sum_{N=-\infty}^{\infty} \exp \left[\frac{2\pi^2}{L^2} \left(-k + \frac{L^2}{2\pi} N \right)^2 + 2i \left(-k + \frac{L^2}{2\pi} N \right) z \frac{\pi}{L} \right] \\
&= \frac{L}{\sqrt{2\pi}} \exp \left(-\frac{2\pi^2 k^2}{L^2} + 2ikz \frac{\pi}{L} \right) \Theta_3 \left[-i\pi k - z \frac{\pi L^2}{4\pi}; i \frac{L^2}{2\pi} \right] \quad (6.13)
\end{aligned}$$

Eqs.(6.6) and (6.13) are used to define the momentum representation

$$|k\rangle_p \rightarrow \pi^{-1/4} \exp \left(-\frac{1}{2} z^2 \right) \Theta_3 \left[\frac{2k\pi^2}{L^2} - iz \frac{\pi}{L}; \frac{i2\pi}{L^2} \right]. \quad (6.14)$$

In the following example, assuming that $d = 5$, the graph of the analytic function $\Phi(z)$ is plotted, using the following coefficients

$$\begin{aligned}
\phi_0 &= 0.6033 - 0.2296i; \quad \phi_1 = 0.3030 + 0.3469i; \quad \phi_2 = 0.0854 + 0.0716i; \\
\phi_3 &= 0.1438 + 0.2450i; \quad \phi_4 = 0.0937 - 0.5189i. \quad (6.15)
\end{aligned}$$

In the first two graphs Figs.(5.1), (5.2) the real and the imaginary parts of the function $\Phi(z)$ are demonstrated, respectively, while its absolute value is shown in Fig.(5.3). From Figs.(5.1-5.3) it can be concluded that the theta function is a periodic function.

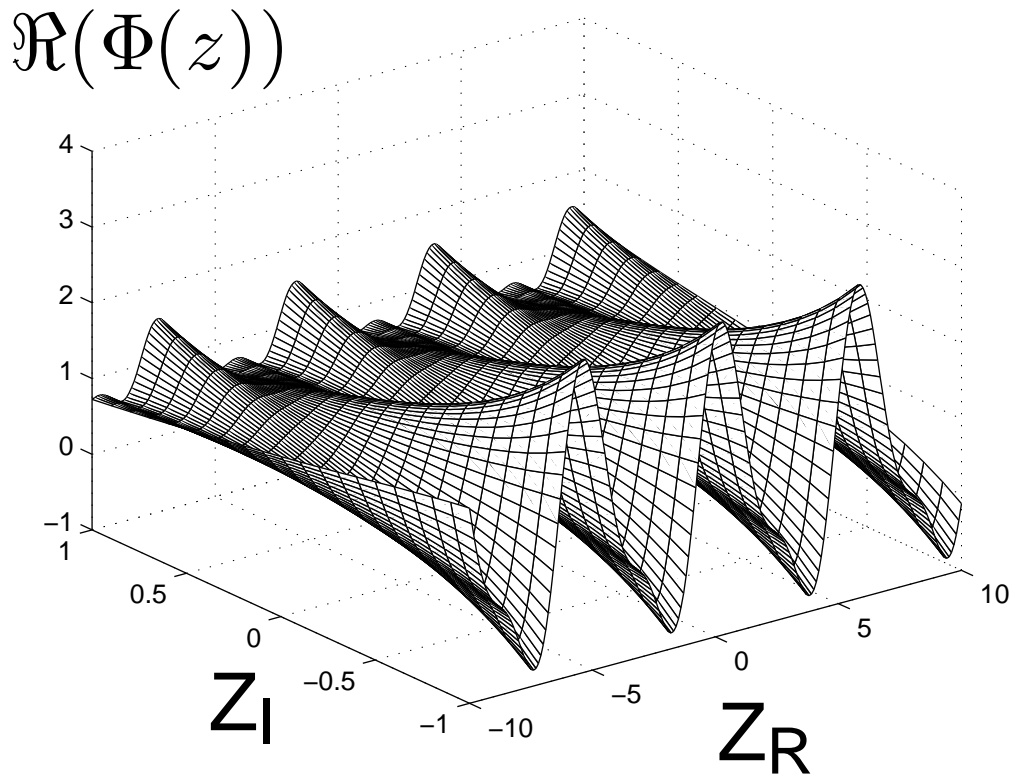


Figure 6.1: Using the coefficients of Eq.(6.15) the real part of function $\Phi(z)$ is demonstrated, for $d = 5$

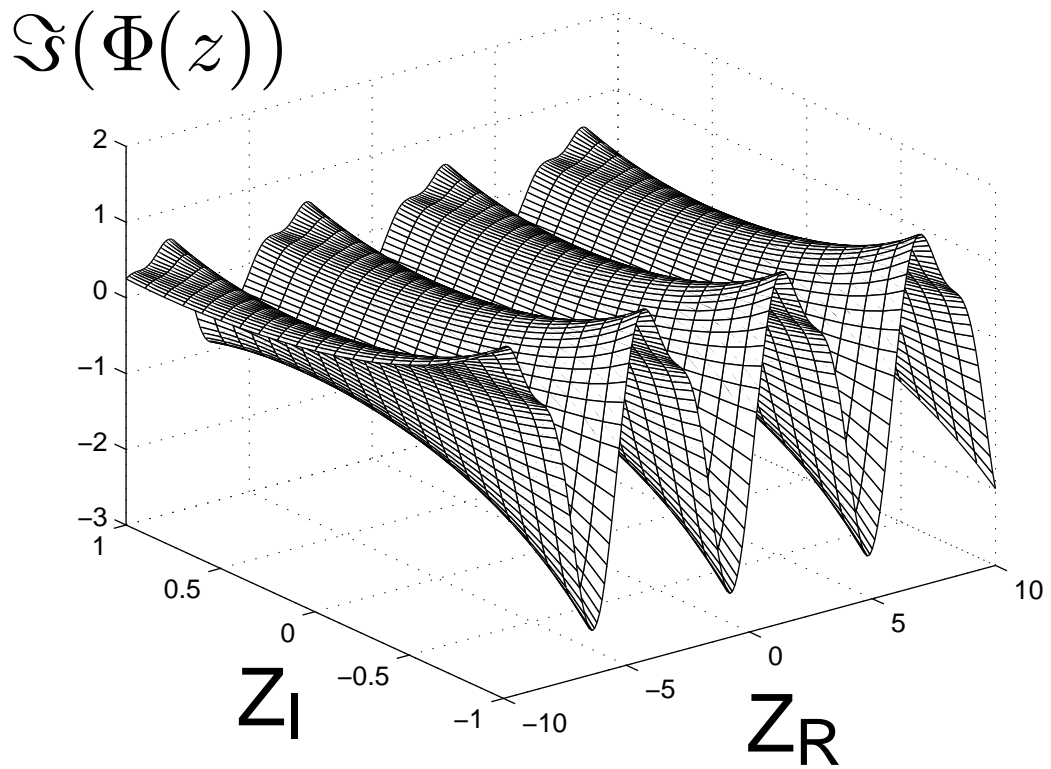


Figure 6.2: Using the coefficients of Eq.(6.15) the imaginary part of function $\Phi(z)$ is shown, for $d = 5$.

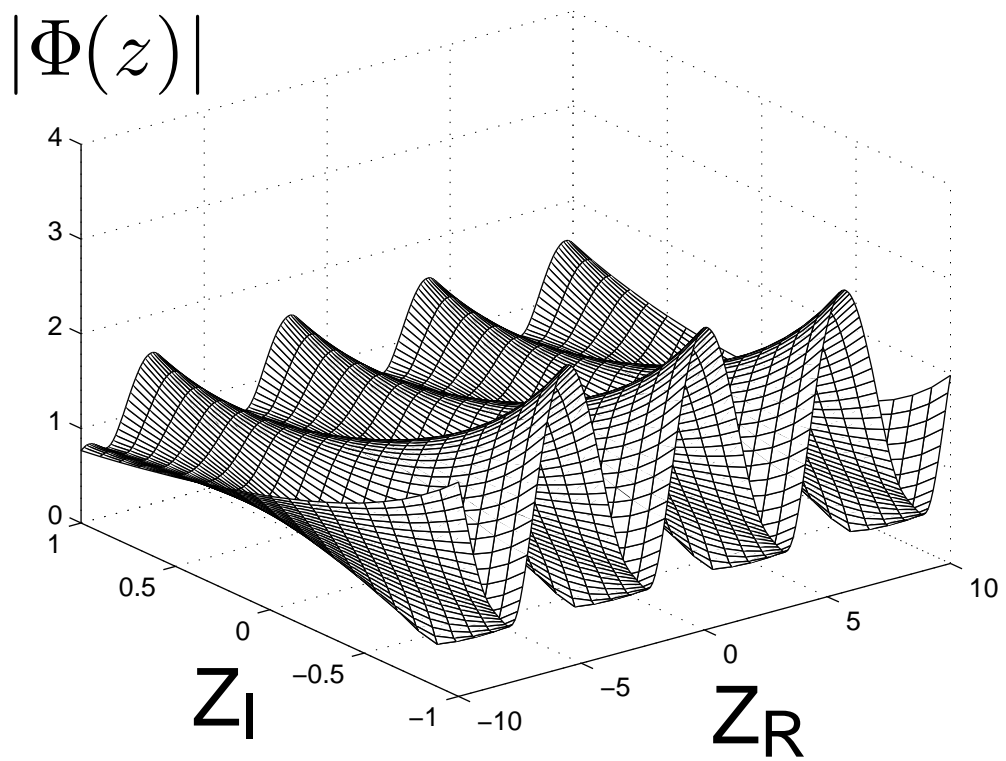


Figure 6.3: Using the coefficients of Eq.(6.15) the absolute value of function $\Phi(z)$ is illustrated, for $d = 5$.

6.3 Zeros of analytic function

Let ζ_n be the zeros of analytic function $\mathfrak{G}(z)$, such as $\mathfrak{G}(\zeta_n) = 0$. The first integral inside the contour γ gives the number of zeros of the function $\mathfrak{G}(z)$, while the second integral gives the sum of the zeros of the analytic function

$$I_0 = \frac{1}{2\pi i} \oint_{\gamma} dz \frac{\mathfrak{G}'(z)}{\mathfrak{G}(z)}; \quad I_1 = \frac{1}{2\pi i} \oint_{\gamma} dz \frac{\mathfrak{G}'(z)}{\mathfrak{G}(z)} z \quad (6.16)$$

where $\mathfrak{G}(z)$ is analytic function and γ is a piecewise continuously differentiable path. The proof has been given in [34, 36, 37].

The quasiperiodicity conditions are used to observe that the analytic function $\Phi(z)$ has exactly d zeros. This is proved using the following contour integral inside the cell \mathcal{S}

$$\frac{1}{2\pi i} \oint_{\mathcal{S}} dz \frac{\Phi'(z)}{\Phi(z)} = d. \quad (6.17)$$

The sum of d zeros of $\Phi(z)$ obeys the following constraint which is proved using the quasiperiodicity relations

$$\frac{1}{2\pi i} \oint_{\mathcal{S}} dz \frac{\Phi'(z)}{\Phi(z)} z = \frac{L^3}{2\pi} (M + iN) + \frac{L^3}{4\pi} (1 + i). \quad (6.18)$$

As a result, the sum of zeros is given as

$$\sum_{\nu=1}^{d-1} \zeta_{\nu} = \frac{L^3}{2\pi} (M + iN) + \frac{L^3}{4\pi} (1 + i) \quad (6.19)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

where ζ_ν are the zeros of analytic function $\Phi(z)$, such as $\Phi(\zeta_\nu) = 0$. The proof has been given in [42, 43, 45, 72].

6.3.1 Construction of the analytic representation from the zeros

Given that the d zeros, of $\Phi(z)$ are denoted by ζ_ν (where $\nu = 0, 1, 2, \dots, d-1$), obey the constraint of Eq.(6.19), it is concluded that, knowing $d-1$ zeros, the last zero can be found. A function which has ζ_ν as zeros is considered

$$y(z) = \prod_{\nu=1}^d \Theta_3 \left[\frac{\pi}{L} (z - \zeta_\nu) + \frac{\pi(1+i)}{2}; i \right]. \quad (6.20)$$

It is evident that the entire function $\frac{\Phi(z)}{y(z)}$ has no zeros. Therefore, the exponential of this entire function is given as

$$\Phi(z) = y(z) \exp [q(z)]. \quad (6.21)$$

The periodicity conditions of $\Phi(z)$ are used, resulting to

$$\begin{aligned} q(z+L) &= q(z) + 2i\pi N \\ q(z+iL) &= q(z) + 2\pi N + 2\pi iK \end{aligned} \quad (6.22)$$

where N has an equal value to that of the constraint of Eq.(6.19), K is an arbitrary integer and $q(z)$ is a polynomial of maximum second degree. This can be explained by the fact that the order of growth of $\Phi(z)$ is equal to 2 and the order of $y(z)$ is equal to 2.

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

From the periodicity equation of $q(z)$, it can be found that

$$q(z) = -\frac{\pi}{L}Nzi + \sigma. \quad (6.23)$$

Substituting Eq.(6.23) into Eq.(6.21) and using Eq.(6.20) resulting in

$$\begin{aligned} \Phi(z) &= \exp(\sigma) \exp\left(-\frac{\pi}{L}Nzi\right) y(z) \\ &= \exp(\sigma) \exp\left(-\frac{\pi}{L}Nzi\right) \prod_{\nu=1}^d \Theta_3\left[\frac{\pi}{L}(z - \zeta_\nu) + \frac{\pi(1+i)}{2}; i\right] \end{aligned} \quad (6.24)$$

where $\exp(\sigma) = C$ is normalization constant and N belongs to \mathbb{Z} .

At this point, a numerical example is considered so that the coefficients ϕ_m are identified and the analytic function is constructed.

In order to find the coefficients ϕ_m , d arbitrary numbers z_0, z_1, \dots, z_{d-1} are inserted, solving one system with d equations and d unknowns. The assumption that the coefficients are normalized is made in this procedure. This is demonstrated by the equation below

$$\Phi(z_n) = \pi^{-1/4} \sum_{m=0}^{d-1} \phi_m \Theta_3\left[\frac{2\pi^2 m}{L} - z_n \frac{\pi}{L}; i \frac{2\pi}{L^2}\right]. \quad (6.25)$$

An example when $d = 3$ is provided below

$$\zeta_0 = 2.96 + 2.33i; \quad \zeta_1 = 2.15 + 2.32i; \quad (6.26)$$

The third zero is equal to $\zeta_2 = 1.4 + 1.85i$, as calculated by the constraint in Eq.(6.19).

Three arbitrary values 0, 1, 2 together with the values of zeros are substituted

into Eq.(6.24) so that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$ are calculated. When these values of $\Phi(\zeta_n)$ are inserted into Eq.(6.25), one system with three equations and three unknowns is derived. In this particular case, the results after solving this system is the following

$$\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0.8043 + 0.0899i \\ 0.4188 - 0.0281i \\ 0.4073 - 0.0542i \end{bmatrix} \quad (6.27)$$

6.4 Coherent states for finite systems

Let $\Psi(z)$ be the analytic representation of a fiducial state $|\psi\rangle$. At this point, the analytic representation in terms of d^2 states $\mathcal{D}_f(a, b)|\psi\rangle$ is considered. In the case that the fiducial vector represents either a position or a momentum state several of the $\mathcal{D}_f(a, b)|\psi\rangle$ differ by a phase factor, and represent the same physical state. Let $|a, b\rangle = \mathcal{D}_f(a, b)|\psi\rangle$ be coherent state for a finite system.

The fiducial vector can be expanded as in Eq.(6.1), then the scalar product is given by

$$\langle -a_1, -b_1 | a, b \rangle = \omega \left[\frac{1}{2} (ab + a_1 b_1) - b a_1 \right] \sum_{n=0}^{d-1} \psi_{n+b-b_1}^* \psi_n \omega[(a - a_1)n] \quad (6.28)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Eq.(6.28) is proved according the following steps

$$\begin{aligned}
 \langle -a_1, -b_1 | a, b \rangle &= \sum_{m,n=0}^{d-1} \psi_m \psi_n^* {}_x \langle m | \mathcal{D}_f(a - a_1, b - b_1) | n \rangle_x \\
 &= \omega \left[\frac{1}{2} (ab + a_1 b_1) - ba_1 \right] \sum_{n=0}^{d-1} \psi_{n+b-b_1}^* \psi_n \omega[(a - a_1)n]
 \end{aligned} \tag{6.29}$$

Definition 1. *The analytic function $F(z; a, b; \psi)$ in terms of $\mathcal{D}_f(a, b)|\psi\rangle$ is defined as follows*

$$F(z; a, b; \psi) = \pi^{-1/4} \sum_{m=0}^{d-1} {}_x \langle m | \mathcal{D}_f(a, b) | \psi \rangle \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \tag{6.30}$$

where $a, b \in \mathbb{Z}(d)$.

Comparing Eq.(6.30) with Eq.(6.4) it can be concluded that

$$F(z; 0, 0; \psi) = \Psi(z). \tag{6.31}$$

Using the properties of Theta functions Eq.(2.2), (2.3) it can be concluded that this function obeys the quasi-periodicity relations in Eq.(6.7)

$$\begin{aligned}
 F(z + L; a, b; \psi) &= F(z; a, b; \psi) \\
 F(z + iL; a, b; \psi) &= F(z; a, b; \psi) \exp(L^2 - iLz).
 \end{aligned} \tag{6.32}$$

The relation between the analytic representation $F(z; a, b; \psi)$ of the coherent state $|a, b\rangle$ and the analytic representation $\Psi(z)$ of the fiducial vector $|\psi\rangle$ is given by the next proposition.

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM
SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Proposition 1.

$$\begin{aligned} F(z; a, b; \psi) &= \exp\left(-\frac{2a^2\pi^2}{L^2} + 2iza\frac{\pi}{L}\right) \omega(-2^{-1}ab) \\ &\quad \times \Psi\left(z + ia\frac{2\pi}{L} - b\frac{2\pi}{L}\right) \end{aligned} \quad (6.33)$$

Proof. Using **Definition 1** and changing $M = m - b$ it can proved that

$$\begin{aligned} F(z; a, b; \psi) &= \pi^{-1/4} \sum_{M=0}^{d-1} \omega(2^{-1}ab + aM) \psi_M \\ &\quad \times \Theta_3\left[\frac{2\pi^2(M+b)}{L^2} - z\frac{\pi}{L}; i\frac{2\pi}{L^2}\right] \end{aligned} \quad (6.34)$$

Using Eq.(6.5) and changing $M = m$, Eq.(6.34) becomes

$$\begin{aligned} F(z; a, b; \psi) &= \pi^{-1/4} \omega(2^{-1}ab) \sum_{m=0}^{d-1} \psi_m \sum_{n=-\infty}^{\infty} \exp\left(\frac{4i\pi^2 mn}{L^2} + \frac{4i\pi^2 ma}{L^2}\right) \\ &\quad \times \exp\left(\frac{i4n\pi^2 b}{L^2} - 2inz\frac{\pi}{L} - \frac{2\pi n^2}{L^2}\right). \end{aligned} \quad (6.35)$$

Letting $N = n + a$, it follows that

$$\begin{aligned} F(z; a, b; \psi) &= \exp\left(-\frac{2a^2\pi^2}{L^2} + 2iza\frac{\pi}{L}\right) \omega(-2^{-1}ab) \\ &\quad \times \Psi\left(z + ia\frac{2\pi}{L} - b\frac{2\pi}{L}\right). \end{aligned} \quad (6.36)$$

□

In Fig.(6.4) the real part of the analytic function $F(z; 2, 1; \psi)$ is shown

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

using the following coefficients [73]

$$\begin{aligned}\psi_0 &= 0.14 + 0.42i; & \psi_1 &= 0.91 + 0.79i & \psi_2 &= 0.95 + 0.65i; \\ \psi_3 &= 0.03 + 0.84i; & \psi_4 &= 0.93 - 0.67i.\end{aligned}\tag{6.37}$$

In Figs.(6.5), (6.6) the imaginary and the absolute value of analytic function $F(z; 2, 1; \psi)$ are plotted, respectively, using the same coefficients. From the graphs it can be concluded that the period of the function $F(z; 2, 1; \psi)$ and the function $\Phi(z)$ is the same.

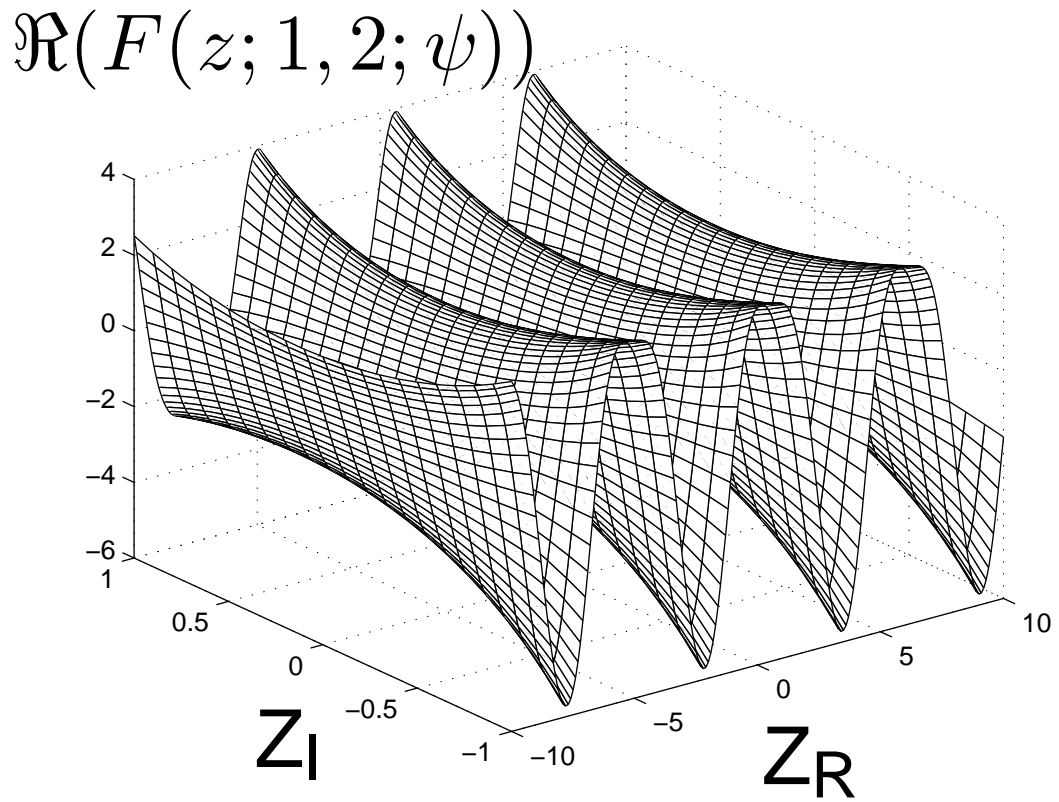


Figure 6.4: The real part of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is demonstrated, for $d = 5$

$$\Im(F(z; 1, 2; \psi))$$

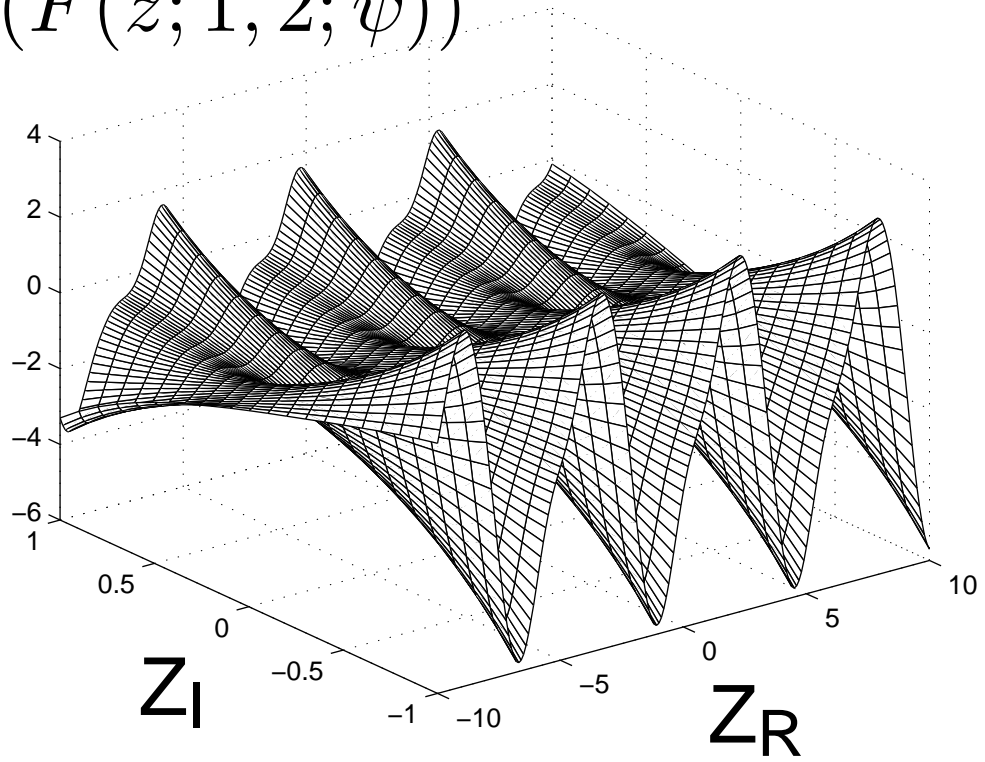


Figure 6.5: The imaginary part of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is presented, for $d = 5$

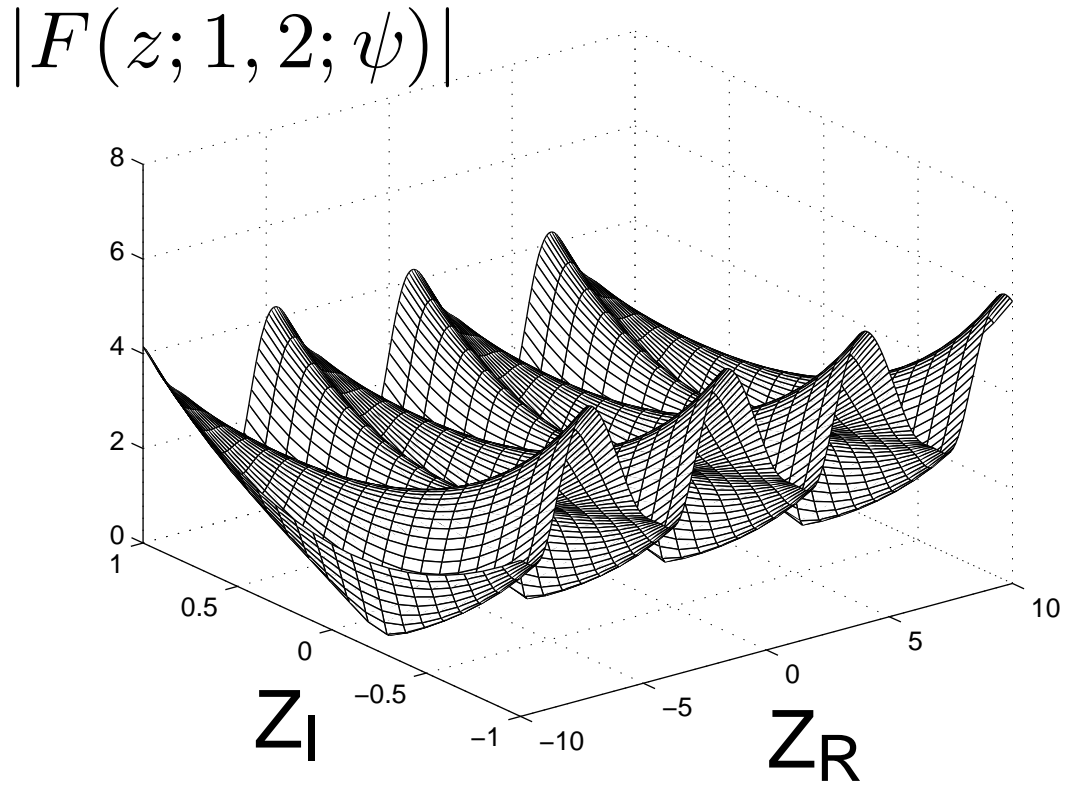


Figure 6.6: The absolute value of $F(z; 2, 1; \psi)$ using the coefficients of fiducial vector in Eq.(6.37) is illustrated, for $d = 5$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Proposition 2. *The zeros ζ_ν of analytic function $\Psi(z)$ is related with the zeros $\zeta_\nu(a, b)$ of $F(z; a, b; \psi)$ as follows*

$$\zeta_\nu(a, b) = \zeta_\nu - i\frac{2\pi a}{L} + \frac{2\pi b}{L}. \quad (6.38)$$

Proof. Eq.(6.38) is immediately consequens of **Proposition 1**. \square

Proposition 3. *The $F(z; \gamma, \delta; \psi)$ is a two-dimensional Fourier transform of $F(-z; \gamma, \delta; \psi)$*

$$F(z; \gamma, \delta; \psi) = \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} \omega(-2^{-1}b\gamma + 2^{-1}a\delta) F(-z; a, b; \psi) \quad (6.39)$$

Proof. The proof is given in Appendix \square

Proposition 4. *The resolution of identity is given by*

$$\frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, b; \psi) F(w; a, b; \psi)^* = \mathfrak{K}(z, w^*) \quad (6.40)$$

and $\mathfrak{K}(z, w^*)$ is the reproducing kernel which is equal to

$$\begin{aligned} \mathfrak{K}(z, w^*) &= \pi^{-1/2} \sum_{m=0}^{d-1} \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \Theta_3 \left[\frac{2\pi^2 m}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\ \mathfrak{K}(z, w^*) &= \mathfrak{K}(w^*, z); \quad \mathfrak{K}(z, w^*) = \mathfrak{K}(-z, w^*). \end{aligned} \quad (6.41)$$

$\mathfrak{K}(z, w^*)$ does not depend on fiducial vector $|\psi\rangle$.

Proof. The resolution of identity is based on the following property

$$\frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} \mathcal{D}_f(a, b) |\psi\rangle \langle \psi| \mathcal{D}_f(-a, -b) = \mathbf{1}. \quad (6.42)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Both sides of Eq.(6.42) are multiplied by $\Theta_3[u; \tau]$ and $|m\rangle_x$ and ${}_x\langle n|$, then Eq.(6.42) becomes

$$\begin{aligned} & \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} {}_x\langle n | \mathcal{D}_f(a, b) | \psi \rangle \langle \psi | \mathcal{D}_f(-a, -b) | n \rangle_x \\ & \times \Theta_3 \left[\frac{2\pi^2 n}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\ & = \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] {}_x\langle n | \mathbf{1} | m \rangle_x \Theta_3 \left[\frac{2\pi^2 n}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right]. \end{aligned} \quad (6.43)$$

Taking the summation over n, m leads to

$$\begin{aligned} & \frac{2\pi}{L^2} \sum_{a,b,n,m=0}^{d-1} \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] {}_x\langle n | \mathcal{D}_f(a, b) | \psi \rangle \\ & \times \langle \psi | \mathcal{D}_f(-a, -b) | m \rangle_x \Theta_3 \left[\frac{2\pi^2 n}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\ & = \sum_{m=0}^{d-1} \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \Theta_3 \left[\frac{2\pi^2 m}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (6.44)$$

Hence,

$$\begin{aligned} & \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, b; \psi) F(w; a, b; \psi)^* \\ & = \pi^{-1/2} \sum_{m=0}^{d-1} \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \Theta_3 \left[\frac{2\pi^2 m}{L^2} - w^* \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (6.45)$$

□

Proposition 5. *The reproducing kernel property is*

$$\Phi(z) = \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) \mathfrak{K}(z, w^*) \Phi(w); \quad d\alpha(w) = d^2 w \exp(-w_I^2) \quad (6.46)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Proof. In order to prove Eq.(6.46), Eq.(6.41) is substituted into Eq.(6.46)

$$\begin{aligned}\Phi(z) &= \pi^{-1/2} \sum_{m=0}^{d-1} \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{i}{d} \right] \\ &\quad \times \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) \Theta_3 \left[\frac{2\pi^2 m}{L^2} - w^* \frac{\pi}{L}; \frac{i}{d} \right] \Phi(w)\end{aligned}\quad (6.47)$$

Both sides of Eq.(6.47) are multiplied by $\frac{2\pi^{3/4}}{L^3}$ and using Eq.(6.10) it can be concluded that

$$\Phi(z) = \pi^{-1/4} \sum_{m=0}^{d-1} \phi_m \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{i}{d} \right] = \Phi(z). \quad (6.48)$$

□

Proposition 6. $\Phi(z)$ can be expressed as

$$\begin{aligned}\Phi(z) &= \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, b; \psi) \phi(a, b; \psi) \\ &= \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} F(-z; \gamma, \delta; \psi) \tilde{\phi}(\gamma, \delta; \psi)\end{aligned}\quad (6.49)$$

where $\phi(a, b; \psi)$ is given by

$$\phi(a, b; \psi) = \langle \psi | \mathcal{D}_f(-a, -b) | \phi \rangle \quad (6.50a)$$

$$= \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) [F(w; a, b; \psi)]^* \Phi(w). \quad (6.50b)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM
SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

and $\tilde{\phi}(\gamma, \delta; \psi)$ is equal to

$$\tilde{\phi}(\gamma, \delta; \psi) = \langle \psi | \mathcal{D}_f(-\gamma, -\delta) \mathcal{P}_f(0, 0) | \phi \rangle \quad (6.51a)$$

$$= \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) [F(-w; a, b; \psi)]^* \Phi(w). \quad (6.51b)$$

Proof. Inserting Eq.(6.50a) into Eq.(6.49) and using Eqs.(6.10), (6.42) leads to

$$\begin{aligned} \Phi(z) &= \frac{2\pi}{L^2} \pi^{-1/4} \sum_{a,b,m=0}^{d-1} {}_x\langle m | \mathcal{D}_f(a, b) | \psi \rangle \langle \psi | \mathcal{D}_f(-a, -b) | \phi \rangle \\ &\quad \times \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\ &= \pi^{-1/4} \sum_{m=0}^{d-1} {}_x\langle m | \phi \rangle \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] = \Phi(z). \end{aligned} \quad (6.52)$$

Inserting Eq.(6.50b) into Eq.(6.49) and using Eq.(6.40), it follows that

$$\begin{aligned} \Phi(z) &= \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, b; \psi) \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) [F(w; a, b; \psi)]^* \Phi(w) \\ &= \frac{2\pi}{L^3} \int_{\mathcal{S}} d\alpha(w) \mathfrak{K}(z, w^*) \Phi(w) = \Phi(z). \end{aligned} \quad (6.53)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Eq.(6.51a) inserting into Eq.(6.49) and changing the variables lead to

$$\begin{aligned}
 \Phi(z) &= \frac{2\pi}{L^2} \pi^{-1/4} \sum_{a,b,m=0}^{d-1} {}_x\langle m | \mathcal{D}_f(a, b) | \psi \rangle \langle \psi | \mathcal{D}_f(-a, -b) \mathcal{P}_f(0, 0) | \phi \rangle \\
 &\quad \times \Theta_3 \left[\frac{2\pi^2 m}{L^2} + z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\
 &= \pi^{-1/4} \sum_{m=0}^{d-1} {}_x\langle m | \phi \rangle \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] = \Phi(z). \tag{6.54}
 \end{aligned}$$

Inserting Eq.(6.51b) into Eq.(6.49) and using Eq.(6.40), it follows that

$$\begin{aligned}
 \Phi(z) &= \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(-z; a, b; \psi) \frac{2\pi}{L^3} \int_S d\alpha(w) [F(-w; a, b; \psi)]^* \Phi(w) \\
 &= \frac{2\pi}{L^3} \int_S d\alpha(w) \mathfrak{K}(z, w^*) \Phi(w) = \Phi(z). \tag{6.55}
 \end{aligned}$$

□

Proposition 7. *The $\tilde{\phi}(\gamma, \delta; \psi)$ is a two-dimensional Fourier transform of $\phi(a, b; \psi)$*

$$\tilde{\phi}(\gamma, \delta; \psi) = \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} \phi(a, b; \psi) \omega(-2^{-1}b\gamma + 2^{-1}a\delta). \tag{6.56}$$

Proof. Eq.(6.56) is proved by inserting Eq.(6.50b) into Eq.(6.56) and using Eqs.(6.39), (5.57b)

$$\begin{aligned}
 \tilde{\phi}(\gamma, \delta; \psi) &= \frac{4\pi^2}{L^6} \sum_{a,b=0}^{d-1} \omega(-2^{-1}b\gamma + 2^{-1}a\delta) \int_S d\alpha(w) [F(w; a, b; \psi)]^* \Phi(w) \\
 &= \frac{2\pi}{L^3} \int_S d\alpha(w) [F(-w; \gamma, \delta; \psi)]^* \Phi(w) = \tilde{\phi}(\gamma, \delta; \psi). \tag{6.57}
 \end{aligned}$$

□

Proposition 8. *The marginal properties are*

$$\begin{aligned}
 \frac{2\pi}{L^2} \sum_{a=0}^{d-1} F(z; a, 2b; \psi) &= \pi^{-1/4} \psi_{-b} \Theta_3 \left[\frac{2\pi^2 b}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\
 \frac{2\pi}{L^2} \sum_{b=0}^{d-1} F(z; 2a, b; \psi) &= \pi^{-1/4} \tilde{\psi}_{-a} \exp \left(-\frac{z^2}{2} \right) \Theta_3 \left[\frac{2\pi^2 a}{L^2} - iz \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\
 \frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, 2b; \psi) &= \Psi(-z).
 \end{aligned} \tag{6.58}$$

where $\Theta_3 \left[\frac{2\pi^2 b}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right]$ represents position states and $\exp \left(-\frac{z^2}{2} \right) \Theta_3 \left[\frac{2\pi^2 a}{L^2} - iz \frac{\pi}{L}; i \frac{2\pi}{L^2} \right]$ represents momentum states (see Eqs.(6.12), (6.14)).

Proof. The first property can be proved using Eq.(6.34), after taking the summation over a and applying it to both sides

$$\begin{aligned}
 \frac{2\pi}{L^2} \sum_{a=0}^{d-1} F(z; a, 2b; \psi) &= \pi^{-1/4} \sum_{m=0}^{d-1} \psi_m \frac{2\pi}{L^2} \sum_{a=0}^{d-1} \omega(ab + am) \\
 &\quad \times \Theta_3 \left[\frac{2\pi^2 (m + 2b)}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\
 \frac{2\pi}{L^2} \sum_{a=0}^{d-1} \omega(ab + am) &= \delta(b, -m)
 \end{aligned} \tag{6.59}$$

$$\begin{aligned}
 \frac{2\pi}{L^2} \sum_{a=0}^{d-1} F(z; a, 2b; \psi) &= \pi^{-1/4} \sum_{m=0}^{d-1} \psi_m \Theta_3 \left[\frac{2\pi^2 (m + 2b)}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\
 &\quad \times \delta(b, -m) \\
 &= \pi^{-1/4} \psi_{-b} \Theta_3 \left[\frac{2\pi^2 b}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right].
 \end{aligned} \tag{6.60}$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Applying the Fourier transform into Eq.(6.34), leads to

$$\begin{aligned} \frac{2\pi}{L^2} \sum_{b=0}^{d-1} F(z; 2a, b; \psi) &= \pi^{-1/4} \sum_{k,m,b=0}^{d-1} \tilde{\psi}_k \frac{2\pi}{L^2} \omega(ab + 2am + mk) \\ &\times \Theta_3 \left[\frac{2\pi^2(m+b)}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \end{aligned} \quad (6.61)$$

Since there is a bijective map from $\mathbb{Z}(d) \times \mathbb{Z}(d)$ to $\mathbb{Z}(d) \times \mathbb{Z}(d)$, the variables m, b are substituted by $\mu = m + b$ and $\lambda = m - b$. To prove this, it is assumed that the substitution is a bijective map for odd values of d , which allows the use of 2^{-1} . As a result,

$$\begin{aligned} \frac{2\pi}{L^2} \sum_{b=0}^{d-1} F(z; 2a, b; \psi) &= \pi^{-1/4} \frac{1}{d^{3/2}} \sum_{k,\mu=0}^{d-1} \tilde{\psi}_k \omega(2^{-1}3a\mu + 2^{-1}k\mu) \\ &\times \sum_{\lambda=0}^{d-1} \omega(2^{-1}a\lambda + 2^{-1}k\lambda) \Theta_3 \left[\frac{2\pi^2\mu}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \end{aligned} \quad (6.62)$$

$$\begin{aligned} \frac{2\pi}{L^2} \sum_{b=0}^{d-1} F(z; 2a, b; \psi) &= \pi^{-1/4} \frac{1}{d^{3/2}} \sum_{k,\mu=0}^{d-1} \tilde{\psi}_k \delta(a, k) \omega(2^{-1}3a\mu + 2^{-1}k\mu) \\ &\times \Theta_3 \left[\frac{2\pi^2\mu}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\ &= \pi^{-1/4} \frac{1}{d^{3/2}} \tilde{\psi}_{-a} \sum_{\mu=0}^{d-1} \omega(a\mu) \Theta_3 \left[\frac{2\pi^2\mu}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\ &= \pi^{-1/4} \frac{1}{d^{3/2}} \tilde{\psi}_{-a} \exp\left(-\frac{z^2}{2}\right) \Theta_3 \left[\frac{2\pi^2 a}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right]. \end{aligned} \quad (6.63)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

The third property is proved with an extra summation $\sum_{b=0}^{d-1}$ and then

$$\frac{2\pi}{L^2} \sum_{a,b=0}^{d-1} F(z; a, 2b; \psi) = \pi^{-1/4} \sum_{b=0}^{d-1} \psi_{-b} \Theta_3 \left[\frac{2\pi^2 b}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \quad (6.64)$$

Changing $b = -B$, then Eq.(6.64) becomes

$$\begin{aligned} \frac{2\pi}{L^2} \sum_{a,B=0}^{d-1} F(z; a, -2B; \psi) &= \pi^{-1/4} \sum_{b=0}^{d-1} \psi_B \Theta_3 \left[-\frac{2\pi^2 B}{L^2} - z \frac{\pi}{L}; \frac{2\pi i}{L^2} \right] \\ &= \Psi(-z). \end{aligned} \quad (6.65)$$

□

6.5 Analytic representation in terms of coherent states using $\mathcal{F}_f|\psi\rangle$ as a fiducial vector

Definition 2. *The analytic representation of a fiducial vector $\mathcal{F}_f|\psi\rangle$ is given by*

$$\mathfrak{F}(z; a, b; \psi) = \pi^{-1/4} \sum_{m=0}^{d-1} {}_x\langle m | \mathcal{F}_f(a, b) | \psi \rangle \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \quad (6.66)$$

Using a different fiducial vector, a different set of d^2 coherent states is expected, but in this case the result is the same set of d^2 coherent states as the next proposition indicates.

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM
SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Proposition 9.

$$\begin{aligned} \mathfrak{F} \left[z; -\frac{1}{2}(a-b), -\frac{1}{2}(a+b); \psi \right] &= \omega \left[\frac{1}{4}(a^2 + b^2) \right] \exp \left(-\frac{z^2}{2} \right) \\ &\quad \times F(iz; a, b; \psi). \end{aligned} \quad (6.67)$$

Proof. In order to prove Eq.(6.67), Eq.(3.15) is used

$$\mathfrak{F}(z; a, b; f) = \pi^{-1/4} \sum_{m=0}^{d-1} {}_x\langle m | \mathcal{F}(a, b) | f \rangle \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \quad (6.68)$$

Eq.(3.15) is inserted into Eq.(6.68), then Eq.(6.68) becomes

$$\begin{aligned} \mathfrak{F}(z; a, b; \psi) &= \frac{\sqrt{2\pi}}{L} \pi^{-1/4} \omega \left[2^{-1}(a^2 + b^2) \right] \sum_{m,n=0}^{d-1} {}_x\langle n | \mathcal{D}_f(-a-b, a-b) | \psi \rangle \\ &\quad \times \omega(mn) \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (6.69)$$

Using Eq.(6.14), it can be concluded that

$$\begin{aligned} \mathfrak{F}(z; a, b; \psi) &= \pi^{-1/4} \omega(a^2) \exp \left(-\frac{1}{2} z_I^2 \right) \sum_{n=0}^{d-1} {}_x\langle n-a+b | \psi \rangle \omega(-an-bn) \\ &\quad \times \Theta_3 \left[\frac{2\pi^2 n}{L^2} - iz \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\ &= \pi^{-1/4} \omega(a^2) \exp \left(-\frac{1}{2} z_I^2 \right) \sum_{n=0}^{d-1} \psi_{n-a+b} \omega(-an-bn) \\ &\quad \times \Theta_3 \left[\frac{2\pi^2 n}{L^2} - iz \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (6.70)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Letting $M = n - a + b$ it can be concluded that

$$\begin{aligned} \mathfrak{F}(z; a, b; \psi) &= \pi^{-1/4} \omega(b^2) \exp\left(-\frac{1}{2} z_I^2\right) \sum_{M=0}^{d-1} \psi_M \exp\left(-\frac{4i\pi^2 a M}{L^2}\right) \\ &\quad \times \sum_{n=-\infty}^{\infty} \exp\left(-\frac{4i\pi^2 b M}{L^2} \frac{4i\pi^2 n M}{L^2} + \frac{4i\pi^2 a n}{L^2} - \frac{4i\pi^2 b n}{L^2}\right) \\ &\quad \times \exp\left(2zn \frac{\pi}{L} - \frac{2\pi^2}{L^2} n^2\right) \end{aligned} \quad (6.71)$$

Letting $N = -a - b + n$

$$\begin{aligned} \mathfrak{F}(z; a, b; \psi) &= \pi^{-1/4} \omega(a^2) \exp\left(-\frac{1}{2} z_I^2 + 2z \frac{\pi}{L} a + 2z \frac{\pi}{L} b - \frac{\pi}{d} (a + b)^2\right) \\ &\quad \times \sum_{M=0}^{d-1} \psi_M \sum_{N=-\infty}^{\infty} \exp\left(\frac{4i\pi^2 N M}{L^2} + \frac{4i\pi^2 a N}{L^2}\right) \\ &\quad \times \exp\left(-\frac{4i\pi^2 b N}{L^2} + 2z \frac{\pi}{L} N - 4 \frac{\pi^2}{L^2} b N - \frac{2\pi^2}{L^2} N^2 - \frac{2\pi^2}{L^2} a N\right) \end{aligned} \quad (6.72)$$

Letting $a = \frac{1}{2}(a + b)$ and $b = -\frac{1}{2}(a + b)$

$$\begin{aligned} \mathfrak{F}\left[z; -\frac{1}{2}(a - b), -\frac{1}{2}(a + b); \psi\right] &= \omega\left[\frac{1}{4}(a^2 + b^2)\right] \exp\left(-\frac{z^2}{2}\right) \\ &\quad \times F(iz; a, b; \psi). \end{aligned} \quad (6.73)$$

□

6.6 Entropic uncertainty relations

The entropic uncertainty relation in a finite systems [65–67] of an arbitrary state $|\phi\rangle$ is given in terms of

$$S_x + S_p - \log d \geq 0 \quad (6.74)$$

S_x and S_p are described by

$$S_x = - \sum_{m=0}^{d-1} |\phi_m|^2 \log |\phi_m|^2; \quad S_p = - \sum_{m=0}^{d-1} |\tilde{\phi}_m|^2 \log |\tilde{\phi}_m|^2. \quad (6.75)$$

A numerical example is considered, where $d = 3$. Given a fiducial vector

$$\begin{aligned} \psi_0 &= 0.1890 + 0.1094i; \quad \psi_1 = 0.3821 - 0.0404i; \\ \psi_2 &= 0.4077 - 0.0588i. \end{aligned} \quad (6.76)$$

Using this fiducial vector the entropic uncertainties $S_x + S_p$ for the coherent states $\mathcal{D}_f(a, b)|\psi\rangle$ can be calculated. The results are shown in the matrix below (the base e was used for the logarithms):

$S_x + S_p - \ln(d)$				
a				
		0	1	2
b	0	0.1276	0.2845	0.4148
	1	0.3286	0.3306	0.2613
	2	0.1276	0.3305	0.6183

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

In the case of Glauber coherent states, the result must be equal to the minimum possible value. However, several generalized coherent states do not obey this property, as it happens here.

6.7 Wigner and Weyl functions for finite quantum systems

Wigner and Weyl functions can be represented for an arbitrary state $|\phi\rangle$ as

$$\begin{aligned}\widetilde{W}(\phi; a, b) &= \langle \phi | \mathcal{D}_f(a, b) | \phi \rangle; & W(\phi; a, b) &= \langle \phi | \mathcal{P}_f(a, b) | \phi \rangle \\ W(\phi; \gamma, \delta) &= \sum_{a, b=0}^{d-1} \omega(\beta\gamma - a\delta) \widetilde{W}(\phi; a, b).\end{aligned}\tag{6.77}$$

The Wigner function is given in terms of $\phi(a, b; \psi)$ as

$$\begin{aligned}W(\phi; a, b) &= \frac{2\pi}{L^2} \sum_{\gamma, \delta, \epsilon, \zeta=0}^{d-1} \omega(a\delta - b\gamma + 2^{-1}\zeta\gamma - a\zeta - 2^{-1}\epsilon\delta + b\epsilon) \\ &\quad \times \phi(\epsilon, \zeta; \psi) \phi^*(\gamma, \delta; \psi).\end{aligned}\tag{6.78}$$

The Weyl function in terms of $\phi(a, b; \psi)$ can be expressed as

$$\widetilde{W}(\phi; a, b) = \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} \phi(\gamma, \delta; \psi) \phi^*(\gamma - a, \delta - b; \psi) \omega[2^{-1}(a\delta - b\gamma)]. \tag{6.79}$$

Proof. The expression (6.78) for the Wigner function is proved by inserting

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Eq.(6.42) into Eq.(6.77), leading to

$$W(\phi; a, b) = \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} \langle \phi | \mathcal{D}_f(\gamma, \delta) | \psi \rangle \langle \psi | \mathcal{D}_f(-\gamma, -\delta) \mathcal{P}_f(a, b) | \phi \rangle \quad (6.80)$$

Using Eqs.(6.50a), (6.51a), (6.56), Eq.(6.80) can be written as

$$\begin{aligned} W(\phi; a, b) &= \frac{2\pi}{L^2} \sum_{\gamma, \delta, \epsilon, \zeta=0}^{d-1} \omega(a\delta - b\gamma + 2^{-1}\zeta\gamma - a\zeta - 2^{-1}\epsilon\delta + b\epsilon) \\ &\quad \times \phi(\epsilon, \zeta; \psi) \phi^*(\gamma, \delta; \psi). \end{aligned} \quad (6.81)$$

In order to prove Eq.(6.79) for the Weyl function, Eq.(6.42) is inserted into Eq.(6.79)

$$\widetilde{W}(\phi; a, b) = \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} \langle \phi | \mathcal{D}_f(\gamma, \delta) | \psi \rangle \langle \psi | \mathcal{D}_f(-\gamma, -\delta) \mathcal{D}_f(a, b) | \phi \rangle \quad (6.82)$$

Using Eq.(6.50a), Eq.(6.82) can be written as

$$\widetilde{W}(\phi; a, b) = \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} \phi(\gamma, \delta; \psi) \phi^*(\gamma - a, \delta - b; \psi) \omega[2^{-1}(a\delta - b\gamma)] \quad (6.83)$$

□

An example with the following pure state $|\psi\rangle$ for $d = 5$ is considered.

$$\begin{aligned} |\psi\rangle &= \sum_{m=0}^4 \psi_m |m\rangle_x; \quad \psi_0 = 0.6822 + 0.2762i; \quad \psi_1 = 0.4959 - 0.0323i; \\ \psi_2 &= 0.4016 - 0.1597i \quad \psi_3 = 0.1016 - 0.0921i; \quad \psi_4 = 0.044 + 0.0535i; \end{aligned} \quad (6.84)$$

CHAPTER 6. ANALYTIC REPRESENTATIONS FOR QUANTUM SYSTEMS ON $\mathbb{Z}(D)$ WITH THETA FUNCTIONS

Fig.(5.7) shows the real part of the Weyl function of the state of Eq.(6.84). In Fig.(5.8), the imaginary part of the Weyl functions of the state of Eq.(6.84) is presented. In Fig.(5.9) the absolute value of the Weyl function of the same state is shown. In Fig.(5.10) the Wigner function for the state of Eq.(6.84), is demonstrated.

$$\Re(\widetilde{W}(\alpha, \beta))$$

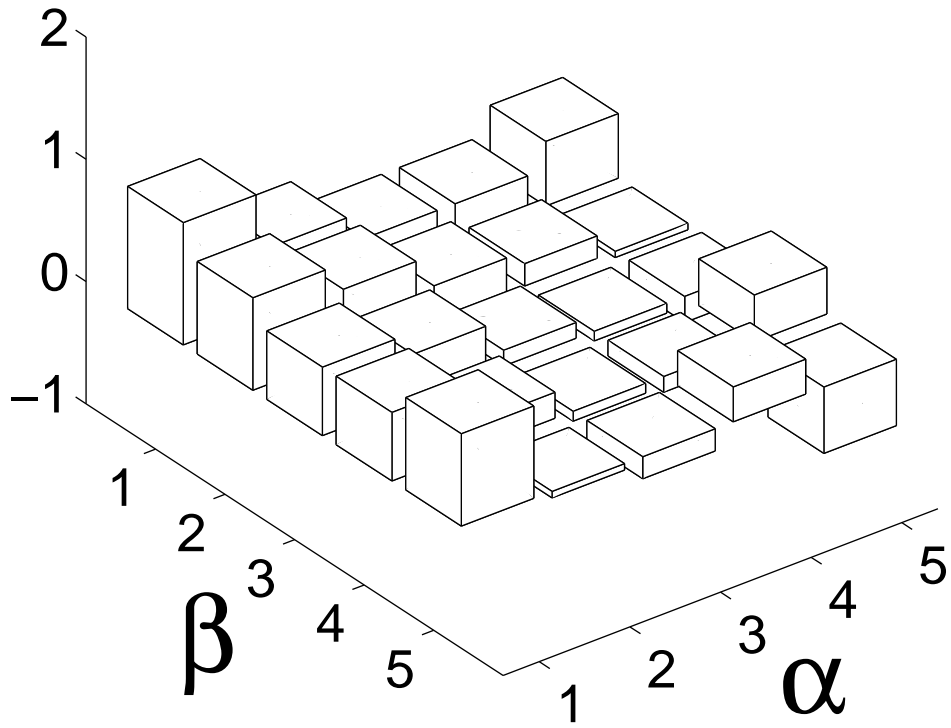


Figure 6.7: Real part of Weyl function of the coefficients of Eq.(6.84)

$$\Im(\widetilde{W}(\alpha, \beta))$$

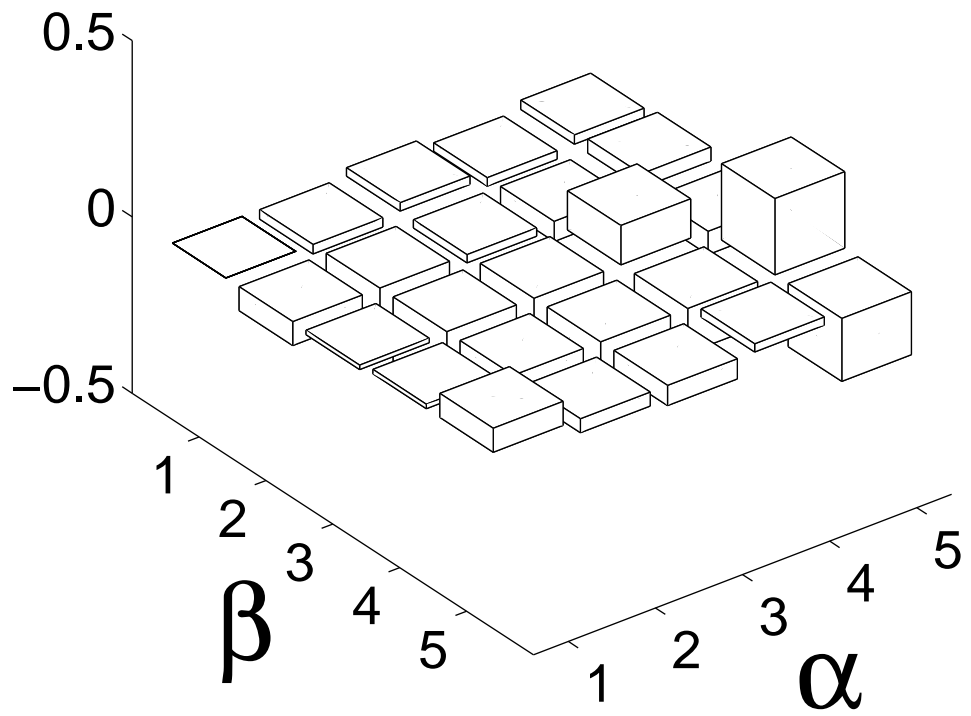


Figure 6.8: Imaginary part of Weyl function of the coefficients of Eq.(6.84)

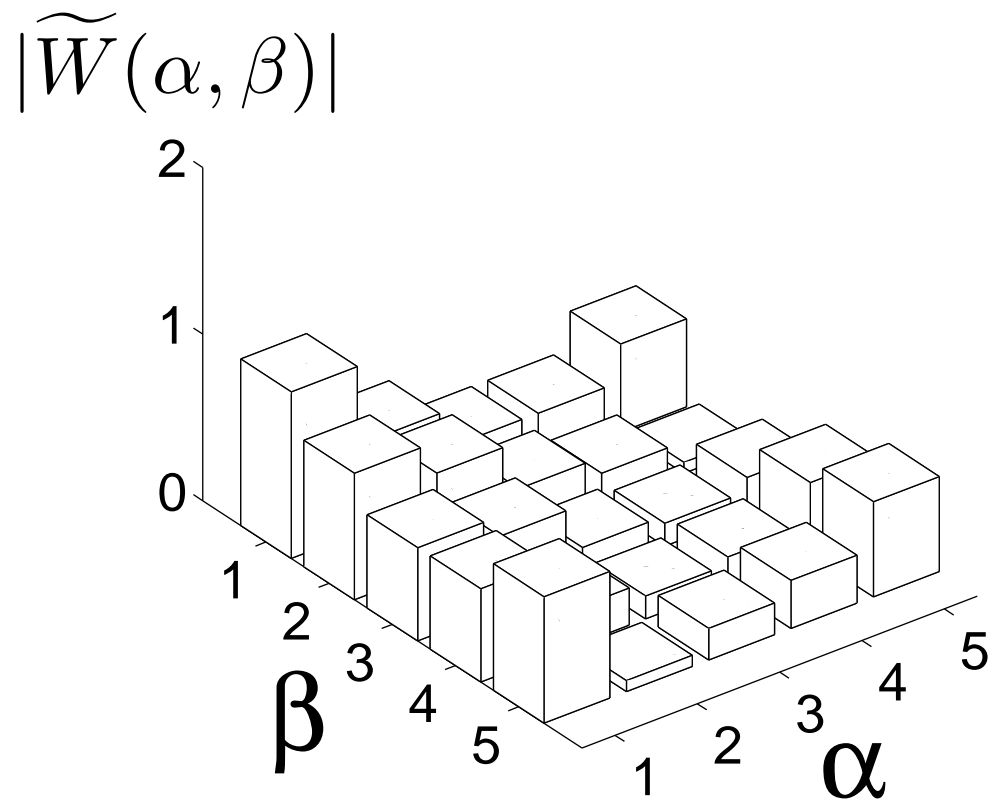


Figure 6.9: Absolute value of Weyl function using the coefficients of Eq.(6.84)

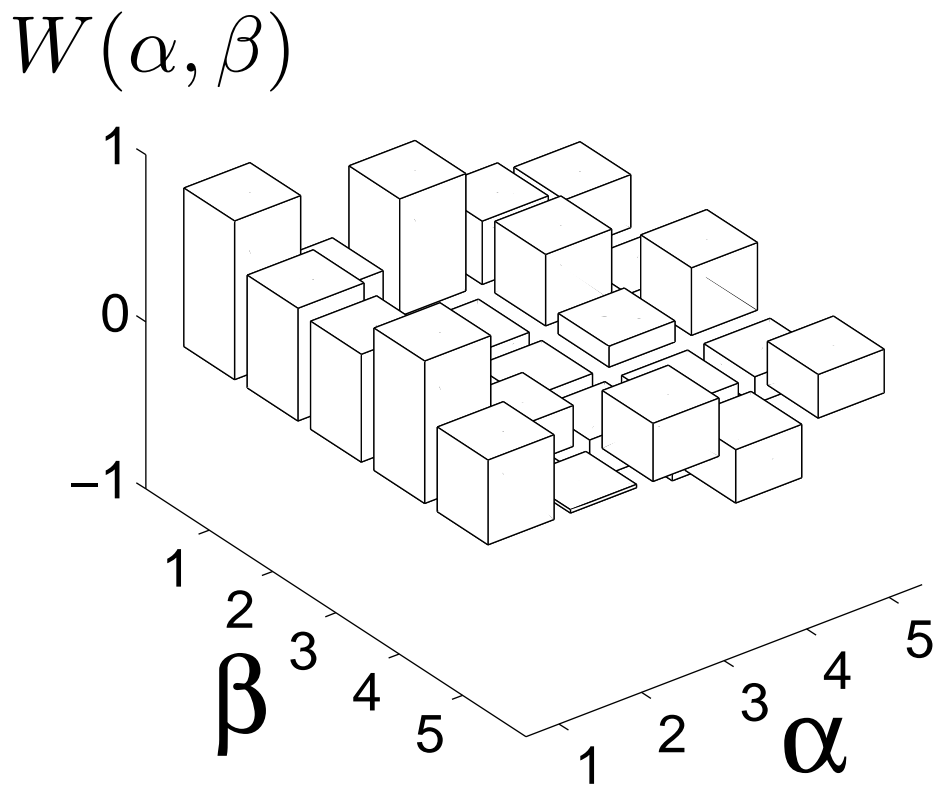


Figure 6.10: Wigner function using the coefficients of Eq.(6.84)

6.8 Discussion and Conclusion

In this chapter, quantum systems with finite Hilbert space for odd values of d are considered. The states of such systems are represented by the analytic function in Eq.(6.4), which obeys the quasiperiodicity conditions of Eq.(6.7), and therefore it is effectively defined on a torus. The scalar product is given in Eq.(6.9). Using the theory of analytic functions it can be proved that the scalar product of an arbitrary state in terms of coherent state is not zero since the zeros of analytic functions are isolated .

The following results are the novel work in this chapter. An analytic representation in terms of coherent states $\mathcal{D}_f(a, b)|\psi\rangle$ is given in Eq.(6.30). The reproducing kernel Eq.(6.41) is important in this formalism since plays the role of the resolution of identity in the language of analytic functions. Wigner and Weyl functions are expanded in Eqs.(6.78), (6.79), respectively.

Chapter 7

Finite quantum systems: Periodic paths of zeros

7.1 Introduction

In this chapter, the time evolution of a finite quantum system is considered. The paths of the zeros with Hamiltonians with a rational ratio of the eigenvalues (such that there exists t with $\exp(i\mathcal{H}t) = \mathbf{1}$) are studied. Using the analytic representation in Eq.(6.4) and the time evolution the zeros follows closed paths. The paths of d zeros are investigated. In some cases, a number N of the zeros follow the same path, and we say that the path has multiplicity N . Some examples of the paths of d zeros using different Hamiltonians are used in order to analyse the behaviour of the paths of d zeros on a torus. In Section 7.2 the time evolution operator in periodic systems is defined. In Section 7.3 several examples are presented.

7.2 Periodic systems using time evolution operator

Let

$$|\phi(0)\rangle = \sum_{m=0}^{d-1} \phi_m(0)|m\rangle_x \quad (7.1)$$

If Eq.(7.1) is time dependent then

$$|\phi(t)\rangle = \exp(-it\mathcal{H})|\phi(0)\rangle = \sum_{m=0}^{d-1} \phi_m(t)|m\rangle_x \quad (7.2)$$

where \mathcal{H} is the Hamiltonian.

$$\Phi(t; z) = \pi^{-1/4} \sum_{m=0}^{d-1} \phi_m(t) \Theta_3 \left[\frac{2m\pi^2}{L^2} - z \frac{\pi}{L}; \frac{i2\pi}{L^2} \right] \quad (7.3)$$

When the ratio of eigenvalues of \mathcal{H} is rational, the time evolution of quantum system is periodic. That is, when the eigenvalues of \mathcal{H} are denoted by N_j (where $j = 0, 1..d-1$) and $\frac{N_j}{N_0}$ is a rational number, then the system is periodic.

In order to plot the paths of the zeros Eq.(7.3) is used. Given the zeros the coefficients f_m can be found using Eq.(6.25) and inserting into Eq.(7.3) the zeros follows closed curve.

Using different Hamiltonians and different set of zeros of [73] the behaviour of the paths of the zeros is considered.

7.2.1 Hamiltonians with rational ratio of the eigenvalues

Using the following Hamiltonian

$$\mathcal{H} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (7.4)$$

with eigenvalues 0, 3, 4 and period $\mathcal{T} = 2\pi$.

Assuming that the paths of the zeros are denoted by $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$, where $t = 0$, the zeros of the analytic function are $\zeta_0(0)$, $\zeta_1(0)$, $\zeta_2(0)$, respectively.

Using the same Hamiltonian with varying values of zeros, a different multiplicity of the paths of zeros is obtained.

The zeros are equal to [73]

$$\zeta_0(0) = 1.01 + 2i; \quad \zeta_1(0) = 2.15 + 2.56i; \quad \zeta_2(0) = 3.35 + 1.95i. \quad (7.5)$$

In a period of 2π , using the zeros of Eq.(7.5) and Hamiltonian of Eq.(7.4), the zeros follow their own path with a multiplicity equal to 1, (M=1). In Fig.(6.1) the paths of these zeros are demonstrated.

$$\zeta_0(0) = 1.4 + 2.33i; \quad \zeta_1(0) = 2.15 + 2.32i; \quad \zeta_2(0) = 2.95 + 1.85i. \quad (7.6)$$

In Fig.(6.2) the paths of the zeros of Eq.(7.6) using the same Hamiltonian are illustrated. When multiplicity is equal to 2, (M=2), it is illustrated two of the zeros follow the same path while the first is used in order to follow its

CHAPTER 7. FINITE QUANTUM SYSTEMS:PERIODIC PATHS OF ZEROS

own path such that $\zeta_0(t) \xrightarrow{2\pi} \zeta_0(t)$, $\zeta_1(t) \xrightarrow{2\pi} \zeta_2(t)$, $\zeta_2(t) \xrightarrow{2\pi} \zeta_1(t)$.

In Fig.(6.2) it can be observed that there is a loop of the paths of the zeros during the period. Therefore,there are singular points for the zeros.

$$\zeta_0(0) = 1.56 + 2.49i; \quad \zeta_1(0) = 1.99 + 2.16i; \quad \zeta_2(0) = 2.95 + 1.85i. \quad (7.7)$$

In Fig.(6.3) using the zeros of Eq.(7.7) with same Hamiltonian, it can be observed that the zeros follow a closed path, resulting to $\zeta_0(t) \xrightarrow{2\pi} \zeta_1(t)$, $\zeta_1(t) \xrightarrow{2\pi} \zeta_2(t)$, $\zeta_2(t) \xrightarrow{2\pi} \zeta_0(t)$. As a result, the multiplicity of the path of the zeros equals to 3, (M=3).

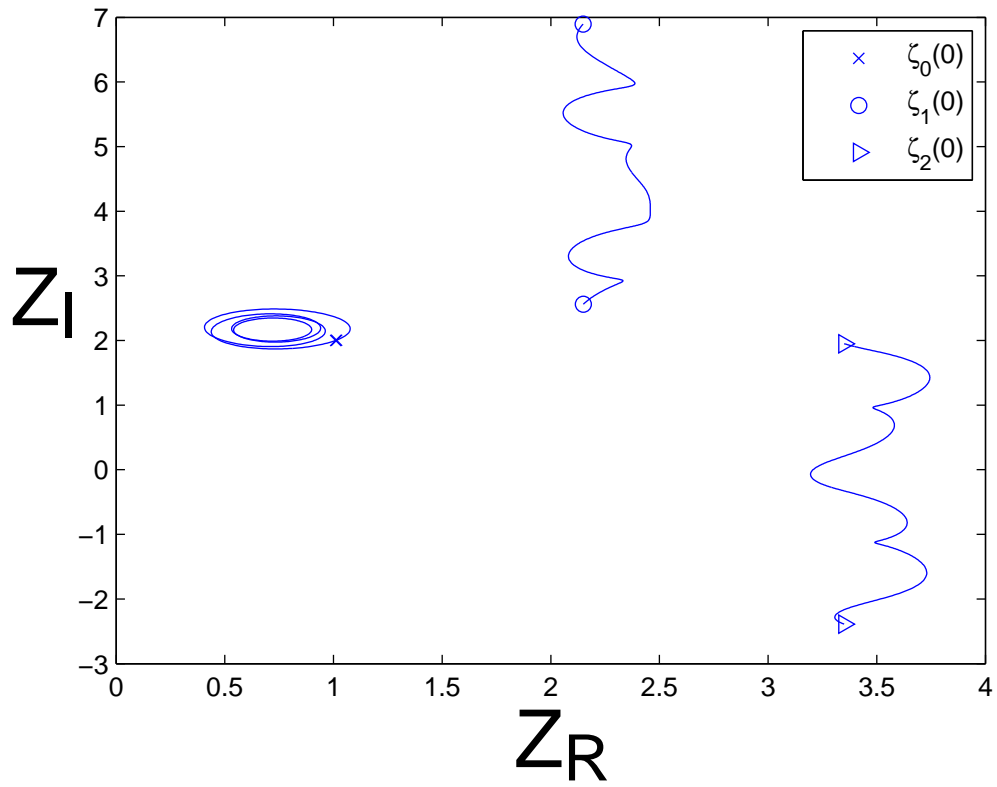


Figure 7.1: Paths of the zeros $\zeta_0(t), \zeta_1(t), \zeta_2(t)$ of Eq.(7.4) using the Hamiltonian of Eq.(7.3)

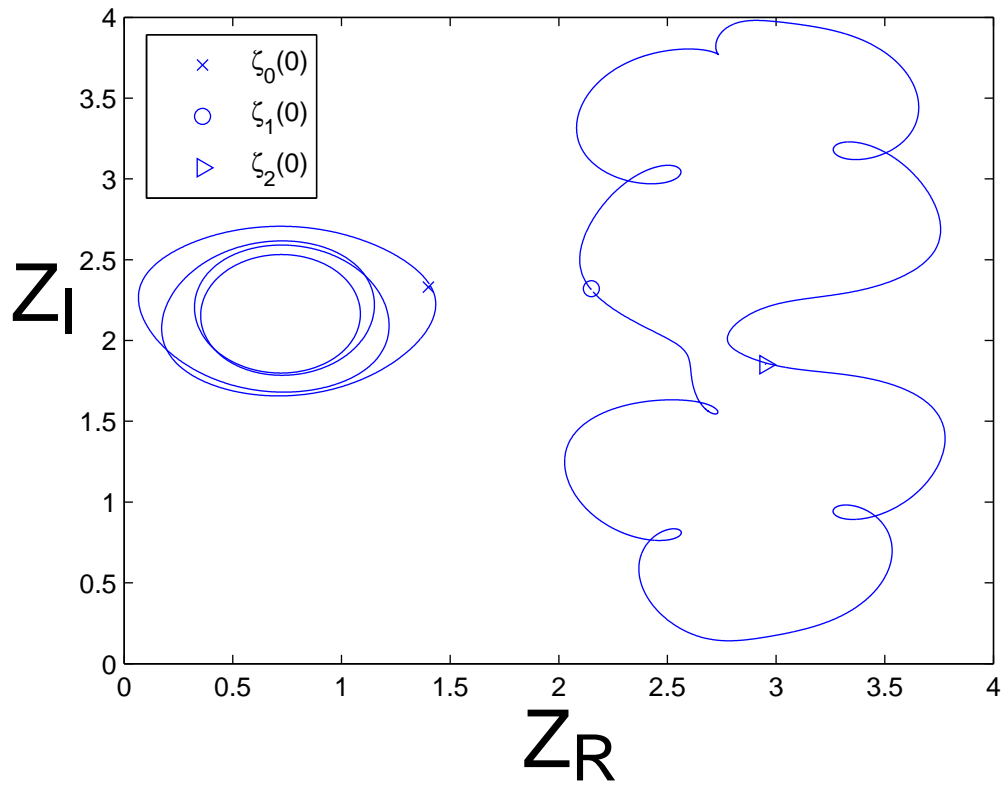


Figure 7.2: Paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ of Eq.(7.5) using the Hamiltonian of Eq.(7.3)

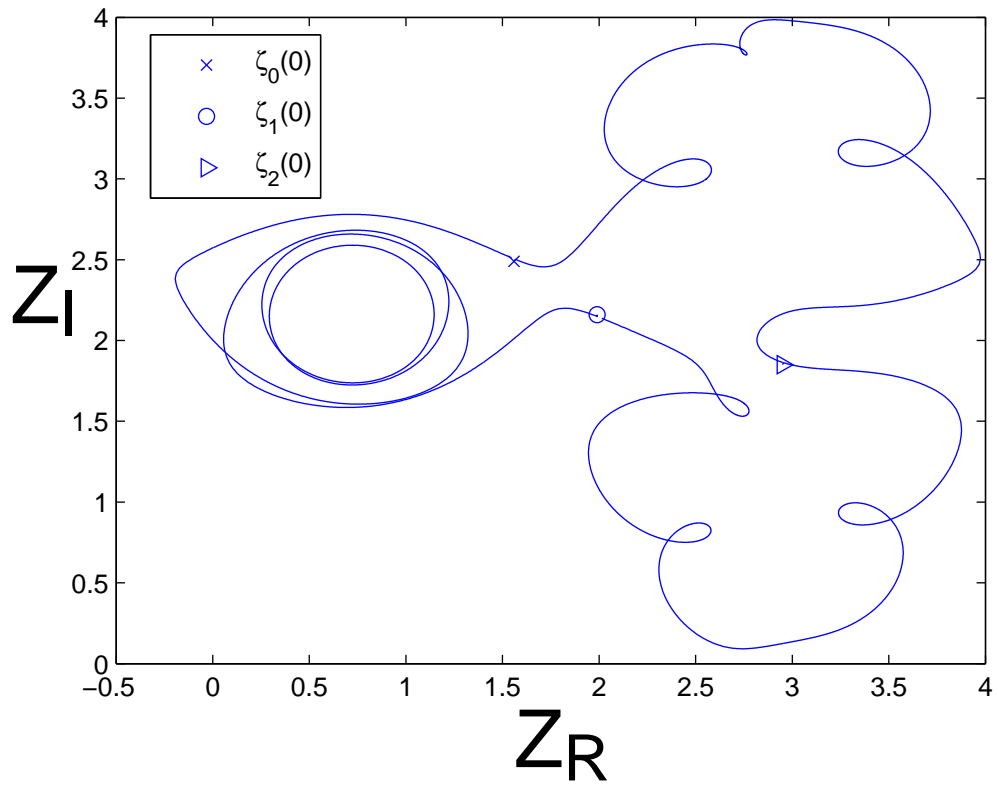


Figure 7.3: Paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ of Eq.(7.6) using the Hamiltonian of Eq.(7.3)

CHAPTER 7. FINITE QUANTUM SYSTEMS:PERIODIC PATHS OF ZEROS

Also, the following Hamiltonian is considered

$$\mathcal{H} = \begin{bmatrix} 2 & -3i & 0 \\ 3i & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (7.8)$$

with eigenvalues $-1, 3, 5$ and period $\mathcal{T} = 2\pi$.

Using the zeros of Eqs.(7.5), (7.6), (7.7) and the Hamiltonian of Eq.(7.8) after a period $\mathcal{T} = 2\pi$, two of the zeros exchange their positions and the third return to its initial position, leading to $\zeta_0(t) \xrightarrow{2\pi} \zeta_1(t)$, $\zeta_1(t) \xrightarrow{2\pi} \zeta_0(t)$, $\zeta_2(t) \xrightarrow{2\pi} \zeta_2(t)$, therefore, the paths of the zeros has a multiplicity of 2, (M=2). The paths of these zeros is presented in Figs.(7.4),(7.5) (7.6), respectively.

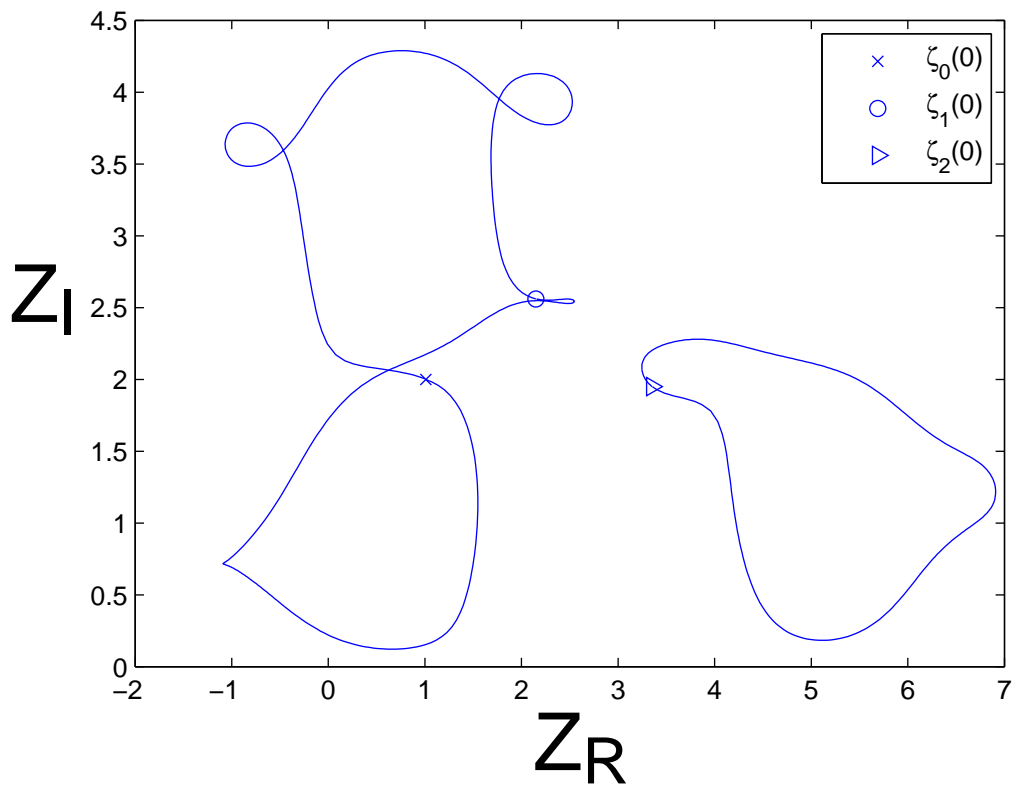


Figure 7.4: Paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ of Eq.(7.4) using the Hamiltonian of Eq.(7.7)

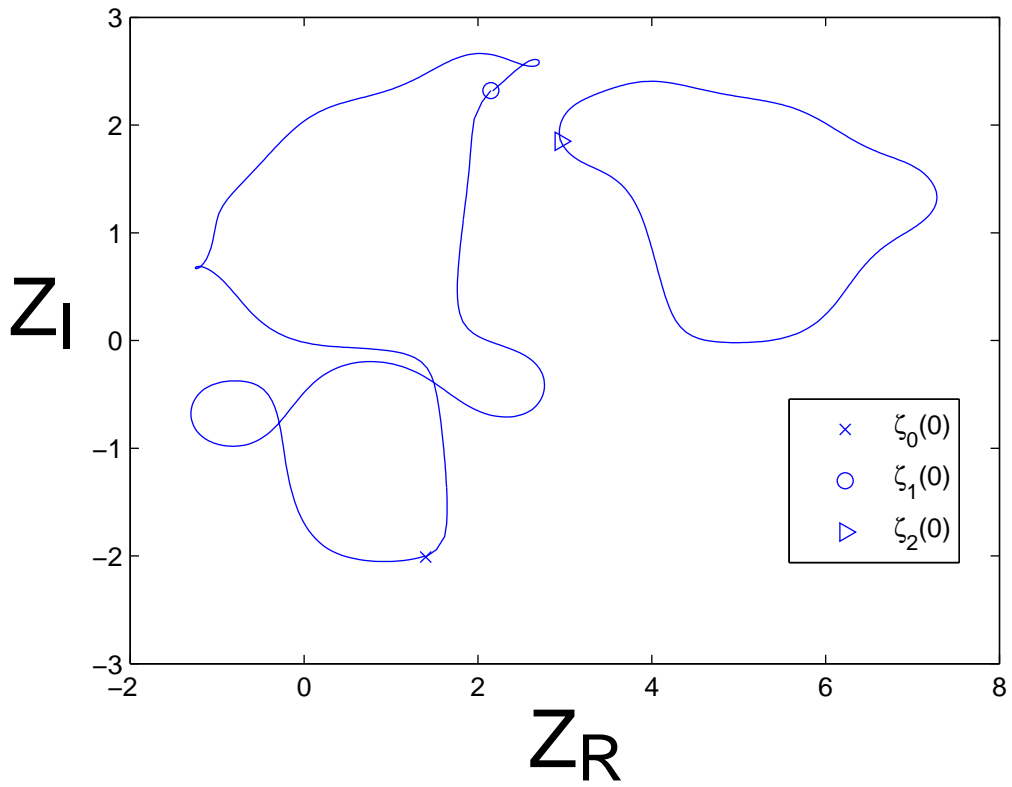


Figure 7.5: Paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ of Eq.(7.5) using the Hamiltonian of Eq.(7.7)

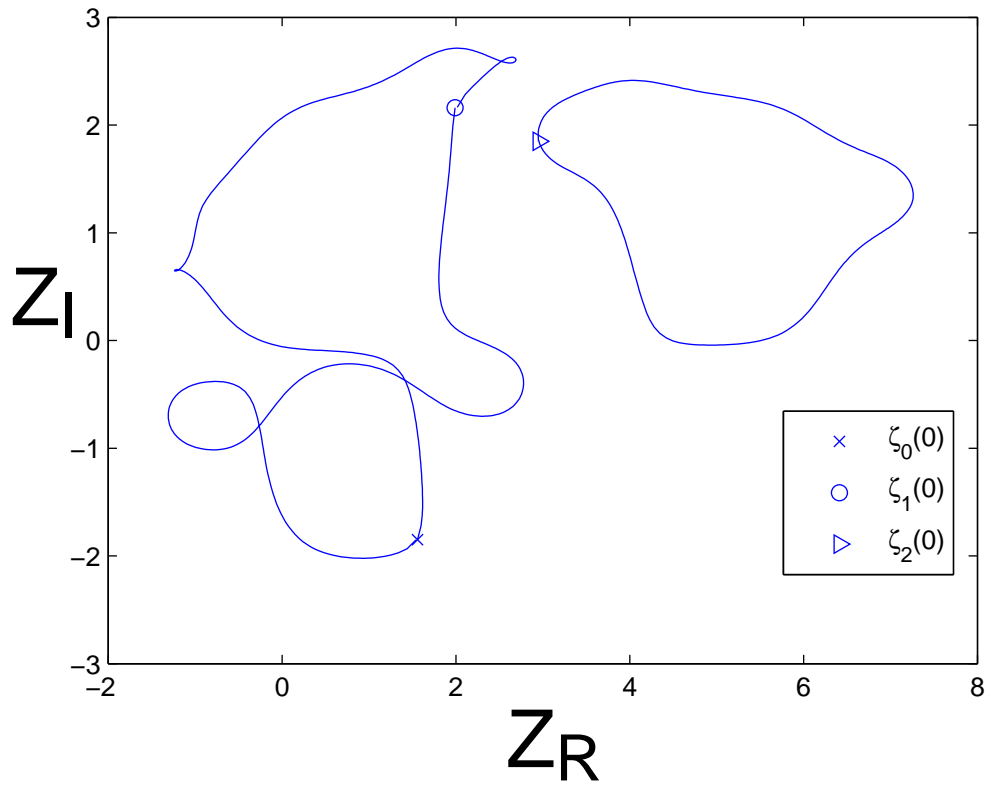


Figure 7.6: Paths of the zeros $\zeta_0(t)$, $\zeta_1(t)$, $\zeta_2(t)$ of Eq.(7.6) using the Hamiltonian of Eq.(7.7)

CHAPTER 7. FINITE QUANTUM SYSTEMS:PERIODIC PATHS OF ZEROS

After, comparing the result of two Hamiltonians in Eqs.(6.3) and (6.7) it can be concluded that:

- 1) Using the zeros of Eqs.(7.6) with the Hamiltonians of Eqs.(6.3) and (6.7), the multiplicity of the path of the zeros remains constant.
- 2) Using the Hamiltonian of Eq.(6.3) and the zeros of Eq.(7.7), it is concluded that the multiplicity of the path of the zeros is equal to 3, while using the same zeros with different Hamiltonian in Eq.(6.7) the multiplicity of the path of the zeros is equal to 2.
- 3) Using the Hamiltonian of Eq.(6.3) and the zeros of Eq.(7.5), it is concluded that the multiplicity of the path of the zeros is equal to 1, while using the same zeros with different Hamiltonian in Eq.(6.7) the multiplicity of the path of the zeros is equal to 2.
- 4) It can be observed that when $M=2$ the period of the zeros in order to return to their initial positions is equal to $2\mathcal{T}$.
- 5) It can be observed that when $M=3$ the period of the zeros in order to return to their initial positions is equal to $3\mathcal{T}$.

7.3 Discussion

In this chapter, the paths of the zeros is studied. The two cases under consideration are firstly when the d zeros return to their initial position and secondly, when they exchange positions where the multiplicity of the paths

CHAPTER 7. FINITE QUANTUM SYSTEMS:PERIODIC PATHS OF ZEROS

of the zeros is equal to 1 and d , respectively. Some examples of the paths of the zeros are provided in order to explore their behaviour.

Chapter 8

Analytic representation of quantum systems on \mathbb{S} using Theta functions

8.1 Introduction

This chapter begins by defining an analytic representation of an arbitrary state $|v\rangle$ on a strip \mathcal{A} . Coherent state on a circle are also studied. An analytic representation in space \mathcal{A} in terms of these coherent states is studied. Wigner and Weyl functions are considered. In Section 8.2, an analytic representation on a circle is defined. Furthermore, coherent states on a circle as well as Wigner and Weyl functions in terms of an arbitrary state $|v\rangle$ are examined in Sections 8.3 and 8.4, respectively.

8.2 Analytic representation for quantum systems on a circle

Let $|v\rangle$ be an arbitrary normalized state on a circle \mathbb{S}

$$|v\rangle = \frac{1}{2\pi} \int_0^{2\pi} dx v(x)|x\rangle; \quad v(x) = \langle x|v\rangle \quad (8.1)$$

with the following normalization condition

$$\frac{1}{2\pi} \int_0^{2\pi} dx |v(x)|^2 = 1. \quad (8.2)$$

The analytic representation of a state $|v\rangle$ is defined as

$$\Upsilon(z) = \int_0^{2\pi} dx v(x) \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \quad (8.3)$$

with periodicity relation

$$\Upsilon(z + 2\pi) = \Upsilon(z). \quad (8.4)$$

$\Upsilon(z)$ is defined on a strip $\mathcal{A} = [0, 2\pi] \times \mathbb{R}$

Proof. The periodicity can be proved using Eq.(2.2). The analytic function $\Upsilon(z)$ can be shown to be periodic along the real axis as follows

$$\begin{aligned} \Upsilon(z + 2\pi) &= \int_0^{2\pi} dx v(x) \Theta_3 \left[\frac{x}{2} - \frac{1}{2}(z + 2\pi); \frac{i}{2\pi} \right] \\ \Upsilon(z) &= \int_0^{2\pi} dx v(x) \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \end{aligned} \quad (8.5)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Therefore, $\Upsilon(z)$ is defined on a strip $\mathcal{A} = [0, 2\pi] \times \mathbb{R}$. \square

The following states are considered

$$\begin{aligned} |x\rangle &\rightarrow 2\pi\Theta_3\left[\frac{x-z}{2}; \frac{i}{2\pi}\right] \\ |N\rangle_p &\rightarrow 2\pi \exp\left(iNz - \frac{1}{2}N^2\right) \end{aligned} \quad (8.6)$$

Proof. Using Eqs.(8.1), (8.3) the position representation is given by

$$|x\rangle \rightarrow 2\pi\Theta_3\left[\frac{x-z}{2}; \frac{i}{2\pi}\right] \quad (8.7)$$

In order to prove the momentum representation Eq.(5.8) is used

$$\begin{aligned} |N\rangle_p &= \int_0^{2\pi} dx \exp(iNx) \Theta_3\left[\frac{x-z}{2}; \frac{i}{2\pi}\right] \\ &= \int_0^{2\pi} dx \exp(iNx) \sum_{M=-\infty}^{\infty} \exp\left(iMx - iMz - \frac{1}{2}M^2\right) \\ &= 2\pi\delta(N+M, 0) \sum_{M=-\infty}^{\infty} \exp\left(-iMz - \frac{1}{2}M^2\right) \\ &= 2\pi \exp\left(iNz - \frac{1}{2}N^2\right). \end{aligned} \quad (8.8)$$

\square

Proposition 8.2.1. *Orthogonality relation*

$$\int_{\mathcal{A}} dm_c(z) \Theta_3\left[\frac{x-z}{2}; \frac{i}{2\pi}\right] \Theta_3\left[\frac{y-z^*}{2}; \frac{i}{2\pi}\right] = \delta_c(x-y) \quad (8.9)$$

where $dm_c(z) = \frac{1}{4\pi^{5/2}} \exp(-z_I^2) d^2z$.

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Proof. Using Eq.(6.5), Eq.(8.9) can be written as follows

$$\begin{aligned}
 &= \frac{1}{4\pi^{5/2}} \sum_{N,K=-\infty}^{\infty} \exp(iNx + iKy) \exp\left[-\frac{1}{2}(N^2 + K^2)\right] \\
 &\quad \times \int_{-\infty}^{\infty} dz_I \exp(-z_I^2 + Nz_I - Kz_I) \\
 &\quad \times \int_0^{2\pi} dz_R \exp(-iKz_R - iNz_R) \tag{8.10}
 \end{aligned}$$

Calculating the integral over dz_R in Eq.(8.10)

$$\begin{aligned}
 &= \frac{1}{2\pi^{3/2}} \sum_{N,K=-\infty}^{\infty} \exp(iNx + iKy) \\
 &\quad \times \int_{-\infty}^{\infty} dz_I \exp(-z_I^2 + Nz_I - Kz_I) \\
 &\quad \times \exp\left[-\frac{1}{2}(N^2 + K^2)\right] \delta(K, -N). \tag{8.11}
 \end{aligned}$$

Let $N = -K$ into Eq.(8.11)

$$\begin{aligned}
 &= \frac{1}{2\pi^{3/2}} \sum_{K=-\infty}^{\infty} \exp(-iKx + iKy) \\
 &\quad \times \int_{-\infty}^{\infty} dz_I \exp(-z_I^2 + Kz_I + K^2). \tag{8.12}
 \end{aligned}$$

The integral over dz_I is a gaussian integral, therefore Eq.(8.12) becomes

$$\begin{aligned}
 &= \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \exp(-iKx + iKy) \\
 &= \delta_c(x - y). \tag{8.13}
 \end{aligned}$$

□

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

The scalar product of two states in terms of the analytic representations of these states is given by

$$\langle v_1^* | v_2 \rangle = \frac{1}{2\pi} \int_{\mathcal{A}} dm_c(z) \Upsilon_1(z^*) \Upsilon_2(z). \quad (8.14)$$

Proof. Using Eq.(8.9), and multiplying both sides by integrals over $dx dy$ and the wave functions $v_1(x)v_2(y)^*$ it can be concluded that

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathcal{A}} dm_c(z) \int_0^{2\pi} \int_0^{2\pi} dx dy v_1(x) v_2(y)^* \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \Theta_3 \left[\frac{y-z^*}{2}; \frac{i}{2\pi} \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} dx v_1(x) v_2^*(x) \end{aligned} \quad (8.15)$$

Hence, the scalar product is given by

$$\langle v_1 | v_2 \rangle = \frac{1}{2\pi} \int_{\mathcal{A}} dm_c(z) \Upsilon_1(z) \Upsilon_2(z)^*. \quad (8.16)$$

□

The wave function $v(x)$ in terms of analytic function $\Upsilon(z)$ is given by

$$v(x) = \int_{\mathcal{A}} dm_c(z) \Theta_3 \left[\frac{x-z^*}{2}; \frac{i}{2\pi} \right] \Upsilon(z) \quad (8.17)$$

Proof. By the use of Eq.(8.9) and multiplying both sides by Theta function

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

and integrating over $dm_c(z)$ it following that

$$\begin{aligned}
 \int_{\mathcal{A}} dm_c(z) \Theta_3 \left[\frac{y - z^*}{2}; \frac{i}{2\pi} \right] \Upsilon(z) &= \int_0^{2\pi} dx v(x) \int_{\mathcal{S}} dm_c(z) \\
 &\quad \times \Theta_3 \left[\frac{x - z}{2}; \frac{i}{2\pi} \right] \Theta_3 \left[\frac{y - z^*}{2}; \frac{i}{2\pi} \right] \\
 &= \int_0^{2\pi} dx v(x) \delta(x - y) = v(y)
 \end{aligned} \tag{8.18}$$

□

8.3 Coherent states on a circle

Let $\mathcal{D}_c(\alpha, N)|\lambda\rangle$ be the definition of coherent state $|\alpha, N\rangle$ [54–58]. The fiducial vector can be expressed as in Eq.(8.1), then the overlap is the following

$$\begin{aligned}
 \langle \beta, M | \alpha, N \rangle &= \int_0^{2\pi} dx \lambda(x) \lambda^*(x + \alpha - b) \exp \left[i(N - M) \left(x + \frac{\alpha - \beta}{2} \right) \right] \\
 &\quad \times \exp \left(\frac{i}{2} N \beta - \frac{i}{2} M a \right).
 \end{aligned} \tag{8.19}$$

Proof. Using Eq.(5.14) it can be concluded that

$$\begin{aligned}
 \langle \beta, M | \alpha, N \rangle &= \int_0^{2\pi} dx \int_0^{2\pi} dy \langle y | \mathcal{D}_c(\alpha - \beta, N - M) | x \rangle r^*(y) r(x) \\
 &\quad \times \exp \left(\frac{i}{2} N \beta - \frac{i}{2} M a \right) \\
 &= \int_0^{2\pi} dx r^*(x + \alpha - \beta) r(x) \exp \left[i(N - M) \left(x + \frac{\alpha - \beta}{2} \right) \right] \\
 &\quad \times \exp \left(\frac{i}{2} N \beta - \frac{i}{2} M a \right).
 \end{aligned} \tag{8.20}$$

□

Proposition 8.3.1. *Resolution of identity*

$$\frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \mathcal{D}_c(\alpha, N) |\lambda\rangle \langle \lambda| \mathcal{D}_c(-\alpha, -N) = \mathbf{1}. \quad (8.21)$$

Proof. In order to prove Eq.(8.21), both sides are multiplied by position states, resulting in

$$\frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \langle x_1 | \mathcal{D}_c(\alpha, N) |\lambda\rangle \langle \lambda| \mathcal{D}_c(-\alpha, -N) | y_1 \rangle = 2\pi \delta(x_1 - y_1). \quad (8.22)$$

Using Eqs.(5.9), (5.14), (8.1) we get

$$\begin{aligned} &= \frac{1}{8\pi^3} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \int_0^{2\pi} dx \int_0^{2\pi} dy \langle x_1 | x + \alpha \rangle \langle y + \alpha | y_1 \rangle \\ &\quad \times \exp(iNx - iNy) \lambda(x) \lambda(y)^* \\ &= \int_0^{2\pi} d\alpha \int_0^{2\pi} dx \int_0^{2\pi} dy \delta(x_1, x + \alpha) \delta(y_1, y + \alpha) \\ &\quad \times \lambda(x) \lambda(y)^* \delta_c(x - y) \\ &= \delta(x_1 - y_1) \int_0^{2\pi} dx \lambda(x) \lambda(x)^* = 2\pi \delta(x_1 - y_1). \end{aligned} \quad (8.23)$$

□

Given a ‘fiducial state’ $|\lambda\rangle$, let $\Lambda(z)$ be its analytic representation.

Definition 1. *The analytic function $G(z; \alpha, N; \lambda)$ based on $D_c(\alpha, N)|\lambda\rangle$ is defined as*

$$G(z; \alpha, N; \lambda) = \int_0^{2\pi} dx \langle x | D_c(\alpha, N) |\lambda\rangle \Theta_3 \left[\frac{x - z}{2}; \frac{i}{2\pi} \right]. \quad (8.24)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

It is easily to proved that

$$\begin{aligned} G(z; 0, 0; \lambda) &= \Lambda(z) \\ G(z; \alpha + 2\pi, N; \lambda) &= (-1)^N G(z; \alpha, N; \lambda). \end{aligned} \quad (8.25)$$

The periodicity relation of Eq.(8.24) is analogous of Eq.(8.4) and is given by

$$G(z + 2\pi; \alpha, N; \lambda) = G(z; \alpha, N; \lambda). \quad (8.26)$$

The relation between the analytic representation $G(z; \alpha, N; \lambda)$ of the coherent states $|\alpha, N\rangle$ and the analytic representation $\Lambda(z)$ of the fiducial vector $|\lambda\rangle$ is given in the next proposition.

Proposition 1.

$$G(z; \alpha, N; \lambda) = \exp\left(-\frac{1}{2}iN\alpha + iNz - \frac{1}{2}N^2\right) \Lambda(z + iN - \alpha). \quad (8.27)$$

Proof. Using Eqs.(8.24), (5.14), leads to

$$\begin{aligned} G(z; \alpha, N; \lambda) &= \int_0^{2\pi} dx \exp\left(iNx - \frac{1}{2}iN\alpha\right) \langle x - \alpha | \lambda \rangle \\ &\quad \times \Theta_3\left[\frac{x - z}{2}; \frac{i}{2\pi}\right] \end{aligned} \quad (8.28)$$

Let $x - \alpha = X$ in Eq.(8.28)

$$\begin{aligned} G(z; \alpha, N; \lambda) &= \int_0^{2\pi} dX \lambda(X) \exp\left(iNX + \frac{1}{2}iN\alpha\right) \\ &\quad \times \Theta_3\left[\frac{X + \alpha - z}{2}; \frac{i}{2\pi}\right] \end{aligned} \quad (8.29)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Using the definition of the Theta function and changing variables as $N + M = K$ in Eq.(8.29) leads to

$$\begin{aligned} G(z; \alpha, N; \lambda) &= \exp\left(-\frac{1}{2}iN\alpha + iNz - \frac{1}{2}N^2\right) \int_0^{2\pi} dX \lambda(X) \\ &\quad \times \sum_{K=-\infty}^{\infty} \exp(iKX) \exp\left(iK\alpha - \frac{1}{2}K^2 + KN - iKz\right) \end{aligned} \quad (8.30)$$

From Eq.(8.30) it can be concluded that

$$G(z; \alpha, N; \lambda) = \exp\left(-\frac{1}{2}iN\alpha + iNz - \frac{1}{2}N^2\right) \Lambda(z + iN - \alpha). \quad (8.31)$$

□

Proposition 2. *The zeros ζ_κ of $\Lambda(z)$ are related to the zeros $\zeta_\kappa(\alpha, K)$ of the $G(z; \alpha, N; \lambda)$ as follows*

$$\zeta_\kappa(\alpha, N) = \zeta_\kappa - iN + \alpha. \quad (8.32)$$

Proof. Eq.(8.32) is a direct consequence of the **Proposition 1** □

Proposition 3. *$G(z; \alpha, N; \lambda)$ is the two-dimensional Fourier transform of the $G(-z; \beta, M; \lambda)$*

$$\begin{aligned} G(z; \alpha, N; \lambda) &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta G(-z; \beta, -N + 2M; \lambda) \\ &\quad \times \exp\left[iM\alpha - \frac{i}{2}N\beta - \frac{i}{2}\alpha N\right]. \end{aligned} \quad (8.33)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Proof. In order to prove **Proposition 3**, it should first be defined that

$$\mathfrak{P}(z; \alpha, N; \lambda) = \int_0^{2\pi} dx \langle x | \mathcal{P}_c(\alpha, N) | \lambda \rangle \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \quad (8.34)$$

where $\mathcal{P}_c(a, K)$ is the displaced parity operator.

$$\begin{aligned} \langle x | \mathcal{P}_c(\alpha, N) &= \exp(iNx - iN\alpha) \langle -x + \alpha | \\ \langle x | \mathcal{D}_c(\alpha, N) &= \exp\left(iNx - \frac{1}{2}iN\alpha\right) \langle x - \alpha |. \end{aligned} \quad (8.35)$$

Using Eqs.(8.24), (8.34), (8.35), it can be proved that

$$\mathfrak{P}(z; \alpha, N; \lambda) = G(-z; -\alpha, -N; \lambda) \quad (8.36)$$

Using Eq.(5.23), including the variable z , it can be concluded that

$$\begin{aligned} \mathfrak{P}(z; \alpha, N; \lambda) &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta G(-z; \beta, N + 2M; \lambda) \\ &\times \exp \left[\frac{i}{2}N\beta - iM\alpha - \frac{i}{2}\alpha N \right] \end{aligned} \quad (8.37)$$

Substituting Eq.(8.36) into Eq.(8.37)

$$\begin{aligned} G(z; \alpha, N; \lambda) &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta G(-z; \beta, -N + 2M; \lambda) \\ &\times \exp \left[iM\alpha - \frac{i}{2}N\beta - \frac{i}{2}\alpha N \right]. \end{aligned} \quad (8.38)$$

□

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Proposition 4. *Resolution of identity*

$$\frac{1}{4\pi^2} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha G(z; \alpha, N; \lambda) [G(w; \alpha, N; \lambda)]^* = \mathfrak{k}(z, w^*) \quad (8.39)$$

where

$$\begin{aligned} \mathfrak{k}(z, w^*) &= \int_0^{2\pi} dx \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \Theta_3 \left[\frac{x-w^*}{2}; \frac{i}{2\pi} \right] \\ \mathfrak{k}(z, w^*) &= \mathfrak{k}(-z, -w^*). \end{aligned} \quad (8.40)$$

Proof. In order to prove the resolution of identity, Eq.(8.24) is used

$$\begin{aligned} & \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha G(z; \alpha, N; \lambda) G(w^*; \alpha, N; \lambda) \\ &= \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \int_0^{2\pi} dx \langle x | \mathcal{D}_c(\alpha, N) | \lambda \rangle \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \\ & \quad \times \int_0^{2\pi} dy \langle \lambda | \mathcal{D}_c(-\alpha, -N) | y \rangle \Theta_3 \left[\frac{y-w^*}{2}; \frac{i}{2\pi} \right] \\ &= \int_0^{2\pi} dx \int_0^{2\pi} dy \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \langle x | \mathcal{D}_c(\alpha, N) | \lambda \rangle \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \\ & \quad \times \langle \lambda | \mathcal{D}_c(-\alpha, -N) | y \rangle \Theta_3 \left[\frac{y-w^*}{2}; \frac{i}{2\pi} \right] \end{aligned} \quad (8.41)$$

Using Eq.(8.22) we get

$$= \int_0^{2\pi} dx \int_0^{2\pi} dy 4\pi^2 \delta(x, y) \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \Theta_3 \left[\frac{y-w^*}{2}; \frac{i}{2\pi} \right] \quad (8.42)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Finally, the resolution of identity of these states is given by

$$\begin{aligned} & \frac{1}{4\pi^2} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha G(z; \alpha, N; \lambda) G(w^*; \alpha, N; \lambda) \\ &= \int_0^{2\pi} dx \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \Theta_3 \left[\frac{x-w^*}{2}; \frac{i}{2\pi} \right]. \end{aligned} \quad (8.43)$$

□

Proposition 5. *Reproducing Kernel relation*

$$\Upsilon(z) = \int_{\mathcal{A}} dm_c(w) \mathfrak{k}(z, w^*) \Upsilon(w). \quad (8.44)$$

where $\Upsilon(z)$ is given in Eq.(8.3).

Eq.(8.44) can be written as

$$\Upsilon(z) = \frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha G(z; \alpha, N; \lambda) v(\alpha, N; \lambda) \quad (8.45)$$

and

$$\Upsilon(z) = \frac{1}{2\pi} \sum_{M=\infty}^{\infty} \int_0^{2\pi} d\beta G(-z; \beta, M; \lambda) \tilde{v}(\beta, M; \lambda) \quad (8.46)$$

where

$$v(\alpha, N; \lambda) = \langle \lambda | \mathcal{D}_c(-\alpha, -N) | v \rangle. \quad (8.47)$$

The $v(\alpha + 2\pi, N; \lambda)$ is periodic and is given by

$$v(\alpha + 2\pi, N; \lambda) = (-1)^N v(\alpha, N; \lambda). \quad (8.48)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM
SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

and $\tilde{v}(\beta, M; \lambda)$ is given by

$$\tilde{v}(\beta, M; \lambda) = \langle \lambda | \mathcal{D}_c(-\beta, -M) \mathcal{P}_c(0, 0) v \rangle = \langle \lambda | \mathcal{P}_c(-\beta, -M) | v \rangle. \quad (8.49)$$

The $\tilde{v}(\beta, M; \lambda)$ is periodic and is given by

$$\tilde{v}(\beta + 2\pi, M; \lambda) = (-1)^M \tilde{v}(\beta, M; \lambda). \quad (8.50)$$

$v(\alpha, N; \lambda)$ is also given by

$$v(\alpha, N; \lambda) = \frac{1}{2\pi} \int_{\mathcal{A}} dm_c(w) G(w; \alpha, N; \lambda)^* \Upsilon(w) \quad (8.51)$$

and $\tilde{v}(\beta, M; \lambda)$ is also equal to

$$\tilde{v}(\beta, M; \lambda) = \frac{1}{2\pi} \int_{\mathcal{A}} dm_c(w) G(-w; \beta, M; \lambda)^* \Upsilon(w). \quad (8.52)$$

Proof. Eq.(8.44) is proved by inserting Eq.(8.40) into Eq.(8.44) and using Eq.(8.18)

$$\begin{aligned} \Upsilon(z) &= \int_{\mathcal{A}} dm_c(w) \Upsilon(w) \int_0^{2\pi} dx \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \\ &\quad \times \Theta_3 \left[\frac{x-w^*}{2}; \frac{i}{2\pi} \right] \\ &= \int_0^{2\pi} dx \int_{\mathcal{A}} dm_c(w) \Theta_3 \left[\frac{x-w^*}{2}; \frac{i}{2\pi} \right] \Upsilon(w) \\ &\quad \times \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] = \Upsilon(z). \end{aligned} \quad (8.53)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Eq.(8.45) is proved by inserting Eq.(8.47) into Eq.(8.45) and using Eq.(8.24)

$$\begin{aligned} \Upsilon(z) = & \frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha \int_0^{2\pi} dx \langle x | \mathcal{D}_c(\alpha, N) | \lambda \rangle \langle \lambda | \mathcal{D}_c(-\alpha, -N) | v \rangle \\ & \times \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] \end{aligned} \quad (8.54)$$

Based on Eq.(8.21), it can be concluded that

$$= \int_0^{2\pi} dx \langle x | v \rangle \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] = \Upsilon(z). \quad (8.55)$$

In order to prove Eq.(8.51), Eq.(8.51) is inserted into Eq.(8.45)

$$\Upsilon(z) = \frac{1}{4\pi^2} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} da \, G(z; \alpha, N; \lambda) \int_{\mathcal{A}} dm_c(w) G(w; \alpha, N; \lambda)^* \Upsilon(w) \quad (8.56)$$

Using Eq.(8.39) it can be easily proved that

$$= \int_{\mathcal{A}} dm_c(w) \mathfrak{k}_c(z, w^*) \Upsilon(w) = \Upsilon(z). \quad (8.57)$$

Eq.(8.46) is proved using Eqs.(8.24), (8.49)

$$\begin{aligned} \Upsilon(z) = & \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta \int_0^{2\pi} dx \langle x | \mathcal{D}_c(\beta, M) | \lambda \rangle \langle \lambda | \mathcal{D}_c(-\beta, -M) \mathcal{P}_c(0, 0) | v \rangle \\ & \times \Theta_3 \left[\frac{x+z}{2}; \frac{i}{2\pi} \right] \end{aligned} \quad (8.58)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Therefore, using Eq.(8.21)

$$= \int_0^{2\pi} dx v(-x) \Theta_3 \left[\frac{x+z}{2}; \frac{i}{2\pi} \right] \quad (8.59)$$

Let $X = -x$ into Eq.(8.59)

$$= \int_0^{2\pi} dX v(X) \Theta_3 \left[\frac{x-z}{2}; \frac{i}{2\pi} \right] = \Upsilon(z). \quad (8.60)$$

In order to prove Eq.(8.52), Eq.(8.52) is substituted into Eq.(8.46)

$$\begin{aligned} \Upsilon(z) &= \frac{1}{4\pi^2} \int_{\mathcal{A}} dm_c(w) \Upsilon(w) \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta G(-z; \beta, M; \lambda) G(-w; \beta, M; \lambda)^* \\ &= \int_{\mathcal{A}} dm_c(w) \Upsilon(w) K_c(z, w^*) = \Upsilon(z). \end{aligned} \quad (8.61)$$

□

Proposition 6. $\tilde{v}(\beta, M; \lambda)$ is the two-dimensional Fourier transform of $v(\alpha, N; \lambda)$

$$\begin{aligned} \tilde{v}(\beta, M; \lambda) &= \frac{1}{2\pi} \sum_{N=-\infty}^{\infty} \int_0^{2\pi} d\alpha v(-\alpha, M - 2N; \lambda) \\ &\quad \times \exp \left[-\frac{i}{2} M\alpha + 2iN\beta - \frac{i}{2} \beta M \right] \end{aligned} \quad (8.62)$$

Proof. Multiplying both sides Eq.(5.23) by $\langle \lambda |$ and $|v\rangle$ it can be proved that

$$\begin{aligned} \langle \lambda | \mathcal{P}_c(\alpha, N) | v \rangle &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta \langle \lambda | \mathcal{D}_c(\beta, N + 2M) | v \rangle \\ &\quad \times \exp \left[\frac{i}{2} N\beta - iM\alpha - \frac{i}{2} \alpha N \right] \end{aligned} \quad (8.63)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Using Eqs.(8.47), (8.49) it can be concluded that

$$\begin{aligned}\tilde{v}(\alpha, N; \lambda) &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta |v(-\beta, -N - 2M; \lambda) \\ &\times \exp \left[\frac{i}{2} N\beta - iM\alpha - \frac{i}{2} \alpha N \right]\end{aligned}\quad (8.64)$$

Let $\alpha = \beta$ and $N = M$ then

$$\begin{aligned}\tilde{v}(\beta, M; \lambda) &= \frac{1}{2\pi} \sum_{K=-\infty}^{\infty} \int_0^{2\pi} d\alpha v(-\alpha, M - 2N; \lambda) \\ &\times \exp \left(-\frac{i}{2} M\alpha + 2iN\beta - \frac{i}{2} \beta M \right).\end{aligned}\quad (8.65)$$

□

Proposition 7. *Marginal properties*

$$\begin{aligned}\sum_{N=-\infty}^{\infty} G(z; \alpha, N; \lambda) &= 2\pi \lambda \left(-\frac{1}{2} \alpha \right) \Theta_3 \left[\frac{2^{-1} \alpha - z}{2}; \frac{i}{2\pi} \right] \\ \int_0^{2\pi} d\alpha G(z; \alpha, -2N; \lambda) &= 4\pi^2 \lambda_N \exp \left(-izN - \frac{1}{2} N^2 \right).\end{aligned}\quad (8.66)$$

where $2\pi \Theta_3 \left[\frac{2^{-1} \alpha - z}{2}; \frac{i}{2\pi} \right]$ represents position states and $2\pi \exp \left(-izN - \frac{1}{2} N^2 \right)$ represents momentum states (see Eq.(8.6)).

Proof. Using Eq.(8.29) with an extra $\sum_{N=-\infty}^{\infty}$ leads to

$$\begin{aligned}\sum_{N=-\infty}^{\infty} G(z; \alpha, N; \lambda) &= \int_0^{2\pi} dx \lambda(x) \sum_{N=-\infty}^{\infty} \exp \left(iNx + \frac{1}{2} iN\alpha \right) \\ &\times \Theta_3 \left[\frac{x + \alpha - z}{2}; \frac{i}{2\pi} \right]\end{aligned}\quad (8.67)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

$\sum_{N=-\infty}^{\infty} \exp(iNx + \frac{1}{2}iN\alpha) = \delta_c(x + \frac{1}{2}\alpha)$. Therefore,

$$\begin{aligned} \sum_{N=-\infty}^{\infty} G(z; \alpha, N; \lambda) &= 2\pi \int_0^{2\pi} dx \lambda(x) \delta_c\left(x + \frac{1}{2}\alpha\right) \\ &\quad \times \Theta_3\left[\frac{x + \alpha - z}{2}; \frac{i}{2\pi}\right] \end{aligned} \quad (8.68)$$

Using the dirac comb delta function it can be concluded that

$$\sum_{N=-\infty}^{\infty} G(z; \alpha, N; \lambda) = 2\pi \lambda\left(-\frac{1}{2}\alpha\right) \Theta_3\left[\frac{-\frac{1}{2}\alpha - z}{2}; \frac{i}{2\pi}\right]. \quad (8.69)$$

Integrating both sides of Eq.(8.29) by $\int_0^{2\pi} d\alpha$ leads to

$$\begin{aligned} \int_0^{2\pi} d\alpha G(z; \alpha, -2N; \lambda) &= \int_0^{2\pi} dx \int_0^{2\pi} d\alpha \lambda(x) \exp(-i2Nx - iN\alpha) \\ &\quad \times \Theta_3\left[\frac{x + \alpha - z}{2}; \frac{i}{2\pi}\right] \\ &= \sum_{K=-\infty}^{\infty} \int_0^{2\pi} d\alpha \exp\left(iK\alpha - iN\alpha - iKz - \frac{1}{2}K^2\right) \\ &\quad \times \int_0^{2\pi} dx \lambda(x) \exp(iKx - 2iNx) \quad (8.70) \end{aligned}$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

The integral over $d\alpha$ is equal to Dirac comb delta function, therefore

$$\begin{aligned}
&= 2\pi \sum_{K=-\infty}^{\infty} \delta(K - N, 0) \exp\left(-iKz - \frac{1}{2}K^2\right) \\
&\quad \times \int_0^{2\pi} dx \lambda(x) \exp(iKx - 2iNx) \\
&= 2\pi \int_0^{2\pi} dx \lambda(x) \exp(iNx - 2iNx) \\
&\quad \times \exp\left[-izN - \frac{1}{2}(N)^2\right]. \tag{8.71}
\end{aligned}$$

Using Eq.(5.5) it can be concluded that

$$\int_0^{2\pi} d\alpha G(z; \alpha, -2N; \lambda) = 4\pi^2 \lambda_N \exp\left[-izN - \left(-\frac{1}{2}N\right)^2\right]. \tag{8.72}$$

□

8.4 Wigner and Weyl functions for systems on a circle

The Wigner function of a state $|v\rangle$ on a circle is given by

$$W(\alpha, N; v) = \langle v | \mathcal{P}_c(\alpha, N) | v \rangle \tag{8.73}$$

and the Weyl function on a circle is equal to

$$\widetilde{W}(\alpha, N; v) = \langle v | \mathcal{D}_c(\alpha, N) | v \rangle. \tag{8.74}$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

The Wigner function in terms of $v(a, N; \lambda)$ is written as

$$\begin{aligned}
 W(\alpha, N; v) &= \frac{1}{4\pi^2} \sum_{M, K=-\infty}^{\infty} \int_0^{2\pi} d\chi \int_0^{2\pi} d\beta v(-\chi, -N + M - 2K; \lambda) \\
 &\quad \times v(\beta, M; \lambda)^* \exp\left(\frac{i}{2}\chi N - \frac{i}{2}M\chi + iN\beta\right) \\
 &\quad \times \exp\left(-\frac{i}{2}\beta M - iK\alpha + i\beta K\right). \quad (8.75)
 \end{aligned}$$

The Weyl function in terms of $v(a, N; \lambda)$ is given by

$$\begin{aligned}
 \widetilde{W}(\alpha, N; v) &= \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta v^*(\alpha, M; \lambda) v(-\alpha + \beta, -N + M; \lambda) \\
 &\quad \times \exp\left[\frac{1}{2}(-\alpha M + N\beta)\right]. \quad (8.76)
 \end{aligned}$$

Proof. In order to prove the Wigner function Eq.(8.21) is inserted into Eq.(8.73)

$$\begin{aligned}
 W(\alpha, N; v) &= \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta \langle v | \mathcal{D}_c(\beta, M) | \lambda \rangle \\
 &\quad \times \langle \lambda | \mathcal{D}_c(-\beta, -M) \mathcal{D}_c(\alpha, N) \mathcal{P}_c(0, 0) | v \rangle \quad (8.77)
 \end{aligned}$$

Using Eq.(8.47), the Wigner function can be expanded as

$$W(\alpha, N; v) = \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta v(\beta, M; r)^* \langle \lambda | \mathcal{D}_c(-\beta, -M) \mathcal{D}_c(\alpha, N) \mathcal{P}_c(0, 0) | v \rangle \quad (8.78)$$

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

Using the Eqs.(8.78), (8.49) it can be proved that

$$W(\alpha, N; v) = \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta v(\beta, M; \lambda)^* \tilde{v}(-\alpha + \beta, -N + M) \times \exp \left[\frac{i}{2} (-M\alpha + N\beta) \right] \quad (8.79)$$

Using Eq.(8.62), Eq.(8.79) becomes Eq.(8.75).

The Weyl function is proved by inserting Eq.(8.21) into Eq.(8.74)

$$\widetilde{W}(\alpha, N; v) = \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta \langle v | \mathcal{D}_c(\beta, M) | \lambda \rangle \langle \lambda | \mathcal{D}_c(-\beta, -M) \mathcal{D}_c(\alpha, N) | v \rangle \quad (8.80)$$

Using the Eq.(8.80), (8.47) it can be concluded that

$$\widetilde{W}(\alpha, N; v) = \frac{1}{2\pi} \sum_{M=-\infty}^{\infty} \int_0^{2\pi} d\beta v^*(\alpha, M; \lambda) v(-\alpha + \beta, -N + M; \lambda) \times \exp \left[\frac{1}{2} (-\alpha M + N\beta) \right]. \quad (8.81)$$

□

8.5 Discussion and Conclusion

An analytic representation of an arbitrary state $|v\rangle$ on a strip \mathcal{A} is defined in Eq.(8.3). The novel part of this chapter is the following. The resolution of identity of the coherent states when displacement operator acts on fiducial vector $|\lambda\rangle$ is given in Eq.(8.21). An analytic representation corresponding to the coherent states $\mathcal{D}_c(a, N)|\lambda\rangle$ is also given in Eq.(8.24), which is part of

CHAPTER 8. ANALYTIC REPRESENTATION OF QUANTUM SYSTEMS ON \mathbb{S} USING THETA FUNCTIONS

the novel work. We also study some properties of this analytic representation such as the reproducing kernel in Eq.(8.40), which plays a central role in this formalism. The reproducing kernel relations are defined in Eq.(8.44). Wigner and Weyl functions are given in Eqs.(8.75) and (8.76), respectively.

Chapter 9

Conclusion

In this work, quantum systems with d dimensional Hilbert space are studied. An analytic representation on a torus in terms of Theta functions is considered. The quantum states of Eq.(6.1) are represented by the analytic function in Eq.(6.4) on a torus. In finite quantum systems, the number of zeros of this analytic function is equal to d , Eq.(6.17) and the zeros obey the constraint of Eq.(6.19). In d dimensional Hilbert space the zeros define the state uniquely.

The novel work in this thesis is described by the following paragraphs. An analytic representation based on d^2 coherent states is given in Eq.(6.30). The reproducing kernel of Eq.(6.41) plays a central role in this formalism. An arbitrary state $|\phi\rangle$ can be expanded in terms of d^2 coherent states as in Eq.(6.49), and can be represented by the $\phi(a, b; \psi)$. The Wigner and Weyl functions for this state can be calculated from the coefficients $\phi(a, b; \psi)$, as in Eqs.(6.78), (6.79), respectively.

The d zeros are used to describe the time evolution of these systems. The

d paths of the zeros define completely the state. The time evolution operator of the system in Eq.(7.2), as well as, the motion of the d zeros on a torus are examined. For the time dependent system, the d zeros follow closed paths on a torus, which define the Hamiltonian. Some examples of the paths of the zeros using different Hamiltonians are given.

Also, the analogous formalism for systems with positions in a circle \mathbb{S} and momenta in \mathbb{Z} defined. The quantum states on a circle Eq.(8.1) are represented by the analytic function of Eq.(8.3) on a strip \mathcal{A} and obey the periodicity relation Eq.(8.4). The scalar product is given in Eq.(8.14).

Coherent states on a circle \mathbb{S} are also considered. An analytic representation based on these coherent states in Eq.(8.24) is studied. The reproducing kernel in this language is given in Eq.(8.44). An arbitrary state $|\nu\rangle$ can be written in terms of these coherent states as in Eq.(8.45), which is defined as $v(\alpha, N; \lambda)$. Using these coefficients, Wigner and Weyl functions can be also calculated as in Eqs.(8.75), (8.76), respectively.

9.1 Future work

The work may be extended in order to study the behaviour of the paths of the zeros using quantum systems which is not periodic. The work may be also extend to consider the analytic representations using entanglement, symplectic transformations and various unitary transformations.

Bibliography

- [1] A.M. Perolomov, *Generalized coherent states and their applications*, Springer, Berlin 1986.
- [2] S. Bergman, *The kernel function and conformal mapping*, Amer. Math.Soc, Rhode island, 1970.
- [3] B.C. Hall, *Holomorphic methods in analysis and mathematical physics*, Contemp. Math. 260,1. 2000.
- [4] A. Vourdas, *Analytic representations in quantum mechanics*, J. Phys. A.Math. Gen., 39:R65, 2006.
- [5] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform Part II*, Commun. Pure Appl. Math., 10.1002, Vol.20, 1967
- [6] A. Vourdas and R. F. Bishop, *Thermal coherent states in the Bargmann representation*,. Phys. Rev. A, 50:3331, 1994.
- [7] V. Bargmann, *Pure and Applied mathematics*, Commun.Pure Appl. Math. 14, 187. 1961.

BIBLIOGRAPHY

- [8] A.M. Perelomov, *Generalized coherent states and some of their applications*, Soviet physics Usp 20 703, 1977.
- [9] John R. Klauber, Bo-Sture Skagerstam, *Coherent states, applications in Physics and Mathematical Physics*, World scientific, Singapore.
- [10] S. Twareque Ali, J-P. Antoine, J-P Gazeau, U. A. Muller, *Coherent states and their generalizations: A Mathematical overview*, Rev. Math. Phys. 07, 1013.
- [11] J-P Gazeau, *Coherent states in quantum physics*, Wiley-Vch Verlag Gmbh and Co. KGaA, 2009.
- [12] A. M. Perelomov, *On the completeness of a system of a coherent states*. Theoretical maths and physics, 1971.
- [13] J.R. Glauber, *Coherent states and Incoherent states in radiation field*, Cambridge, 1963.
- [14] J.P. Gazeau, *Coherent states in quantum physics*, Wiley-VCH, Berlin, 2009.
- [15] J. Klauder, *Coherent states*, World scientific, Singabore, 1985.
- [16] P. Stovicek, J. Tolar, *Quantum mechanics in a discrete space-time*, Rep. math. Phys. 20, 157, 1984.
- [17] J. Tolar, G. Hadjitaskos, *Quantization on $Z(M)$ and coherent states over $Z(M) \times Z(M)$* , J. Physs. A30, 2509, 1997.

BIBLIOGRAPHY

- [18] A. Vourdas, *Quantum systems with finite Hilbert space*, Rep. Prog. Phys. 67 267, 2004.
- [19] A. Vourdas, *Quantum systems with finite Hilbert space: Galois fields in quantum mechanics*, J. Phys. A:Math. Theor. 40, R285, 2007.
- [20] M. Kibler, *An angular momentum approach to quadratic Fourier transform, Hadamard matrices, Gauss sums, mutually unbiased bases, the unitary group nad Pauli group*, J. Phys. A42, 35:3001, 2009
- [21] N. Cotfas, J.P Gazeau, *Finite tight frames and some applications*, J. Phys. A43, 193001, 2010.
- [22] G. Hadjitaskos, J. Tolar, *Feynman path-integral and ordering rules on discrete finite space*, Int. J. Theor. Phys. Vol.32,, 1993
- [23] T. Durt, B.G. Englert, I. Bengtsson, K. Zyczkowski *On mutually unbiased bases*, J. Quantum Information, 8, 535, 2010.
- [24] A.Vourdas, *Analytic representation in quantum mechanics*, J. Phys A:Math. Gen. 39 R65.
- [25] D. Mumford, *Tata lectures on Theta, Vols 1,2,3*, Birkhauser, Boston, 1983.
- [26] J. Igusa, *Theta functions*, Springer, Berlin, 1972.
- [27] J. Fay, *Theta functions on Riemann surfaces*, Springer, Berlin, 1973.

BIBLIOGRAPHY

- [28] R. Bellman, *A Brief Introduction to Theta functions*, Holt, Rinehart and Winston, New York, 1961.
- [29] H.M Farkas, I. Kra, *Riemann surfaces*, Springer, Berlin, 1980.
- [30] I.S Gradsteyn, I.M Ryzhik, *Table of integrals Series and Products*, London: Academic, 2000.
- [31] A. Vourdas, *Analytic representation in Quantum mechanics*, J. Physics. A39, R65, 2006.
- [32] N. Cotfas, J-P Gazeau, A. Vourdas, *Finite-dimensional Hilbert space and frame quantization*, J.Physics. A44, 175303, 2011.
- [33] J. Harkness, F. Morley, *Introduction to the theory of analytic functions*, G. E. Stechert and Co, New York, 1924.
- [34] R.P.Boas, *Entire functions*, Academic, New York,1954.
- [35] R. Yarlagadda, *A note on the eigenvectors of DFT matrices*, IEEE Trans. ASSP-25, 586, 1978.
- [36] B.Ja.Levin, *Lectures on Entire functions*, American Math.Soc, Rhode Island, 1996.
- [37] B.Ja.Levin, *Distribution of zeros of entire functions*, American Math.Soc, Rhode Island,1964.
- [38] L. Ahfols, *Complex analysis*, McGraw-Hill, New York, 1966.

BIBLIOGRAPHY

- [39] J. Bak, D. J. Newman, *Complex analysis*, Springer-Verlag, New York, 1982.
- [40] E. Hille,, *Analytic function theory, Vols.1-2*, Chelsea, New York, 1962.
- [41] W.Dickinson, K.Steiglitz, *Eigenvectors and functions of the discrete Fourier transform*, IEEE Trans. ASSP-30, 1, 1982
- [42] S. Zhang, A. Vourdas, *Analytic representation on finite systems*, J. Physics. A, Math. Gen, 37, 8349, 2004.
- [43] S. Zhang, A. Vourdas, *Analytic representation on finite systems*, corrigendum, J. Physics. A38, 1197, 2005.
- [44] N. Aronzajin, *Theory of reproducing Kernel*, Trans. Amer. Math. Soc. 68, 1950.
- [45] M. Tubani, A. Vourdas, S. Zhang, *Zeros in analytic representation of finite quantum systems on a torus*, Phys. Scr. 82, 038107, 2010.
- [46] L.S Schulman, *Techniques and applications of path integral*, Willey, New York, 1981.
- [47] L.S Schulman, *Approximate topologies*, J. Math. Physics, 12, 304, 1971.
- [48] J.S. Dowker, *Quantum mechanics and field theory on multiply connected and on homogenous spaces*, J.Phys. A: Gen. Phys. 5, 936, 1972.

BIBLIOGRAPHY

- [49] S. Zhang, A. Vourdas, *Phase space methods for particle on a circle*, J.Math. Physics, 44, 5084, 2003
- [50] F. Acerbi, G. Morchio and F. Strocchi, *Infrared singular fields and non-regular representations of CCR algebras*, J. Math. Phys. 34, 899, 1993.
- [51] F. Acerbi, G. Morchio and F. Strocchi, *Theta vacua, charge confinement and charged sectors from non-regular representations of CCR algebras*, Lett. Math. Phys. 27, 1, 1993.
- [52] M. G. G. Laidlaw, C. Morette - De Witt, *Feynman functional integrals for systems of indistinguishable particles*, Phys. Rev, D3, 1375, 1971.
- [53] H. Narnhofer, W. Thirring *KMS states for Weyl algebra*, Lett. Math. Phys. 27, 133, 1993.
- [54] K. Kowalski, J. Rembielinski, L.C. Papaloucas, *Coherent states for a quantum particle on a circle*, J.Phys. A29, 4149, 1996.
- [55] J. Gonzalez, M.A. Del Olmo, *Coherent states on a circle*, J.Phys. A31, 8841, 1998.
- [56] G. Chadzitaskos, P. Luft, J. Tolar, *Quantizations on the circle and coherent states*, J.Phys. A:Math. Theor. 45 244027, 2012.
- [57] G. Chadzitaskos, P. Luft, J. Tolar, *Coherent states on a circle*, J.Phys: Conf. Series, 284, 012016 , 2011.

BIBLIOGRAPHY

- [58] K. Kowalski, J. Rembielinski, *On the uncertainty relations and squeezed states for the quantum mechanics on a circle*, J.Phys. A35, 1405, 2002.
- [59] C. K. Zachos, D. B. Fairlie, T. L. Curtright, *Quantum mechanics in Phase space*, World Scientific, Singabore, 2005.
- [60] Roland Omnes, *Understanding Quantum mechanics*, Princeton University Press, Chichester, 1999.
- [61] Linus Pauling and E. Bright. Wilson, *Introduction to Quantum mechanics*, McGraw-Hill Book Co, New York, 1935.
- [62] Thomas F. Jordan, *Linear operators for Quantum mechanics*, Dover edition, 2006.
- [63] R.N. Bracewell, *The Fourier Transform and its application*, second edition, McGraw-Hill Book Co, New York, 1978.
- [64] C. C. Chong, A. Vourdas, *Fractional Fourier transforms for quasi-periodicity Bloch functions*, J. Opt. Soc. Am. A, 18, 2478, 2001.
- [65] D. Deutsch, *Uncertainty in quantum measurements*, Phys. Rev. Lett. 50, 631, 1983.
- [66] M. H. Partovi, *Entropic formulation and quantum measurements*, Phys. Rev. Lett. 50, 1883.
- [67] I. Bialynicki-Birula, *Entropic uncertainty relations*, Phys. Lett. 103, A253, 1984.

BIBLIOGRAPHY

- [68] J. Zak, *Finite translations in solid physics*, Phys. Rev. Lett. 19, 1385, 1967.
- [69] M. Boon, J. Zak, *Amplitudes on Von Neumann lattices*, J. Math. Phys. 22, 1090, 1981.
- [70] A. J. E. M Janssen, *Bargmann transform, Zak transform, and coherent states*, J. Math. Phys. 23, 720, 1982.
- [71] P. Evangelides, C. Lei, A. Vourdas, *Analytic representations using Theta functions for quantum systems on $\mathbb{Z}(d)$ and on \mathbb{S}* , arXiv.1502.05309.
- [72] P. Leboeuf, A. Voros, *Chaos-revealing multiplicative representation of quantum eigenstates*, J.Physics, A: Math. Gen.23, 1765, 1990.
- [73] M.Jabuni, *Analytic representations of finite quantum systems on a torus*, Thesis, 2010.

Chapter 10

Appendix

In order to prove Eq.(6.39), it should be first defined that

$$\begin{aligned} Y(z; a, b; \psi) &= \pi^{-1/4} \sum_{m=0}^{d-1} {}_x\langle m | \mathcal{P}_f(a, b) | \psi \rangle \\ &\quad \times \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \\ &= \pi^{-1/4} \sum_{m=0}^{d-1} \omega(-2ab + 2am) \psi_{-m+2b} \\ &\quad \times \Theta_3 \left[\frac{2\pi^2 m}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (10.1)$$

Substituting the variable m in Eq.(10.1) with $M = -m + 2b$, Eq.(10.1) becomes

$$\begin{aligned} Y(z; a, b; \psi) &= \pi^{-1/4} \sum_{M=0}^{d-1} \omega(2ab - 2aM) \psi_M \\ &\quad \times \Theta_3 \left[\frac{2\pi^2(-M + 2b)}{L^2} - z \frac{\pi}{L}; i \frac{2\pi}{L^2} \right] \end{aligned} \quad (10.2)$$

Using Eq.(6.5), Eq.(6.34) becomes

$$\begin{aligned}
 Y(z; a, b; \psi) &= \pi^{-1/4} \omega(2ab) \sum_{M=0}^{d-1} \psi_M \sum_{n=-\infty}^{\infty} \exp\left(-\frac{4i\pi^2 Mn}{L^2} - \frac{8i\pi^2 Ma}{L^2}\right) \\
 &\times \exp\left(\frac{i8n\pi^2 b}{L^2} - 2inz\frac{\pi}{L} - \frac{2\pi^2 n^2}{L^2}\right). \quad (10.3)
 \end{aligned}$$

Letting $n = N - 2a$, it follows that

$$\begin{aligned}
 Y(z; a, b; \psi) &= \pi^{-1/4} \omega(2ab) \sum_{M=0}^{d-1} \psi_M \exp\left(-\frac{4i\pi^2 M}{L^2}\right) \\
 &\times \sum_{N=-\infty}^{\infty} \exp\left[\frac{i8(N-2a)\pi^2 b}{L^2}\right] \\
 &\times \exp\left[-2i(N-2a)z\frac{\pi}{L} - \frac{2\pi^2(N-2a)^2}{L^2}\right] \quad (10.4)
 \end{aligned}$$

$$\begin{aligned}
 Y(z; a, b; \psi) &= \pi^{-1/4} \omega(2ab) \exp\left(-4iaz\frac{\pi}{L} - \frac{i16N\pi^2 ab}{L^2} + \frac{8\pi^2 a^2}{L^2}\right) \\
 &\times \sum_{N=-\infty}^{\infty} \sum_{M=0}^{d-1} \psi_M \exp\left(-\frac{4i\pi^2 M}{L^2} - 2iNz\frac{\pi}{L}\right) \\
 &\times \exp\left(\frac{i8N\pi^2 b}{L^2} - \frac{2\pi^2 N^2}{L^2} + \frac{8\pi^2 Na}{L^2}\right) \quad (10.5)
 \end{aligned}$$

Using Eq.(2.1), Eq.(10.5) becomes

$$\begin{aligned}
 Y(z; a, b; \psi) &= \pi^{-1/4} \omega(-2ab) \exp\left(-4iaz\frac{\pi}{L} + \frac{8\pi^2 a^2}{L^2}\right) \\
 &\times \sum_{M=0}^{d-1} \psi_M \Theta_3\left[-\frac{2\pi^2 m}{L^2} - z\frac{\pi}{L} + b\frac{4\pi^2}{L^2} - ia\frac{4\pi^2}{L^2}; i\frac{2\pi}{L^2}\right] \quad (10.6)
 \end{aligned}$$

Using Eq.(2.5), Eq.(10.6) becomes

$$\begin{aligned}
 Y(z; a, b; \psi) &= \pi^{-1/4} \omega(-2ab) \exp\left(-4iaz \frac{\pi}{L} + \frac{8\pi a^2}{L^2}\right) \\
 &\times \sum_{M=0}^{d-1} \psi_M \Theta_3 \left[\frac{2\pi^2 m}{L^2} + z \frac{\pi}{L} - b \frac{4\pi^2}{L^2} + ia \frac{4\pi^2}{L^2}; i \frac{2\pi}{L^2} \right] \quad (10.7)
 \end{aligned}$$

$$\begin{aligned}
 Y(z; a, b; \psi) &= \exp\left(\frac{8a^2\pi^2}{L^2} - 4iza \frac{\pi}{L}\right) \omega(-2ab) \\
 &\times \Psi\left(-z - ia \frac{4\pi}{L} + b \frac{4\pi}{L}\right) = F(-z; -2a; -2b; \psi) \quad (10.8)
 \end{aligned}$$

In this case the Eq.(3.23) is used, including the variable z and fiducial state $|\psi\rangle$

$$F(z; a, b; \psi) = \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} Y(z; \gamma, \delta; \psi) \omega(-b\gamma + a\delta). \quad (10.9)$$

Taking into account Eqs.(10.8), (10.9) and changing variables then

$$\begin{aligned}
 F(z; a, b; \psi) &= \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} F(-z; -2\gamma, -2\delta; \psi) \omega(-b\gamma + a\delta) \\
 &= \frac{2\pi}{L^2} \sum_{\gamma, \delta=0}^{d-1} F(-z; \gamma, \delta; \psi) \omega(2^{-1}b\gamma - 2^{-1}a\delta). \quad (10.10)
 \end{aligned}$$